Modification of chaotic systems limit sets by multiparametrical optimal correction

Yuri V. Talagaev
Department of Physics and Mathematics, Saratov State University Balashov Branch
Karl Marx, 29, Balashov, 412300, Russia
shangyi@narod.ru

Andrey F. Tarakanov
Department of Physics and Mathematics, Borisoglebsk State Teachers Training Institute
Narodnaya, 43, Borisoglebsk, 397160, Russia
aft777@mail.ru

***Abstract*—In the paper we investigate the problem when the aim of control is the modification of the system limit set (chaotic attractor) into the stable invariant set. This problem is on a joint of chaos control and bifurcation control methods, and the complete understanding of stabilization peculiarities requires the development of means of multiparametrical analysis. Hence the dynamic correction technique of parametric space of chaotic systems is offered. Thus the demand of small parametric changes naturally allows formulating the problem of optimal correction. Based on Pontryagin's maximum principle the corrective functions and necessary conditions of achievement of the invariant stable set are found. The efficiency of correction for chaos suppression is demonstrated on Lorenz system.**

I. Introduction

Chaos control is an intensively developing field of investigation that has showed its utility in a number of applications [1-2]. One of the ways of increasing the efficiency of chaos control techniques is involving the devices of optimal control theory [3]. In this direction some well-known control techniques were modified and new ones were offered.

The optimal chaos control is understood as the transition of a system from a given initial state $x(t_0)$ to the terminal one $x^*(t)$ either with the least inputs of energy at the system perturbation (energy-optimal control [4]), or in minimal time (time-optimal control [5]). Combined cost functions [6] can also be applied. A number of works in this field is devoted to the direct application of variational principles and their generalization in form of Pontryagin Maximum Principle [7] for the problem of chaotic dynamics. In [8] this apparatus was used to find unconstrained energy-optimal control function that provides the transition between co-existing in the phase space chaotic attractor and stable limit cycle. Without an additional performance functional in [9] for additive and scalar control subject to inequality constraints closed-loop system on the basis of maximum principle Bang-Bang control is achieved and an ingenious algorithm of stabilization of Lorenz system is presented. The possibility of stabilization of the periodically driven oscillator via optimal correction of system parameters based on chaos control techniques that use perturbations of an accessible system parameter [10] is demonstrated in [11].

In this paper we offer and prove the solution technique of the problem frequently appearing in applications, when the modification of the system's limit set (chaotic attractor) is the aim of the control. This very class of aims of the control is found in the works on chaos control [12] and bifurcations [13]. Its difference from the problems of classical control theory is that the quantitative characteristics of terminal state are not given beforehand. In the absence of information can be postulated only the desirable type of the limit set (that must be stable). In this very case for the situation of system parametric perturbation the term "corrective influence" is the most suitable one. Under the correction parameters of the system cease to be fixed and turn into new variables. Consequently qualitative changes of the mode of operation of the corrected nonlinear dynamic system will significantly depend on the bifurcation parameters values captured in the course of correction.

The typical way of investigation of dynamic system bifurcations providing strict analytical results is the study of differential equations depending on a single parameter. On this ground the methods based on the control bifurcation phenomenon were developed [14]. However the full picture of appearance (disappearance) of chaotic dynamics is possible only via multiparametric analysis. Having focused on evolution of the given unstable limit set, dynamic correction of the space of system parameters is one of the ways of progress in this direction. Thus the demand of small parametric changes naturally leads us to strengthening the correction problem via the demand of optimal modification of the system's limit set. In studies of the given class of issues special attention is paid to invariant properties and conditions of stability of corrected systems.

II. Problem statement and its solvability

Consider a nonlinear dynamic system in the following form

$$\dot{x} = f(x, p), \tag{1}$$

where $x \in R^n$ is the state vector, $p \in R^m$ $(m \le n)$ is a vector of parameters, depending on the parameter values the system can exist either in chaotic or in regular mode. Let under the given values of parameters p and initial state

$$x(0) = x_0 \in M_0 = \{(x_0 \ne 0) \in B_A\}$$

the system (1) have chaotic attractor A_p as a limit set (B_A denotes the basin of attractor $B_A(A_p) \subseteq R^n$ and attractor A_p restricted i.e. $\| x(t) \| < D$, $D > 0$). Let also chaotic attractor of the system (1) include one or several unstable equilibrium states x_e such that $\dot{x}_e = f(x_e, p) = 0$, which taken altogether define the set $E = \{x \in B_A : f(x,p) = 0\}$.

