SOME ASPECTS OF A PARAMETRIC SIMPLE PENDULUM DYNAMICS

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Abstract

By constructing basins of attractions and by numerical continuation techniques, some unexpected aspects of parametric resonances in a single pendulum are described. Odd resonances not predictable by the Melnikov method for vertical excitation have been numerically investigated for degenerate period-3 attractors. For excitations tilted up to $7\pi/180$, there are two different types of period-3 attractors, for $\phi>7\pi/180$ just one type of period-3 attractor is observed.

Key words

Parametric pendulum, Numerical continuation, basins of attraction.

1 Introduction

The studies of parametric resonances started with the first observation done by Faraday[Faraday, 1831] of a resonance in surface waves of a fluid submitted to vertical excitation exhibiting twice the period of the excitation.

Parametric resonances are related to the temporal modulation of one or more of system physical parameters. For single or multiple arm pendulums one of the most striking effects is the stabilization of arms in the inverted position [Stephenson, 1908], [Sartorelli, Lacarbonara, 2012]; [Depetri, Sartorelli, Marin, Baptista, 2016] and references therein. [Koch, Leven, 1985] applied the Melnikov theorem for subharmonic bifurcations and obtained the parameter ranges for the existence of oscillations with period that is an even multiple of the perturbation, but they could not obtain the ranges for the odd oscillations.

[Depetri, Sartorelli, Marin, Baptista, 2016] also applied the Menilkov method and showed that for the tilted case the odd resonances are predictable. Here, we are presenting some aspects that cannot be predicted by such method.

2 Equations of motion

The equation of the motion is given by Eq. 1, see also Fig. 1.

$$\ddot{\theta} = -\Omega_0^2(1-\frac{\ddot{y}_p}{g})\sin(\theta) - b\dot{\theta} - \Omega_0^2\frac{\ddot{x}_p}{g}\cos(\theta) \qquad (1)$$

where $\Omega_0=2\pi f_0$ related to the fundamental frequency, g the gravity acceleration, b the viscous dissipation and $\mathbf{s} = (x_p,y_p)$ is the pivot pendulum position. The pivot, attached to slider car in a rail which makes an angle ϕ with the vertical direction, can be put to oscillate periodically with amplitude A and frequency f_0

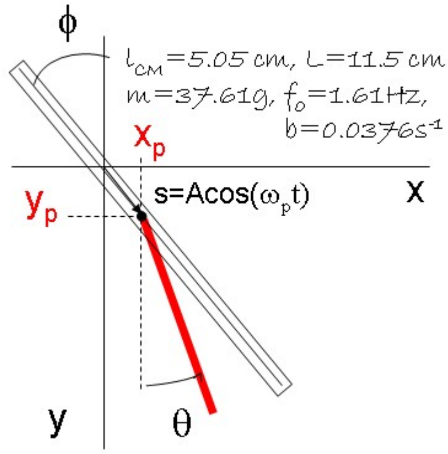

Figure 1. Diagram of a parametric pendulum under a generic tilted direction ϕ. l_{CM} is the center of mass position, L is the total length, m is the mass, b the viscous dissipation and f_0 the fundamental frequency.

exciting the pendulum dynamics. The absolute value of the pendulum angular speed ω can be measured with the help an optical rotary encoder while the pivot speed using an optical linear encoder. In this way we can obtain the amplitude A and the excitation frequency. To obtain basins of attraction we have integrated the autonomous dimensionless equations Eq. (2) using initial conditions described below and by measuring the attractor period.

$$\begin{aligned}
&\dot{\theta} = \omega \\
&\dot{\omega} = -\sin\theta - PX\sin(\theta-\phi) - \beta\omega \\
&\dot{X} = X - \Omega_p Y - X(X^2+Y^2) \\
&\dot{Y} = Y + \Omega_p X - Y(X^2+Y^2) \\
&X(0) = 1, \quad Y(0) = 0 \\
&\Omega_p = \frac{\omega_p}{\omega_0}, \qquad P = \frac{\omega_p^2 A}{g}, \qquad \beta = \frac{b}{\omega_0}
\end{aligned} \tag{2}$$

3 Results and discussion

In Fig 2(A) we show basins of attraction for the vertically excited pendulum (ϕ=0), with f_p=5Hz and A=2.02 cm.

