ON PROBLEM OF STABILITY WITH RESPECT TO A PART OF THE VARIABLES

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Abstract
A stability problem with respect to a part of variables of the zero equilibrium position is considered for nonlinear non-stationary systems of ordinary differential equations with the continuous right-hand side. As compared to known assumptions, more general assumptions are made on the initial values of variables non-controlled in the course of studying stability. Conditions of stability and asymptotic stability of this type are obtained within the method of Lyapunov functions and generalize a number of existing results. The results are applied to the stability problem with respect to a part of variables of equilibrium positions of nonlinear holonomic mechanical systems.

Key words
Partial stability, Lyapunov function method.

1 Introduction

There exists a great number of works on the stability problem with respect to a part of variables (rather than with respect to all variables that determine the system's state) stated by V. V. Rumyantsev [1]. The problem arises frequently and naturally in applications; moreover, advances in detectability of dynamic systems make it efficient at the first stage of studying stability with respect to all variables. A survey on the problem and rich bibliography can be found in [2-5].

The classical definition of stability with respect to a part of variables of the zero equilibrium position of the system of ordinary differential equations [1] assumes the domain of initial perturbations to be a sufficiently small neighborhood of the zero equilibrium position. Along with this statement, the cases of arbitrary [2-5] or large (belonging to an arbitrary compact set) [4, 5] initial perturbations for a part of variables that are non-controlled when studying stability are investigated.

However, in analysis of complex nonlinear systems, it turns out to be interesting to study more general cases when stability with respect to a part of variables of the zero equilibrium position presumes that initial perturbations, being small with respect to a part of variables studied for stability, can be simultaneously large with respect to one part and arbitrary with respect to the other part (the rest) of non-controlled variables.

Indeed, assuming that initial perturbations of the stated variables are large (as opposed to arbitrary), we arrive at much softer requirements on the Lyapunov functions. In this sense, the combined statement of the problem can be an admissible tradeoff between the meaning of stability and the respective requirements on the Lyapunov functions.

We obtain stability and asymptotic stability conditions of the stated type within the method of Lyapunov functions, which generalize a number of existing results. We apply the results to the stability problem with respect to a part of variables of equilibrium positions of holonomic mechanical systems.

2 STABILITY PROBLEMS WITH RESPECT TO A PART OF VARIABLES

Suppose we have a nonlinear system of ordinary differential equations

$$\mathbf{x}' = \mathbf{X}(t, \mathbf{0}),\ \mathbf{X}(t, \mathbf{0}) \equiv \mathbf{0}. \tag{1}$$

We divide the variables in system (1) that belong to the phase vector $\mathbf{x}$ that gives its state into two groups: (1) $\mathbf{y}$-variables used to study stability of the equilibrium position $\mathbf{x} = \mathbf{0}$, and (2) other (non-controlled) $\mathbf{z}$-variables. We also divide the variables included in the subvector $\mathbf{z}$ into two subgroups so that $\mathbf{x} = (\mathbf{y}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}$; $\mathbf{z} = (\mathbf{z}_1^{\mathrm{T}}, \mathbf{z}_2^{\mathrm{T}})^{\mathrm{T}}$.

In a way common to partial stability ($\mathbf{y}$-stability) theory [2-8, 11, 12], we assume that the vector function $\mathbf{X}$

is continuous in the domain

$$t \geq 0, \|\mathbf{y}\| \leq h, \|\mathbf{z}\| < \infty,$$
$$\|\mathbf{x}\| = (\|\mathbf{y}^2\| + \|\mathbf{z}^2\|)^{1/2} = (x_1^2 + \ldots + x_n^2)^{1/2}, \quad (2)$$
$$n = \dim(\mathbf{x}),$$

and solutions of system (1) are unique and $\mathbf{z}$-extendable (i.e., any solution $\mathbf{x}(t)$ is defined for all $t \geq 0$ such that $\|\mathbf{y}(t)\| \leq h$). We use $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ to denote the solution of system (1) given by the initial condition $\mathbf{x}_0(t) = \mathbf{x}(t_0; t_0, \mathbf{x}_0)$.

We use the designations D_δ is the domain of the initial values x_0 such that $\|\mathbf{y}_0\| < \delta, \|\mathbf{z}_{10}\| \leq L$, and $\|\mathbf{z}_{20}\| < \infty$; the domain D_Δ is obtained by replacing δ by Δ; and $K_\mathbf{y}$ and $K_{\mathbf{z}1}$ are arbitrary compacts in the $\mathbf{y}$- and $\mathbf{z}_1$-spaces, respectively.