The essence of the main problem is the following. How can we provide the stability of the system (1) via minimal correction of its parameters? There are two ways of solution for this problem. The first one presupposes the formulation of the optimization problem for fixed measuring the parameters $\widetilde{p}_j = p_j + \delta h_j$, $\delta h_j = const$ and after the system (1) is linearized, may by investigated via well-known Hurwitz criterion [15]. In the paper we study the second way of solution when corrective perturbations have dynamic character.

Let us transform the system (1) to the form

$$\dot{x} = f(x, p^*), \qquad (2)$$

where accessible parameters p_j are corrected according to the rule

$$p_j^*(t) = p_j(1 + h_j(t)),\ j = \overline{1,r},\ r \le m. \qquad (3)$$

As the natural demand $\| p^* - p \| \to \min$ is equivalent to $\| h \| \to \min$ the dynamic correction problem is reduced to finding the vector-function $h(t) = (h_1(t),...,h_r(t))$ with the constraint set

$$U = \left\{ h(\cdot) \in C[0,T] \mid \| h(t) \| \le a, t \in [0,T], T >> T^* \right\},$$

($C[0,T]$ is the class of continuous constrained functions defined at the time interval $[0,T]$, T – a final, but not fixed moment of time definable from the duration of transient process T^*), which for $x_0 \in M_0$ with the condition of minimum energy expenses

$$\frac{1}{2}\int_0^T \| h(t) \|^2 \, dt \to \min_{h \in U} \qquad (4)$$

provides the stability of the system (2). As the trajectories of an uncorrected system (1) are globally bounded and locally unstable, the demand of stability is considered as providing optimal dynamic modification of an unstable limit set (chaotic attractor) to a stable one. Finally, correctable parameters define the structure of the system, so that the result of stabilization will depend on inherent properties of a correctable system.

Note that the choice of the function $h(t)$, $t \in [0,T]$, is simultaneously constrained by the restriction $\| h(t) \| \le a$ and the demand (4). The first considers obeying the demand of small perturbation on system parameters and presupposes the choice of the minimal quantity $a_{\min}$. The second shows natural want to carry out the dynamic modification of chaotic attractor into the stable set with minimal energy costs on correction.

After the correction of the system (2) on rule (3) we get a system linear at h. Therefore the system (2) can be rewritten in the form

$$\dot{x} = f(x,p) + g(x,p)h, \qquad (5)$$

where the corrective part $g(x,p)h$ is the n-dimensional vector-function, its r component looks like $g_j(x, p_j)h_j$ and $n - r$ are equal to zero.

The requirement for providing stability for the system (5) leads us to the target set

$$M_E = \{(x,h) : f(x,p) + g(x,p)h = 0\}.$$

Note that the structure of the set $M_E \supseteq E$ depends on the number and place of unperturbed system equilibrium points x_e.

Definition 1. *Optimal modification of the limit set* A_P *into the set* M_E *(*$A_P \mapsto M_E$*) is a process* $(x(t), h(t))$, $t \in [0,T]$, *such that for all the initial states* $x_0 \in M_0$ *in case the condition* (4) *is met, it provides the achievement in some final time* T^* *(moment of stabilization) the set* M_E *and remaining of the corrected trajectory for all* $t \ge T^*$ *in it.*

As result, for the controlled object (5) our problem is reduced to Lagrange optimization problem: *it is necessary to find an admissible function* $h(t)$, $t \in [0,T]$, *that transfers at the interval* $[0,T]$ *the corrected system* (5) *with initial state* $x_0 \in M_0$ *into the set* M_E *providing demand* (4) *and the fulfillment of the condition* $A_P \mapsto M_E$ (see. Def. 1). Solution conditions of the above problem can be obtained with Bellman Dynamic Programming method (for the details see [3]).

