CYBERNETICS AND PHYSICS, VOL. 14, NO. 1, 2025, 5–12

ON THE ASYMPTOTIC STABILITY WITH RESPECT TO A PART OF VARIABLES FOR A CLASS OF NONLINEAR SYSTEMS WITH DISTRIBUTED DELAY

Alexander Aleksandrov
Saint Petersburg State University, Saint Petersburg, Russia
a.u.aleksandrov@spbu.ru

Article history:
Received 15.02.2025, Accepted 17.06.2025

Abstract

This paper studies the stability problem for a class of nonlinear systems with distributed delay. Using special constructions of Lyapunov–Krasovskii functionals and the differential inequalities method, sufficient conditions are obtained ensuring that zero solutions of the investigated systems are stable with respect to all variables and asymptotically stable with respect to a part of variables. This result is an extension of well-known Lyapunov–Malkin Theorem corresponding to the critical (in the Lyapunov sense) case where the matrix of the associated linear approximation system admits several zero eigenvalues. In addition, some scenarios are considered for which the derived stability conditions can be relaxed. Two examples of applications of the developed theory to the stability analysis and control synthesis for mechanical systems are provided.

Key words

Nonlinear system, distributed delay, sector constraints, partial asymptotic stability, Lyapunov–Krasovskii functional, differential inequalities.

1 Introduction

The problem of partial stability was formulated by A. M. Lyapunov, see [Lyapunov, 1992]. In such a setting, the stability with respect to a function depending on the state vector of a system is studied.

The methods of partial stability theory are widely used in mechanics, economics, electrodynamics, biology, etc. [Fradkov, Miroshnik and Nikiforov, 1999; Halanay and Safta, 2020; Martynyuk, 2007; Rumyantsev and Oziraner, 1987; Vorotnikov, 1998]. During past decades, the growing interest to these methods is due to their applications in formation control problems [Fridman, 2014; Michiels, Morarescu and Niculescu, 2009; Network Control Problems, 2015; Sharma and Lather, 2024]. They are also used for control of cyber-physical systems. In particular, in [Fradkov, Miroshnik and Nikiforov, 1999], problems of partial stabilization of the energy of Hamiltonian systems were considered. The concept of partial stability takes on especial significance when studying output stability [Chaillet, Karafyllis, Pepe and Wang, 2023; Sontag and Wang, 2000].

A special case of the notion of partial stability is that of stability with respect to a part of variables. The foundations of the theory of stability with respect to a part of variables were developed by V. V. Rumyantsev [Rumyantsev and Oziraner, 1987]. The counterparts of the Lyapunov first and second (direct) methods for such a problem were proposed. The results of [Rumyantsev and Oziraner, 1987] have got deep development in numerous papers and monographs (see, e.g., [Chellaboina and Haddad, 2002; Costa and Astolfi, 2009; Fradkov, Miroshnik and Nikiforov, 1999; Martynyuk, 2007; Miroshnik, 2004; Michel, Molchanov and Sun, 2003; Vorotnikov, 1998] and the bibliography cited therein).

One of the first but interesting and important results on partial stability is well known Lyapunov–Malkin Theorem [Malkin, 1963] that provides us sufficient conditions of the asymptotic stability with respect to a part of variables for the zero solution of a nonlinear system in the case where the matrix of the corresponding linear approximation system admits several zero eigenvalues. This theorem was further developed by A. S. Oziraner, V.I. Vorotnikov and some others authors, see [Rumyantsev and Oziraner, 1987; Vorotnikov, 1998] and the references therein. In recent paper [Halanay and Safta, 2020], a counterpart of Lyapunov–Malkin Theorem was obtained for systems with constant delay.

In [Aleksandrov, 2000; Aleksandrov, Aleksandrova, Zhabko and Chen, 2017; Aleksandrov, 2022], Lyapunov–Malkin Theorem was extended to some classes of delay-free systems with strongly nonlinear systems of the first approximation. Furthermore, the case of a strongly nonlinear system of the first approximation with time-varying delay was investigated in [Aleksandrov, Aleksandrova, Zhabko and Chen, 2017].

However, in various applied models, instead of constant or time-varying delays, distributed ones are used, see, for instance, [Fridman, 2014; Michiels, Morarescu and Niculescu, 2009; Formal'sky, 1997; Imangazieva, 2024]. Therefore, the objective of this paper is an extension of results of [Aleksandrov, 2000; Aleksandrov, Aleksandrova, Zhabko and Chen, 2017] to the case of nonlinear systems with distributed delays. We will study a complex system describing the interaction of two subsystems. It is assumed that the first subsystem belongs to the class of Persidskii type system [Kazkurewicz and Bhaya, 1999], and its zero solution is asymptotically stable, whereas the zero solution of the second subsystem is stable. With the aid of the differential inequalities method and the Lyapunov direct method, conditions will be derived providing that the zero solution of the interconnected system is stable with respect to all variables and asymptotically stable with respect to a part of variables. In addition, some scenarios will be considered for which the obtained stability conditions can be relaxed.