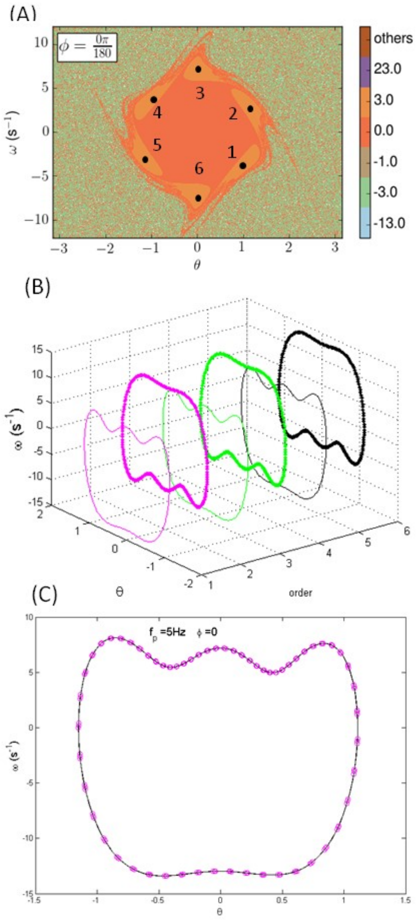

Figure 2. In (A), basins of attraction for the vertically excited pendulum (ϕ=0). Each color corresponds to attractors of different periodicity, with period 0 indicating fixed points while negative periods denote rotations. In (B), phase portraits obtained with the initial conditions indicated by dots in (A). There are two kinds of period-3 attractors (odd and even), conjugated by inversion symmetry as depicted in (C) , where (θ,ω) are plotted for odd order attractors while ($-\theta$,$-\omega$) are plotted for even ones.

Each colour corresponds to attractors of different periodicities. Initial conditions in the dark orange region are attracted to a fixed point. The six light orange regions, ordered in counterclockwise, constitute the basin for period-3 oscillations. It should be noticed the inversion symmetry of the six priod-3 regions. The negative periods denote rotations with initial conditions distributed sparsely. In Fig 2(B), we depict phase portraits obtained by integrating autonomous dimensionless equations Eq. (2) from the initial conditions indicated by dots in 2(A). There are two classes of period-3 orbits conjugated by (θ, ω) inversion as depicted in Fig. 2(C), where we plot (θ, ω) for odd attractors (initial conditions in regions 1, 3 and 5) and *(-θ, -ω)* for even ones (initial conditions 2, 4, 6).

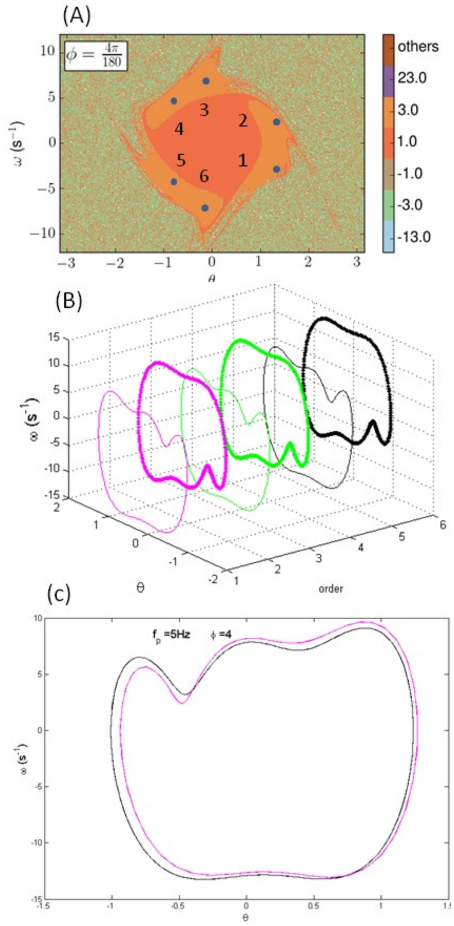

Figure 3. In (A) basins of attraction for tilted excitation (*ϕ=4π/180*). Each color corresponds to attractors of different periodicity, while negative periods denote rotations. In (B), phase portraits obtained by integrating the model from the initial conditions indicated by dots in (A). In (C), phase portrait for the two kinds of attractor present in (B), where (θ,ω) are plotted for odd-order trajectories and *(θ, $-\omega$)* for even ones.