Theorem 1. Let chaotic attractor A_p *be the limit set for the system* (1). *Then for the corrected system* (5) *corresponding to the system* (1) *there exists the process* $(x(t), h(t))$, $t \in [0,T]$, *that provides* $A_P \mapsto M_E$.

Proof. On the basis of Hamilton-Jacobi-Bellman (HJB) equation

$$\min_{h \in U}(V_x(x)(f(x,p) + g(x,p)h) + 0.5 \| h \|^2) = 0$$

(hereafter f_x denotes the partial derivative of the function f at x) we see that along the optimal trajectory there is

$$\dot{V}(x) = V_x(x)(f(x,p) + g(x,p)h) = -0.5 \| h \|^2 < 0. \ (6)$$

Moreover Bellman function, being the solution of HJB equation, is defined as

$$V(x_0) = 0.5 \min_{h \in U} \int_0^T \| h(t) \|^2 \, dt > 0. \qquad (7)$$

The execution of inequalities (6) and (7) for the pair $(x(t), h(t))$ is automatically follows from the peculiarities of target function in (4), which is quadratic and positive definite. Hence, Bellman function $V(x)$ is positive and its total derivative is negative definite (6). It means that Bellman function is at the same time optimal Lyapunov function and there exists a process $(x(t), h(t))$ such that $\lim_{t \to +\infty} x(t) = x_e \in E \subseteq M_E$, that is the set M_E is accessible.

Thus there exists such terminate moment of time T^* that for all $t \geq T^*$ we have $x(t) \in M_E$. This completes the proof.

III. Pontryagin Maximum Principle and the Search of the Stable Invariant Set

In the previous part the correction problem was formulated and it was shown that the process $(x(t), h(t))$, wherein the equilibrium set M_E is accessible, exists. Now we will show that Pontryagin Maximum Principle [7] allows to move further and to get necessary conditions of the limit set modification. Based on the link $\psi(t) = -V_x(x(t))$ let us make Hamilton-Pontryagin function for the system (5)

$$H(x,h,\psi) = \psi^T (f(x,p) + g(x,p)h) - 0.5 \| h \|^2 .$$

To make vector-function $h^*(t) \in U$ and the corresponding trajectory $x^*(t)$ with boundary conditions $x^*(0) \in M_0$, $x^*(T) \in M_E$ optimal in the sense of Lagrange task (4), there should exist such a non-zero vector-function $\psi(t) \in R^n$ satisfying the system $\dot{\psi}(t) = -H_x(x^*, h^*, \psi)$ that function $h^*(t) = h^*(x(t), \psi(t))$ satisfies the maximum condition

$$H(x^*(t), h^*(t), \psi(t)) = \max_{h \in U} H(x^*(t), h, \psi(t)) \equiv 0 \quad (8)$$

and in the points $x^*(0)$ and $x^*(T)$ the conditions of transversality $\psi(0) \perp \Omega(x^*(0))$ and $\psi(T) \perp \Omega(x^*(T))$ are performed, where $\Omega(x^*(0))$ and $\Omega(x^*(T))$ – are tangent manifolds to the sets M_0 and M_E in the points $x^*(0) \in M_0$ and $x^*(T) \in M_E$ respectively.

Optimal corrective function that can be defined from (8) looks like the function of saturation:

$$H_h(x,h,\psi) = 0 \quad \Rightarrow \quad \tilde{h} = \psi^T g(x,p),$$

$$h^*(t) = \begin{cases} \tilde{h}(t), & if \;\; \tilde{h}(t) \in U, \\ a \cdot sign(\tilde{h}(t)), & if \;\; \tilde{h}(t) \notin U, \end{cases} \quad (9)$$

Having defined (9), as a result we get the system

$$\begin{cases} \dot{x} = H_\psi(x, h(x,\psi), \psi), & x(0) = x_0, \\ \dot{\psi} = -H_x(x, h(x,\psi), \psi), & \psi(0) \perp \Omega(x(0)), \end{cases} \quad (10)$$

the integration of which in view of (9) gives the optimal pair $x^*(t)$, $h^*(t) = h(x^*(t), \psi(t))$. Note that condition $\psi(0) \perp \Omega(x(0))$ can easily be performed via corresponding choice of point $x(0) \in M_0$, and condition $\psi(T) \perp \Omega(x^*(T))$ on the set M_E is performed automatically according to the Theorem 1.