D e f i n i t i o n s. The zero equilibrium position $\mathbf{x} = \mathbf{0}$ of system (1) for a large $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$ is

1. $\mathbf{y}$-*stable* if for any $\varepsilon > 0, t_0 \geq 0$ and any given number $L > 0$ one can find $\delta(\varepsilon, t_0, L) > 0$ such that $\mathbf{x}_0 \in D_\delta$ yields $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$;
2. *uniformly* $\mathbf{y}$-*stable with respect to* t_0 if $\delta = \delta(\varepsilon, L)$;
3. *asymptotically* $\mathbf{y}$-*stable* if it is $\mathbf{y}$-stable in terms of definition 1 and one can find $\Delta(t_0, L) > 0$ such that the arbitrary solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of system (1) with $\mathbf{x}_0 \in D_\Delta$ satisfies the limit relation

$$\lim \|\mathbf{y}(t; t_0, \mathbf{x}_0)\| = 0, \; t \to \infty. \quad (3)$$

(for any numbers $\eta > 0$, $t_0 \geq 0$ and any given number $L > 0$, one can find the number $T(t_0, L, \mathbf{x}_0, \eta) > 0$ such that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T, \mathbf{x}_0 \in D_\Delta$);
4. *equiasymptotically* $\mathbf{y}$-*stable* if one can find $\Delta(t_0, L) > 0$ such that relation (3) is met uniformly with respect to $\mathbf{x}_0$ from the domain $t_0 \geq 0, \mathbf{x}_0 \in D_\Delta$ (for any numbers $\eta > 0, t_0 \geq 0$ and any given number $L > 0$ one can find the number $T(t_0, L, \eta) > 0$ such that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T, \mathbf{x}_0 \in D_\Delta$);
5. *uniformly asymptotically* $\mathbf{y}$-*stable* if it is $\mathbf{y}$-stable uniformly with respect to t_0 in terms of definition (2) and one can find $\Delta(L) > 0$ such that relation (3) is met uniformly with respect to $t_0, \mathbf{x}_0$ from the domain $t_0 \geq 0, \mathbf{x}_0 \in D_\Delta$ (for any numbers $\eta > 0, t_0 \geq 0$ and any given number $L > 0$ one can find the number $T(L, \eta) > 0$ such that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T, \mathbf{x}_0 \in D_\Delta$);
6. *uniformly globally asymptotically* $\mathbf{y}$-*stable* if it is $\mathbf{y}$-stable uniformly with respect to t_0 in terms of definition (2) and an arbitrary solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of system (1) is given for all $t \geq 0$, is uniform with respect to $t_0, \mathbf{x}_0$ from the domain $t_0 \geq 0, \mathbf{y}_0 \in K_\mathbf{y}, \mathbf{z}_{10} \in K_{\mathbf{z}1}, \|\mathbf{z}_{20}\| < \infty$ is $\mathbf{y}$-bounded and satisfies limit relation (3) (for any numbers $\eta > 0, t_0 \geq 0$ and any given numbers $L_\mathbf{y} > 0, L_{\mathbf{z}1} > 0$ one can find the numbers $L_1(L_\mathbf{y}, L_{\mathbf{z}1}) > 0$ and $T(L_\mathbf{y}, L_{\mathbf{z}1}, \eta) > 0$ such that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \leq L_1$ for all $t \geq t_0$ and $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T$ if $\|\mathbf{y}_0\| < L_\mathbf{y}, \|\mathbf{z}_{10}\| \leq L_{\mathbf{z}1}, \|\mathbf{z}_{20}\| < \infty$).

R e m a r k 1. The earlier statements of $\mathbf{y}$-stability problems of the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (1) involved three cases (1) $\|\mathbf{x}_0\| < \delta$ (A.M. Lyapunov – V.V. Rumyantsev $\mathbf{y}$-stability [1]), (2) $\|\mathbf{y}_0\| < \delta, \|\mathbf{z}_0\| \leq L$ ($\mathbf{y}$-stability for a large $\mathbf{z}_0$ [4, 5]), and (3) $\|\mathbf{y}_0\| < \delta, \|\mathbf{z}_0\| < \infty$ ($\mathbf{y}$-stability on the whole with respect to $\mathbf{z}_0$ [2-5]). The definitions we propose here are more general and involve the case $\mathbf{x}_0 \in D_\delta$ that was not considered previously.