Taking randomly any initial conditions in the odd regions as well as in the even regions we obtain the same result. Therefore, each one of these attractors presents the same loci of saddle-node bifurcations obtained separately by numerical continuation of each one as shown in Fig. 4(A).

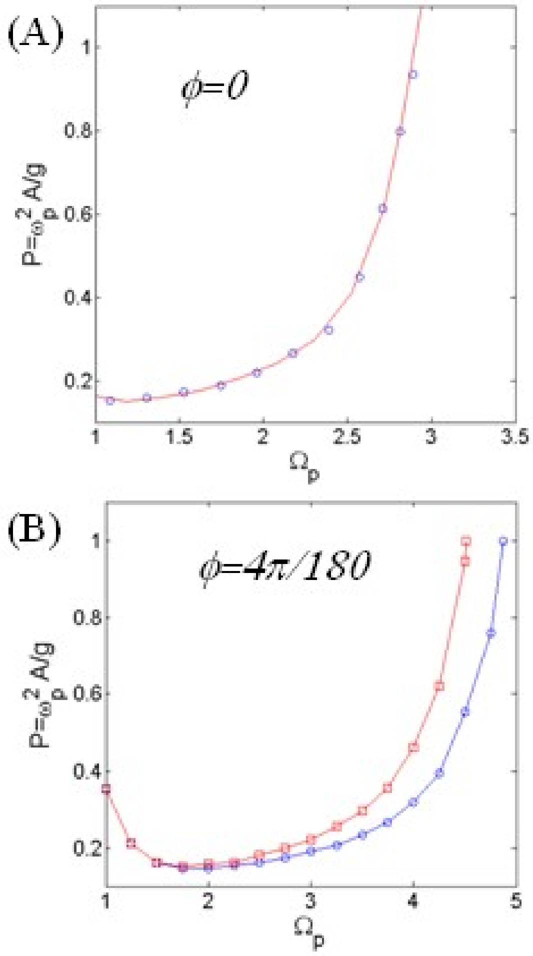

Figure 4. In (A) is shown the common loci of saddle-node bifurcations obtained by numerical continuation technique of each degenerate attractor. In (B) the numerical continuation results for non-degenerate attractors.

We repeated the same procedure for a small angle $\phi=4\pi/180$ breaking the pendulum symmetry, with initial conditions carefully chosen. In Fig. 3(A) apparently now the basin of attraction of period-3 has only three regions but the phase spaces in 3(B) illustrate that we have even and odd attractors as before, and also an apparent reflection symmetry in ω. However, in this case taking randomly any initial conditions in the three regions we necessarily will not obtain the same result, that is, three even attractors and three odd ones but we always obtain one of the two types de attractor shown in Fig. 3(C). Therefore, each one of these attractors presents different loci of saddle-node bifurcations, as shown in Fig. 4(B), showing that period-3 attractors degenerescence were broken.

4 Conclusions

As a first order model, the Melnikov method cannot predict odd resonances of simple planar pendulum with vertical excitation but they indeed exist such as period-3 and period-5 and we studied the first case. The basin of attraction of period-3 resonance presents six regions. The attractors obtained with initial conditions in alternate regions present inversion symmetry effects so they are degenerate and present the same loci of saddle-node bifurcations. Therefore, we named the attractors as in-phase and out-phase mode according to odd and even regions, respectively.

For tilted excitation, the last term in Eq. 1 is an additional torque that is responsible to replace the fixed point of the vertical case by period-1 oscillation. Another effect is to promote the fusion of the six regions into 3 regions, but in these regions there are small dense sub-regions around the black dots in Fig. 3(A) which are related to attractors of different phases. In the same way we can name in-phase and out-phase these different attractors, relate to the breaking degeneracy. This feature is preserved up to $\phi\approx7\pi/180$. Above this value we will have basins of attraction similar to the ones for $\phi\approx4\pi/180$ but no sub-regions and just one class of attractor of period-3 remains. We are still working to get more detailed results.

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