Theorem 2. Let the function $h^*(t)$ for the system (5) *be obtained from the conditions* (8)-(10). *Then there exist a number $a_{\min} \in (0,1)$ and final time $T^* \in [0,T]$ so that the corrective function* (9) *and trajectory $x^*(t)$ corresponding to it create the process $(x^*(t), h^*(t))$, that provides $A_P \mapsto M_E$, so that for the system* (5) *the set M_E is a stable invariant one.*

Proof. First of all note, that the peculiarity of a system like (10) is that it can not be stable on the variables x and ψ at a time. For instance, if the system is stable on x it will necessarily be unstable on ψ. In our case before the correction of parameters of the system (1) its trajectories are locally unstable and globally bounded ($\| x \| < D$). Having achieved some bounded area – chaotic attractor – they stayed in it at $t \to \infty$. Thus the phase space expansion of (5) by introducing conjugate phase variables leaves the condition $\| x(t) \| < D$ in force and leads to the unbounded increase of the norm $\| \psi(t) \|$ at the interval $[0,T]$.

Without restrictions the function $\tilde{h} = \psi^T g(x,p)$ in view of unstable trajectory $\psi(t)$ is growing speedily so that it is difficult to realize any of solution techniques of (10). However, because of (9) corrective function $h^*(t)$ is bounded, that is $\| h^*(t) \| \leq a$ and at $t \geq 0$ satisfies Theorem 1.

In the investigation of invariant features of the set M_E equation (8) plays an important role:

$$\psi^T (f(x^*,p) + g(x^*,p)h^*) - 0.5 \| h^* \|^2 = 0 .$$

As $\psi(t) \neq 0$ then at $t > 0$ with regard to the increase of the norm $\| \psi(t) \|$ at $t \to T$ we get

$$\| f(x^*,p) + g(x^*,p)h^* \| \leq \frac{a^2}{2 \| \psi \|} \to 0 .$$

Based on it and theorem 1 there exist numbers $a_{\min} \in (0,1)$ and $\varepsilon > 0$, as well as the moment of stabilization $T^*(\varepsilon) \in [0,T)$ so that at $\| h^* \| \leq a_{\min}$ and $t \geq T^*(\varepsilon)$ the inequality $\| f(x^*,p) + g(x^*,p)h^* \| < \varepsilon$ is fulfilled. Hence for all $x_0 \in M_0$, $t \geq T^*(\varepsilon)$

$$\inf_{z \in M_E} \| (x^*, h^*) - z \| \to 0 ,$$

that is $(x^*(t), h^*(t)) \in O_\varepsilon(M_E)$, where $O_\varepsilon(M_E)$ – is ε-vicinity of the set M_E. It means that the correction of parameters leads to the modification of chaotic attractor into the stable invariant set M_E. This completes the proof.

IV. Optimal Correction of Parameters of Lorenz System

Consider the correction procedure applied to Lorenz system [16]:

$$\dot{x}_1 = \sigma(x_2 - x_1),$$
$$\dot{x}_2 = r\, x_1 - x_1\, x_3 - x_2,$$
$$\dot{x}_3 = x_1 x_2 - b\, x_3 .$$

With vector of parameters under correction $p = (\sigma, r, b) = (10,\ 28, 8/3)$, all the trajectories with initial state $x_0 \in B_A$ are attracted to the bounded set A_P known as Lorenz attractor. The target set M_E defines three equilibria: $[0,0,0]^T$ and $[\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1]^T$. For the given values of components of vector p they are unstable.