R e m a r k 2. The concept of $\mathbf{y}$-stability of the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (1.1) introduced in [4, 5] for large $\mathbf{z}_0$ can be also treated as [13] $\mathbf{y}$-stability non-uniform with respect to $\mathbf{z}_0$ so that $\delta = \delta(\varepsilon, t_0, \mathbf{z}_0)$: for any $\varepsilon > 0$, $t_0 \geq 0$ and $\mathbf{z}_0$ one can find $\delta(\varepsilon, t_0, \mathbf{z}_0) > 0$ such that $\|\mathbf{y}_0\| < \delta$ yields $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. Note that [13] does not assume system (1) to have the zero equilibrium position $\mathbf{x} = \mathbf{0}$ and in fact deals with non-uniform with respect to $\mathbf{z}_0$ stability of the "partial" equilibrium position $\mathbf{y} = \mathbf{0}$. Stability of the same type called stability for large $\mathbf{z}_0$ of the "partial" equilibrium position $\mathbf{y} = \mathbf{0}$ is considered in [14] independently of and simultaneously with [13]. (Stability conditions of the stated types obtained in [4,5, 13,14] within the method of Lyapunov functions coincide.)

R e m a r k 3. If the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (1) is equiasymptotically $\mathbf{y}$-stable for large $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$, it is $\mathbf{y}$-stable for a large $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$. Indeed, for any numbers $\varepsilon > 0, t_0 \geq 0$ and any given number $L > 0$ one can find the numbers $\delta_1(t_0, L) > 0$, $T(t_0, L, \varepsilon) > 0$ such that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0 + T$, $\mathbf{x}_0 \in D_\delta$. Since $\mathbf{y}(t; t_0, \mathbf{x}_0)$ is continuous on the set $\mathbf{y}_0 = \mathbf{0}$ (since solutions of system (1) continuously depend on the initial conditions), for any numbers $\varepsilon > 0$, $t_0 \geq 0$, any given number $L > 0$ and any given number $T(t_0, L, \varepsilon) > 0$ one can choose $\delta_2(t_0, L) > 0$ such that for $\|\mathbf{y}_0\| < \delta_2$ the inequality $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ holds for $t \in [t_0, t_0 + T]$ as well. Choosing $\delta = \min(\delta_1, \delta_2)$, we can conclude that the inequality $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ holds for all $t \geq t_0$, $\|\mathbf{x}_0\| \in D_\delta$.

R e m a r k 4. The earlier statements of problems of global uniform asymptotic $\mathbf{y}$-stability of the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (1) studied the cases [2,3,6] when $\mathbf{y}$-attraction of solutions is uniform from the domain $t_0 \geq 0, \mathbf{x}_0 \in K_\mathbf{x}$ or the domain $t_0 \geq 0, \mathbf{y}_0 \in K_\mathbf{y}, \|\mathbf{z}_0\| < \infty$. The domain $t_0 \geq 0, \mathbf{y}_0 \in K_\mathbf{y}, \mathbf{z}_{10} \in K_{\mathbf{z}1}, \|\mathbf{z}_{20}\| < \infty$ of the uniform $\mathbf{y}$-attraction of solutions was considered in [2], where the decomposition of the vector $\mathbf{z}_0$ into the subvectors $\mathbf{z}_{10}$ and $\mathbf{z}_{20}$ was not supposed to be initially set and depended on the properties of the appropriate Lyapunov function in the course of solving the problem.

R e m a r k 5. The example from [15] shows that the requirement for global uniform $\mathbf{y}$-boundedness of solutions in definition (6) does not generally follow from other two requirements of this definition, viz. the requirement for uniform $\mathbf{y}$-stability combined with the requirement for uniform global $\mathbf{y}$-attraction of solutions.

2.1 Generalization of Lyapunov–Rumyantsev Theorems

To obtain stability conditions of the stated type, we consider auxiliary functions: (1) the scalar function $V(t,\mathbf{x}), V(t,\mathbf{0}) \equiv \mathbf{0}$, which is continuously differentiable in domain (1), and its derivative V' by system (1); (2) the scalar function $V^*(t,\mathbf{y},\mathbf{z}_1), V^*(t,\mathbf{0},\mathbf{0}) \equiv \mathbf{0}$ and the vector function $\mathbf{W}(t,\mathbf{x}), \mathbf{W}(t,\mathbf{0}) \equiv \mathbf{0}$, which is continuous in domain (2); and (3) $a(r)$, $b(r)$, $c(r)$, which are continuously monotonically increasing for $r \in [0,h]$ or $r \in [0,\infty)$, respectively, in the problem of global asymptotic $\mathbf{y}$-stability such that $a(0) = b(0) = c(0) = 0$ (the Hahn functions for $r \in [0,h]$ or for $r \in [0,\infty)$ [2].

T h e o r e m 1. Suppose we can find the V-function for system (1) in domain (2) such that

$$V(t,\mathbf{x}) \geq a(\|\mathbf{y}\|),\ V'(t,\mathbf{x}) \leq 0. \quad (4)$$

Then, for a large $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$, the equilibrium position $\mathbf{x} = \mathbf{0}$ is

1. $\mathbf{y}$-stable if, in addition,

$$V(t,\mathbf{x}) \leq V^*(t,\mathbf{y},\mathbf{z}_1),\ V^*(t,\mathbf{0},\mathbf{z}_1) \equiv 0; \quad (5)$$

2. $\mathbf{y}$-stable uniformly with respect to t_0 if

$$V(t,\mathbf{x}) \leq V^*(\mathbf{y},\mathbf{z}_1),\ V^*(\mathbf{0},\mathbf{z}_1) \equiv 0; \quad (6)$$

T h e o r e m 2. Suppose we can find V-function and the vector $\mathbf{W}(t,\mathbf{x})$ – function for system (1) in domain (2) such that

$$a(\|\mathbf{y}\|) \leq V(t,\mathbf{x}) \leq b(\|\mathbf{u}\|),\ V'(t,\mathbf{x}) \leq -c(\|\mathbf{u}\|), \\ \mathbf{u} = [\mathbf{y}^{\mathrm{T}}, \mathbf{W}(t,\mathbf{x})^{\mathrm{T}}]^{\mathrm{T}}. \quad (7)$$

Then, for a large $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$, the equilibrium position $\mathbf{x} = \mathbf{0}$ is

1. asymptotically $\mathbf{y}$-stable if conditions (5) are met;
2. uniformly asymptotically $\mathbf{y}$-stable if conditions (6) are met;
3. uniformly globally asymptotically $\mathbf{y}$-stable if conditions (6), (7) are met in the domain $t \geq 0, \|\mathbf{x}\| < \infty$ and, in addition,

$$a(r) \to \infty,\ r \to \infty. \quad (8)$$

We prove Theorems 1 and 2 in the Appendix.

Theorems 1 and 2 generalize A. M. Lyapunov–V. V. Rumyantsev theorems [1, 2] and their complimentary results obtained in [4, 5].

2.2 Addition to the Lagrange–Dirichlet Theorem

Motion of the holonomic mechanical system with n degrees of freedom is described by Lagrange equations of the second kind [16]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial P}{\partial q_i},\ i = \overline{1,n}, \quad (9)$$

T is the kinetic energy, P is the potential energy, and q_i are generalized coordinates. We use $q = (q_1,\ldots,q_n)$ to denote the vector of generalized coordinates and assume that the bounds imposed on the system do not explicitly depend on t and system (9) allows the zero equilibrium position $\mathbf{q} = \mathbf{q}' = \mathbf{0}$.

We divide the components of the vector $\mathbf{q}$ into two groups so that $\mathbf{q} = (\mathbf{q}_\mathbf{y}^{\mathrm{T}}, \mathbf{q}_\mathbf{z}^{\mathrm{T}})^{\mathrm{T}}$. We consider stability problem of the zero equilibrium position of (9) with respect to $(\mathbf{q}_\mathbf{y}, \mathbf{q}')$ (with respect to a part of the generalized coordinates and to all generalized velocities). We divide the generalized coordinates included in the subvector $\mathbf{q}_\mathbf{z}$ into two subgroups $\mathbf{q}_\mathbf{z} = (\mathbf{q}_{\mathbf{z}1}^{\mathrm{T}}, \mathbf{q}_{\mathbf{z}2}^{\mathrm{T}})^{\mathrm{T}}$ as well.

The zero equilibrium position of system (9) is uniformly $(\mathbf{q}_\mathbf{y}, \mathbf{q}')$-stable for a large $\mathbf{q}_{\mathbf{z}10}$ and on the whole with respect to $\mathbf{q}_{\mathbf{z}20}$ if for any $\varepsilon > 0, t_0 \geq 0$ and any given number $L > 0$ one can find $\delta(\varepsilon, L) > 0$ such that for arbitrary solution of system (9) with $\|\mathbf{q}_{\mathbf{y}0}\| < \delta, \|\mathbf{q}_0'\| < \delta, \|\mathbf{q}_{\mathbf{z}10}\| \leq L, \|\mathbf{q}_{\mathbf{z}20}\| < \infty$ for all $t \geq t_0$ the inequalities $\|\mathbf{q}_\mathbf{y}(t;t_0,\mathbf{q}_0,\mathbf{q}_0')\| < \varepsilon$ and $\mathbf{q}'(t;t_0,\mathbf{q}_0,\mathbf{q}_0')\| < \varepsilon$ hold.