Using (5),(8)-(10) we get the system

$$\begin{aligned}
\dot{x}_1 &= \sigma(1+h_1)(x_2 - x_1),\\
\dot{x}_2 &= r(1+h_2)\,x_1 - x_1\,x_3 - x_2,\\
\dot{x}_3 &= x_1 x_2 - b\,(1+h_3)\,x_3,\\
\dot{\psi}_1 &= \psi_1\sigma(1+h_1) - \psi_2(r(1+h_2) - x_3) - \psi_3 x_2,\\
\dot{\psi}_2 &= -\psi_1\sigma(1+h_1) + \psi_2 - \psi_3 x_1,\\
\dot{\psi}_3 &= \psi_2 x_1 + \psi_3 b(1+h_3),
\end{aligned}$$

where the components of the corrective function $h = (h_1, h_2, h_3)$ look like

$$h_j = \begin{cases} \tilde{h}_j, & if \ \ \|\tilde{h}\| \le a,\\ a\,sign(\tilde{h}_j), & if \ \ \|\tilde{h}\| > a, \end{cases}$$

$$\tilde{h}_1 = \sigma\psi_1(x_2 - x_1)\,,\ \tilde{h}_2 = r\psi_2 x_1\,,\ \tilde{h}_3 = -b\,\psi_3 x_3\,.$$

The simulation was carried out with initial states

$$x_0 = (x_{01} = 1, x_{02} = 1, x_{03} = 1)^T\,,$$

$$\psi_0 = (-x_{02}x_{03}, -x_{03}, x_{02} + x_{01}x_{02})^T. \qquad (11)$$

Fig. 1,2 show the results for two variants of correction $h = (h_1, h_2, h_3)$ (fig.1) and $h = (0, h_2, 0)$ (fig.2), obtained for $x_0 = (1,1,1)^T$ and $a = 0.15$. The end of transient process (x, h), $t \in [0, T^*]$, that provides the modification $A_P \mapsto M_E$ at minimum expenses of energy is the choice (by the system (5)) of one of two non-zero positions of equilibrium and attraction of the system trajectory in its vicinity.

In a numerical experiment the time T was chosen a fortiori more than the time of transient process of stabilization. Changes of the boundary size $a \in (0,1)$ showed that with its extension the stabilization comes sooner. The dynamic of corrective functions changes-over in fig. 1,2(b) demonstrate that the system stabilizes in the interval $[0, T^*]$, after which the dynamic of corrective functions either acquires a regular character (h_1 in fig.1(b)), or saturates itself at the edge of the boundary (h_2, h_3 in fig.1,2(b)).

The comparison of fig. 1(b) and 2(b) shows that the character of corrective perturbation on parameter r is preserved. The correction function saturation observable on the lower edge of the boundary displays bifurcational features of the system, arising at the change of the given parameter. Via linearization of the system in the vicinity of the stabilized position of equilibrium it is easy to make sure [14] that the obtained value of the boundary lies in the

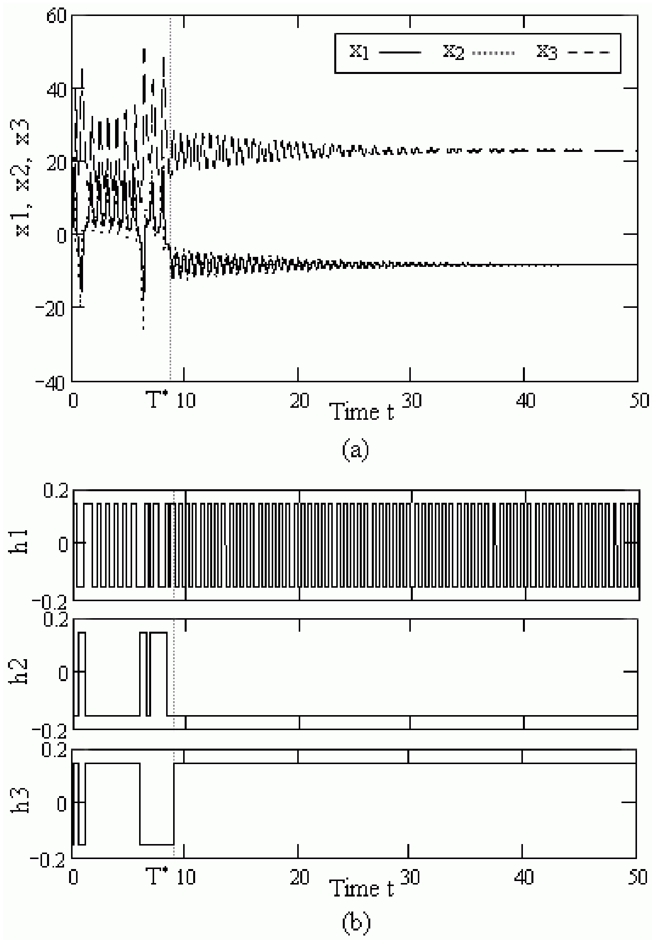

Fig. 1. Correction of Lorenz system on all parameters: (a) time responses; (b) corrective functions.