Suppose T does not explicitly depend on t and the conditions hold in the domain $\|\mathbf{q}_\mathbf{y}\| \leq h, \|\mathbf{q}'\| \leq h, \|\mathbf{q}_\mathbf{z}\| < \infty$

1. T is positive definite with respect to all generalized velocities;
2. P is positive definite with respect to a part of generalized coordinates (with respect to $\mathbf{q}_\mathbf{y}$);
3. the inequalities hold

$$P(\mathbf{q}) \leq P^*(\mathbf{q}_\mathbf{y}, \mathbf{q}_{\mathbf{z}1}),\ P^*(\mathbf{0}, \mathbf{q}_{\mathbf{z}1}) \equiv 0, \\ T(\mathbf{q}, \mathbf{q}') \leq T^*(\mathbf{q}_\mathbf{y}, \mathbf{q}_{\mathbf{z}1}, \mathbf{q}'). \quad (10)$$

In this case, the energy integral $H = T + P = \text{const}$ holds for system (9). We take this integral as the V-Lyapunov function. In addition to classical results [1, 16, 17] and by Theorem 1, we can conclude that if conditions (1) and (2) are met, the zero equilibrium position of system (9) is uniformly $(\mathbf{q}_\mathbf{y}, \mathbf{q}')$-stable for a large $\mathbf{q}_{\mathbf{z}10}$ and on the whole with respect to $\mathbf{q}_{\mathbf{z}20}$.

For instance, conditions (10) are met for the class of mechanical systems, for which P and T are independent of $\mathbf{q}_{\mathbf{z}2}$ and P is a positive definite quadratic form

$P = P(\mathbf{q_y}, \mathbf{q_{z1}}) = \mathbf{q_y}^{\mathrm{T}} A(\mathbf{q_{z1}})\mathbf{q_y}$ of the variables $\mathbf{q_y}$, with its coefficients depending on $\mathbf{q_{z1}}$.

We can also consider the problem of uniform stability with respect to a part of generalized coordinates and a part of generalized velocities, with generalized velocities not necessarily corresponding to the chosen part of generalized coordinates. Along with the decomposition $\mathbf{q} = (\mathbf{q_y}^{\mathrm{T}}, \mathbf{q_z}^{\mathrm{T}})^{\mathrm{T}},\ \mathbf{q_z} = (\mathbf{q_{z1}}^{\mathrm{T}}, \mathbf{q_{z2}}^{\mathrm{T}})^{\mathrm{T}}$, we introduce the decomposition $\mathbf{q}' = (\mathbf{q_u}'^{\mathrm{T}}, \mathbf{q_w}'^{\mathrm{T}})^{\mathrm{T}}$ and consider the stability problem of the zero equilibrium position of system (9) with respect to $(\mathbf{q_y}, \mathbf{q_u}')$ for a large $\mathbf{q_{z10}}$ and on the whole with respect to $\mathbf{q_{z20}}$. The stated property of partial stability takes place if for any $\varepsilon > 0, t_0 \geq 0$ and any given number $L > 0$ one can find $\delta(\varepsilon, L) > 0$ such that for any arbitrary solution of system (9) with $\|\mathbf{q_{y0}}\| < \delta, \|\mathbf{q}_0'\| < \delta, \|\mathbf{q_{z10}}\| \leq L, \|\mathbf{q_{z20}}\| \leq \infty$ for all $t \geq t_0$ inequalities $\|\mathbf{q_y}(t; t_0, \mathbf{q}_0, \mathbf{q}_0')\| < \varepsilon$ and $\|(\mathbf{q_u}'; t_0, \mathbf{q}_0, \mathbf{q}_0')\| < \varepsilon$ hold.

We assume that T does not explicitly depend on t and the conditions are met in the domain $\|\mathbf{q_y}\| \leq h, \|\mathbf{q_u}'\| \leq h, \|\mathbf{q_z}\| < \infty, \|\mathbf{q_w}'\| < \infty$

1. T is positive definite with respect to a part of generalized velocities (with respect to $\mathbf{q_u}'$);
2. P is positive definite with respect to a part of generalized coordinates (with respect to $\mathbf{q_y}$);
3. inequalities (10) hold.