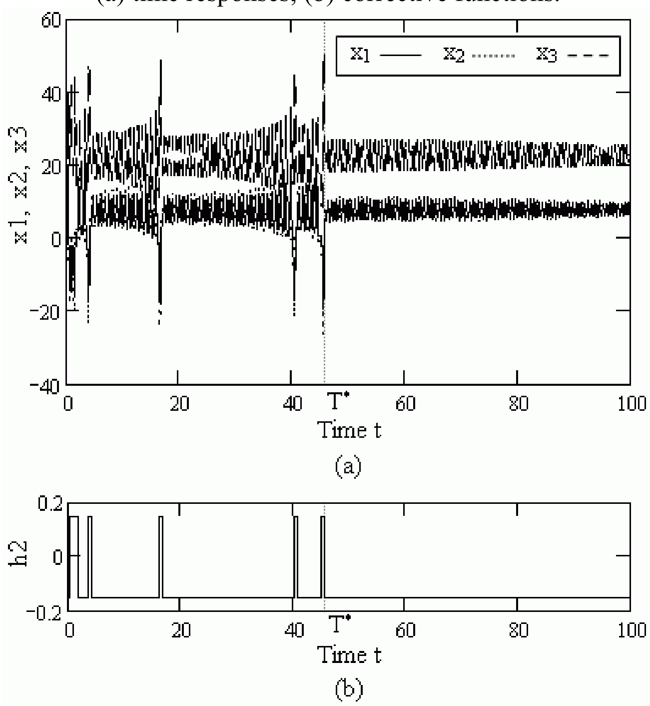

Fig. 2. Correction of Lorenz system on parameter r: (a) time responses; (b) corrective functions.

admissible region of stability, available from Hurwitz criterion.

As it follows from theorem 2, the set M_E is invariant. Thus the condition of transversality $\psi_0 \perp \Omega(x_0)$ can be investigated from two points of view. On the one hand it allows making the optimal choice of initial condition for the pair (x_0, ψ_0). This possibility was exploited above.

On the other hand it may be required to keep the orthogonality of the vectors x and ψ through the correction process. Then according to (11) we have

$$\psi_1 = -x_2 x_3, \psi_2 = -x_3, \psi_3 = x_2 + x_1 x_2 . \quad (12)$$

Note that the link (11),(12) is a variant of the general condition $\sum_{i=1}^{3} x_i \psi_i = 0$. From (12) we have corrective functions

$$\tilde{h}_1 = -\sigma x_2 x_3 (x_2 - x_1) , \ \tilde{h}_2 = -r x_1 x_3 , \ \tilde{h}_3 = -b x_2 x_3 (1 + x_1) .$$

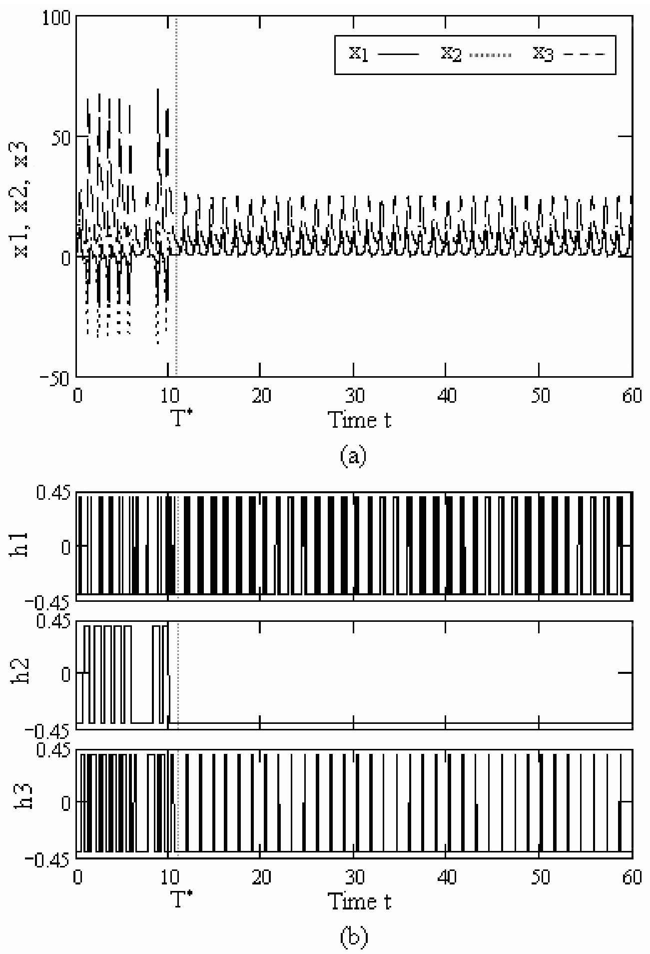

Fig. 3. Correction of Lorenz system on all parameters fulfilling the condition of transversality along the trajectory: (a) time responses; (b) corrective functions.

In this case the realization of correction algorithm is essentially simplified, as it is enough to integrate the corrected system's own equations in the view of (9) without involving the conjugate system. The results for $x_0 = (1,1,1)^T$ and $a = 0.4$ are shown in fig. 3. As can be seen, in this case the result of correction is the limit cycle, localized in the vicinity of one of the equilibrium positions (fig. 3(a)) and a bit different character of the functions h_j (cf. fig. 1(b) and fig. 3(b)).

V. CONCLUSIONS

In the paper theoretical apparatus of chaotic systems parametric space optimal correction is presented. It is analytically proved and numerically confirmed that existence of constrains is the necessary condition of achievement of the invariant stable set. Note that not only chaotic system own trajectories constraints, but also the restrictions put on the corrective functions are essential. Several variants of correction are investigated and the peculiarities of optimal corrective functions $h = h(t)$ are cleared out. There is shown the possibility of synthesis of corrective functions $h = h(x(t))$ based on fulfilling the condition transversality along the trajectory.

Providing optimal modification of the limit set, the correction of the system gives information about physical properties of the described object as system reaction on the total parametric perturbation. Correction technique is applicable to the wide family Lorenz-like chaotic systems and may be used in case of unspecified steady states. Thus the comparison and efficiency rating of perturbation on a single parameter is possible. After that, in practical applications, when the number of parameters available for correction is restricted the perturbation of only one parameter becomes more grounded and its results are predictable. In general, parametric correction may be considered as the link between the methods of chaos control and control bifurcation.

ACKNOWLEDGMENT

The authors would like to thank prof. V.A. Gorelik (from A. A. Dorodnitsin Computing Centre of Russian Academy of Science) for valuable remarks.

REFERENCES

[1] A. L. Fradkov, R. J. Evans, and B. R. Andrievsky, "Control of chaos: methods and applications in mechanics," *Phil. Trans. R. Soc. A*, vol. 364, pp.2279–2307, 2006.

[2] G. Chen and X. Dong, *From chaos to order: perspectives, methodologies and applications.* Singapore: World Scientific, 1998; A. L. Fradkov and A. Yu. Pogromsky, *Introduction to control of oscillations and chaos*. Singapore: World Scientific, 1998.

[3] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems,* Texts in

Applied Mathematics 6. New York: Springer-Verlag, 2nd edition, 1998.

[4] G. Chen, "Optimal control of chaotic systems," *Int. J. Bifurcation and Chaos,* vol. 4, pp. 461-463, 1994; H. D. I. Abarbanel, L. Korzinov, A. I. Mees et al., "Optimal control of nonlinear systems to given orbits," *Syst. Contr. Lett.*, vol. 31, pp. 263-276, 1997; R. Mettin, "Control of chaotic maps by optimized periodic inputs," *Int. J. Bifurcation and Chaos*, vol. 8, no. 8, pp. 1707-1711, 1998;. B. I. Epureanu and E. H. Dowell, "Optimal Multi-Dimensional OGY Controller for Stabilizing Fully Unstable Limit Cycles," *Physica D*, vol. 139, no.1-2, pp. 87-96, 2000.