Unlike the case when T is positive definite with respect to all generalized velocities, in this case Lagrange equations cannot be resolved with respect to the vector $\mathbf{q}''$ of generalized accelerations and cannot be thus reduced to normal form (1). Nevertheless, if we consider the energy integral, we can show, in addition to the result [1] (see also [18]), that if the listed conditions are met, the zero equilibrium position of system (9) is uniformly $(\mathbf{q_y}, \mathbf{q_u}')$-stable for a large $\mathbf{q_{z10}}$ and on the whole with respect to $\mathbf{q_{z20}}$.

E x a m p l e 1. We consider the unit mass moving in the constant gravitational field along the surface $x_1 = f(x_2, x_3)$, where f is a smooth function of x_2, x_3 in the three-dimensional space $Ox_1x_2x_3$ with the axis Ox_1 directed vertically up. In this case, we have $P = gf(x_2, x_3),\ g = \text{const} > 0$.

Suppose the condition $f(0, x_3) = 0$ is met; for instance, in [5], the surface has the form $x_1 = \frac{1}{2}x_2^2(1 + x_3^2)$.

By the addition we made to the Lagrange–Dirichlet theorem, the equilibrium position of the point is stable with respect to a part of coordinates (with respect to x_1, x_2) and with respect to all velocities x_1', x_2', x_3', for a large x_{30}.

E x a m p l e 2. We consider the system of differential equations

$$\begin{aligned} Ax_1' &= (B - C)x_2x_3 + u_1, \\ Bx_2' &= (C - A)x_1x_3 + u_2, \\ Cx_3' &= (A - B)x_1x_2, \end{aligned} \qquad (11)$$

that describe the rotation of the rigid body subjected to the action of two control moments u_1 and u_2. In this system, A, B, C are the principal central moments of inertia of the body, x_1, x_2, and x_3 are the projections of the vector of instantaneous angular velocity of the body onto its principal central axes of inertia i_1, i_2, i_3.

The control moments

$$u_j = \alpha_j x_j,\ \alpha_j = \text{const} < 0,\ j = 1, 2 \qquad (12)$$

are known [4, 5] to ensure asymptotic damping of rotations with respect to variables x_1 and x_2. If $C < A, B$, this means that the body is twisting with respect to the bigger axis i_3.

We use the Lyapunov function

$$V = 1/2A(A - C)x_1^2 + B(B - C)x_2^2,$$

whose derivative V', due to closed system (11), (12), has the form

$$V' = (A - C)\alpha_1 x_1^2 + (B - C)\alpha_2 x_2^2,$$

to refine the nature of the transfer process in system (11), (12).

Denoting $\mathbf{y} = (x_1, x_2),\ z_1 = x_3$, by assumption (3) of Theorem 2, we can conclude that if $C < A, B$, the equilibrium position $x_1 = x_2 = x_3 = 0$ of system (11), (12) is uniformly globally asymptotically $\mathbf{y}$-stable on the whole with respect to z_{10} in terms of definition 6. This means that twisting is ensured for all motions $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of the body uniformly with respect to $t_0, \mathbf{x}_0$ from the domain $t_0 \geq 0, \mathbf{y}_0 \in K_\mathbf{y}$ for the arbitrary value z_{10}.

E x a m p l e 3. We complement the results [2, 19] on stability of the equilibrium position of the rigid body of the mass m with the symmetry axis and the point fixed on this axis. The body is subjected to the force with the potential $U(\psi), U(0) \equiv 0$, and $\partial U/\partial\psi = 0$ for $\psi = 0$.

In this case, we have the energy integral for the "reduced" (with respect to ψ, θ) system

$$\begin{gathered} H^* = 1/2[A(\theta'^2 + \psi'^2 \sin\theta)] + \\ + mgz_0(\cos\theta - 1) + U(\psi) = \text{const}, \end{gathered}$$

where $\psi,\ \theta$ are the precession and nutation angles, is the transverse moment of inertia, and z_0 is the coordinate of the center of gravity.

If the conditions

$$z_0 < 0,\ \partial^2 U/\partial\psi^2 > 0 \text{ for } \psi = 0$$

are met, the function H^* is positive definite with respect to the variables θ, θ', ψ, and $H^* = 0$ for $\theta = \theta' = \psi = 0$, and H^* is independent of φ, φ' (φ is the angle of proper rotation of the body).

Therefore, the equilibrium position $\theta = \theta' = \psi = \psi' = \varphi = \varphi' = 0$ of the body is stable with respect to θ, θ', ψ for a large ψ_0' and on the whole with respect to φ_0, φ_0'.