[5] E. J. Kostelich, C. Grebogi, E. Ott, and J. A. Yorke, "Higher dimensional targeting," *Phys. Rev. E*, vol. 47, pp. 305-310, 1993; M. Paskota, A. I. Mees, and K. L. Teo, "Geometry of targeting of chaotic systems," *Int. J. Bifurcation and Chaos*, vol. 5, pp. 1167-1173, 1995; E. M. Bollt and E. J. Kostelich, "Optimal targeting of chaos," *Phys. Lett. A*, vol. 245. pp. 399-406, 1998; P. Gora and A. Boyarsky, "Optimal Control of Chaotic Systems," *Int. J. Bifurcation and Chaos,* vol. 11, no. 7, pp. 2007-2018, 2001; J. Starrett, "Time-optimal chaos control by center manifold targeting," *Phys. Rev. E*, vol. 66, p. 046206, 2002.

[6] C. Piccardi and S. Rinaldi, "Optimal control of chaotic systems via peak-to-peak maps," *Physica D*, vol.144. pp. 298–308, 2000; H. Richter, "On optimality of local control of chaos," *Int. J. Bifurcation and Chaos*, vol. 11, pp. 871-879, 2001.

[7] L. S. Pontryagin, V. G. Boltyanski, R. V. Gamkrelidze, and E. F. Mischenko, *The Mathematical Theory of Optimal Processes*. New York: John Wiley & Sons, 1962.

[8] I. A. Khovanov, D. G. Luchinsky, R. Mannella, and P. V. E. McClintock, "Fluctuations and the energy-optimal control of chaos," *Phys. Rev. Lett*., vol. 85, no.10, pp. 2100–2103, 2000.

[9] Yu.-C. Tian, M. Tade, and D. Levy, "Constrained control of chaos," *Phys. Let. A*, vol. 296, pp.87-90, 2002.

[10] V. V. Alekseev, A. Y. Loskutov, "Control of a system with a strange attractor through periodic parametric action", *Sov. Phys. Dokl*, vol. 32, pp.1346-1348, 1987; R. Lima, M. Pettini, "Suppression of Chaos by Resonant Parametric Perturbations", *Phys. Rev. A*, vol. 41, pp. 726-733, 1990; R. Chacon, J. Diaz Bejarano, "Routes to suppressing chaos by weak periodic perturbations," *Phys. Rev. Lett.,* vol. 71, pp. 3103-3106, 1993; K. A. Mirus, J. C. Sprott, "Controlling chaos in a high dimensional system with periodic parametric perturbations," *Phys. Lett. A.,* vol. 254, pp. 275-278, 1999.

[11] Yu. V. Talagaev and A. F. Tarakanov, "Suppressing Chaos through Optimum Correction of the Control Parameters in a Duffing–van der Pol Oscillator," *Technical Phys. Lett.*, vol. 32, no.12, pp. 1043–1046, 2006.

[12] E. Ott, C. Grebogi, and J. A.Yorke, "Controlling chaos," Phys. Rev. Lett., vol. 64, no. 11, pp. 1196–1199, 1990.

[13] H. O. Wang and E. H. Abed, "Bifurcation control of a chaotic system," *Automatica*, vol. 31, pp. 1213–1226, 1995.

[14] G. Chen, J. L. Moiola, and H. O. Wang, "Bifurcation control: Theories, Methods, and Applications," *Int. J. Bifurcation and Chaos*, vol. 10, no. 3, pp. 511-548, 2000; A. Krener, W. Kang, and D. E. Chang, "Control bifurcations," *IEEE Trans. on Automatic Control*, vol. 49, no. 8, pp.1231-1246, 2004.

[15] V. A. Gorelik, Y. V. Talagaev, and A. F. Tarakanov, "Optimal parametric correction of dynamics of systems with chaotic attractor," *Modeling, Decomposition and Optimization of Complex Dynamic Processes*. Moscow: A. A. Dorodnitsin Computing Centre of Russian Academy of Science, pp. 34-46, 2006. (in Russian)

[16] E. N. Lorenz, "Deterministic non-period flow," *J. Atoms. Sci.,* vol. 20, pp. 130-141, 1963.