2.3 Applying Differential Inequalities

Suppose the auxiliary V-function satisfies the differential inequality [11]

$$V' \leq \omega(t, V(t, \mathbf{x})) \tag{13}$$

in domain (2) by system (1), where $\omega(t, v)$ is the function continuous for $t \geq 0, v \geq 0$ such that the conditions of existence and uniqueness of solutions are met for the equation

$$v' = \omega(t, v),\ \omega(t, 0) \equiv 0 \tag{14}$$

for each point (t_0, v_0) from the domain of definition.

T h e o r e m 3. Suppose there exists a V-function for system (1) that satisfies the condition $V(t, \mathbf{x}) \geq a(\|\mathbf{y}\|)$ and differential inequality (13) in domain (2). Then, for a large $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$, the equilibrium position $\mathbf{x} = \mathbf{0}$ is

1. $\mathbf{y}$-stable (equiasymptotically $\mathbf{y}$-stable) if conditions (5) are met and the solution $v = 0$ of equation (14) is Lyapunov (asymptotically) stable;
2. uniformly $\mathbf{y}$-stable (uniformly asymptotically $\mathbf{y}$-stable) if conditions (6) are met and the solution $v = 0$ of Eq. (14) is Lyapunov uniformly (uniformly asymptotically) stable.

Theorem 3 complements the known results obtained by Corduneanu [11] and their complementary results obtained in [4].

3 Conclusion

We considered modified stability problems with respect to a part of variables of both the zero equilibrium for nonlinear non-stationary systems of ordinary differential equations. As compared to existing assumptions, we made more general assumptions on the initial values of variables non-controlled in the course of studying stability.

We obtained the conditions of stability and asymptotic stability of this type within the method of Lyapunov functions. We apply the results to the stability problem with respect to a part of variables of equilibrium positions of nonlinear holonomic mechanical systems.

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Appendix

Proof of Theorem 1. For any $\varepsilon > 0, t_0 \geq 0$ and any given number $L > 0$, by continuity of the functions V and V^* and conditions (4), one can find $\delta(\delta, t_0, L) > 0$

such that $\mathbf{x}_0 \in D_\delta$ yields $V(t_0, \mathbf{x}_0) < a(\varepsilon)$. Taking into account the equality

$$V(t, \mathbf{x}(t;\ t_0, \mathbf{x}_0)) = V(t_0, \mathbf{x}_0) + \int\limits_{t_0}^{t} V'(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_0))\, dx \tag{A.1}$$

for the arbitrary solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of system (1) with $\mathbf{x}_0 \in D_\delta$, the relations

$$a(\|\mathbf{y}(t; t_0, \mathbf{x}_0)\|) \le V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \le \\ \le V(t_0, \mathbf{x}_0) < a(\varepsilon)$$

hold by conditions (4) for all $t \ge t_0$. Keeping in mind the properties of the functions $a(r)$, we obtain $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \ge t_0$. The first part of the theorem is proved.

If conditions (6) are met, for any $\varepsilon > 0, t_0 \ge 0$ and any given number $L > 0$, by continuity of the functions $V,\ V^*$ one can find the number $(\varepsilon, L) > 0$ independent of t_0 such that $\mathbf{x}_0 \in D_\delta$ yields $V(t_0, \mathbf{x}_0) < a(\varepsilon)$. The further proof is similar.

Proof of Theorem 2. Incase (1), the equilibrium position $\mathbf{x} = \mathbf{0}$ is $\mathbf{y}$-stable for a large $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$ under the theorem's hypotheses. Therefore, for the number $h > 0$ there exists such $\Delta(t_0) > 0$ that $\mathbf{x}_0 \in D_\delta$ yields $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \le h$ for all $t \ge t_0$.

We show that for $\mathbf{x}_0 \in D_\delta$ the relation

$$\lim V(t, \mathbf{x}(t;\ t_0, \mathbf{x}_0)) = 0, \ \ t \to \infty \tag{A.2}$$

also holds. We assume the contrary. Then, by the condition $V'(t, \mathbf{x}) \le 0$, the inequality $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \ge V_* > 0$ holds for all $t \ge t_0$. Therefore, by $V(t, \mathbf{x}) \le b(\|\mathbf{u}\|),\ V'(t, \mathbf{x}) \le -c(\|\mathbf{u}\|)$, the relation

$$V'(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \le -c(b^{-1}(V_*))$$

holds. Taking into account equality (A.1), we obtain

$$0 \le V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \le \\ \le V(t_0, \mathbf{x}_0) - c(b^{-1}(V_*))(t - t_0),$$

which is impossible for a sufficiently large t.

Thus, for $\mathbf{x}_0 \in D_\Delta$ we have relation (A.2); hence, $\lim a(\|\mathbf{y}(t; t_0, \mathbf{x}_0)\|) = 0,\ t \to \infty$ and, hence, relation (3) holds. In case (2), the equilibrium position $\mathbf{x} = \mathbf{0}$ is uniformly $\mathbf{y}$-stable for a large $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$ under the theorem's hypotheses. Therefore for the number $h > 0$ there exists such $\Delta_0 > 0$ independent of t_0 that $\mathbf{x}_0 \in D_\Delta$ yields $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \le h$ for all $t \ge t_0$.

Let $0 < \varepsilon < \Delta_0$. We put

$$T(\varepsilon) = [a(h) - a(\varepsilon)]/c(b^{-1}(a(\varepsilon)))$$

and show that $V(t_*, \mathbf{x}(t_*; t_0, \mathbf{x}_0)) < a(\varepsilon)$ for some value $t_* \in (t_0, t_0 + T)$. Indeed, otherwise $a(\varepsilon) \le V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \le b(\|\mathbf{u}(t, \mathbf{x}(t; t_0, \mathbf{x}_0))\|)$ for $t_* \in (t_0, t_0 + T)$, hence $\|\mathbf{u}(t, \mathbf{x}(t; t_0, \mathbf{x}_0))\| \ge b^{-1}(a(\varepsilon))$ for the same t. Then,

$$a(\varepsilon) \le V(t_0 + T, \mathbf{x}(t_0 + T; t_0, \mathbf{x}_0)) \le \\ \le a(h) - c(b^{-1}(a(\varepsilon)))T < a(\varepsilon),$$

which is impossible. The existence of t_* is proved.

Since the V-function is non-decreasing, $a(\|\mathbf{y}(t; t_0, \mathbf{x}_0)\|) \le V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \le V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) < a(\varepsilon)$ for all $t \ge t_*$; hence, the inequality $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ holds for all $t \ge t_0 + T > t_*$ if $x_0 \in D_\Delta$.

In case (3), we consider the set of initial values $\mathbf{x}_0$ such that $\mathbf{y}_0 \in K_\mathbf{y}, \mathbf{z}_{10} \in K_{\mathbf{z}1}$ ($K_\mathbf{y}, K_{\mathbf{z}1}$ are arbitrary compacts in the $\mathbf{y}$- and $\mathbf{z}_1$-spaces, respectively), $\|\mathbf{z}_{20}\| < \infty$. Let $L_2 = \max V^*(\mathbf{y}_0, \mathbf{z}_{10})$ for $\mathbf{x}_0 \in M$.

By conditions (6), (7), inequalities $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \le V(t_0, \mathbf{x}_0) \le V^*(\mathbf{y}_0, \mathbf{z}_{10}) \le L_2$ hold; hence, by (8), one can find $L_1 > 0$ such that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \le L_1$ for all $t \ge t_0$ and $\mathbf{x}_0 \in M$. This means that for all $t \ge t_0$ and $\mathbf{x}_0 \in M$ solutions of system (1) are $\mathbf{y}$-bounded uniformly with respect to $t_0, \mathbf{x}_0$ from the domain $t \ge t_0$, $\mathbf{x}_0 \in M$.

Putting $\delta(\varepsilon) = b^{-1}(a(\varepsilon))$ and taking into account the first group of conditions (7), for any $\varepsilon > 0, t_0 \ge 0$ we have $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \le a^{-1}(b(\|\mathbf{u}(t_0, \mathbf{x}_0)\|) < \varepsilon$ for all $t \ge t_0$ if $\|\mathbf{u}(t_0, \mathbf{x}_0)\| < \delta$. Let $T(\varepsilon) = 2L_2/c(\delta(\varepsilon))$ and $\mathbf{x}_0 \in M$. Assuming $\|\mathbf{u}(t, \mathbf{x}(t; t_0, \mathbf{x}_0))\| > \delta(\varepsilon)$ for $t \in (t_0, t_0 + T)$, we obtain the contradictory estimates

$$0 \le V(t_0 + T, \mathbf{x}(t_0 + T; t_0, \mathbf{x}_0)) \le \\ \le L_2 - c(b^{-1}(a(\varepsilon)))T < 0.$$

Therefore, one can find $t_* \in (t_0, t_0 + T)$, for which $\|\mathbf{u}(t_*, \mathbf{x}(t_*; t_0, \mathbf{x}_0))\| < \delta(\varepsilon)$. Then, taking into account the previous reasoning, $\|\mathbf{y}(t, t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \ge t_0 + T > t_*$. The theorem is proved.