2 Preliminaries

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}^k$ and $\mathbb{R}^{k\times l}$ be the vector spaces of k-tuples of real numbers and of $k\times l$ matrices, respectively. The notation $\|\cdot\|$ is used for the Euclidean norm of a vector. A matrix $A\in\mathbb{R}^{k\times l}$ is called nonnegative if all its entries are nonnegative. A matrix $A\in\mathbb{R}^{k\times k}$ is called Metzler if all its off-diagonal entries are nonnegative.

Let $\operatorname{diag}\{\lambda_1,\dots,\lambda_k\}$ be a diagonal matrix with the elements $\lambda_1,\dots,\lambda_k$ along the main diagonal. A matrix $A\in\mathbb{R}^{k\times k}$ is said to be diagonally stable if there exists a positive definite matrix $\Lambda=\operatorname{diag}\{\lambda_1,\dots,\lambda_k\}$ such that the matrix $\Lambda A+A^\top\Lambda$ is negative definite [Kazkurewicz and Bhaya, 1999].

For a given positive number $h>0$, $C([-h,0],\mathbb{R}^k)$ is the space of continuous functions $\phi(\theta):[-h,0]\mapsto\mathbb{R}^k$ with the uniform norm $\|\phi\|_h=\sup_{\theta\in[-h,0]}\|\phi(\theta)\|$.

Consider a time-delay system

$$\dot z(t)=\Xi(t,z_t). \tag{1}$$

Here $t\geq 0$, $z(t)\in\mathbb{R}^k$, the functional $\Xi(t,\phi)$ is continuous for $t\geq 0$, $\phi(\theta)\in\Omega$, where

$$\Omega=\{\phi(\theta)\in C([-h,0],\mathbb{R}^k):\,\|\phi\|_h<\rho\},$$

$h>0$, $0<\rho\leq+\infty$.

Denote by $z(t,t_0,\phi)$ a solution of the system (1) with the initial conditions $t_0\geq 0$, $\phi\in\Omega$. Let $z_t(t_0,\phi)$ be the restriction of the solution to the segment $[t-h,t]$, i.e., $z_t(t_0,\phi):\,\theta\to z(t+\theta,t_0,\phi)$, $\theta\in[-h,0]$. In the cases where the initial conditions are not important, or are well defined from the context, we will use $z(t)$ and z_t, instead of $z(t,t_0,\phi)$ and $z_t(t_0,\phi)$, respectively.

Decompose the vector z as follows:

$$z=(x^\top,y^\top)^\top, \tag{2}$$

$x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, $n+m=k$. Then $z(t,t_0,\phi)=(x^\top(t,t_0,\phi),y^\top(t,t_0,\phi))^\top$. We will assume that (1) has the zero solution and solutions of this system are y-extendable [Rumyantsev and Oziraner, 1987].

Definition 1. *The zero solution of (1) is x-stable (stable with respect to x) if, for any $\varepsilon>0$ and $t_0\geq 0$, there exists $\delta_1>0$ such that $\|x(t,t_0,\phi)\|<\varepsilon$ for $t\geq t_0$, $\|\phi\|_h<\delta_1$.*

Definition 2. *The zero solution of (1) is asymptotically x-stable (asymptotically stable with respect to x) if it is x-stable and, for any $t_0\geq 0$, there exists $\delta_2>0$ such that if $\|\phi\|_h<\delta_2$, then $\|x(t,t_0,\phi)\|\to 0$ as $t\to+\infty$.*

3 Formulation of the Problem

Assume that in result of the decomposition (2) the system (1) takes the form

$$\begin{aligned}\dot x(t)&=PF(x(t))+Q\textstyle\int_{t-h}^{t}F(x(s))ds\\&\quad+D(t,x_t,y(t)),\\\dot y(t)&=L(t,y(t))+G(t,x_t,y(t)).\end{aligned} \tag{3}$$

Here P and Q are constant matrices, $F(x)=(f_1(x_1),\dots,f_n(x_n))^\top$, scalar functions $f_i(x_i)$ are continuous for $|x_i|<\rho$ and satisfy the sector-like constraints: $x_if_i(x_i)>0$ for $x_i\neq 0$, $i=1,\dots,n$, vector function $L(t,y)$ is continuous for $t\geq 0$, $\|y\|<\rho$, functionals $D(t,x_t,y(t))$ and $G(t,x_t,y(t))$ are continuous for $t\geq 0$, $\|x_t\|_h<\rho$, $\|y(t)\|<\rho$.

We can interpret (3) as a complex system describing the interaction of the subsystems

$$\dot x(t)=PF(x(t))+Q\int_{t-h}^{t}F(x(s))ds, \tag{4}$$

$$\dot y(t)=L(t,y(t)) \tag{5}$$

and connections between the subsystems are characterized by the functionals $D(t,x_t,y(t))$, $G(t,x_t,y(t))$.

The subsystem (4) is Persidskii type system with distributed delay. It is worth noticing that Persidskii systems are widely used for modeling automatic control systems, population dynamics, neural networks, opinion dynamics, etc. [Lur'e, 1957; Kazkurewicz and Bhaya, 1999; Mei, Efimov, Ushirobira and Fridman, 2023].

In the present paper, we will study the system (3) with functions $f_i(x_i)$ of a power type.

Assumption 1. *Let $f_i(x_i) = x_i^{\mu_i}$, where $\mu_i \geq 1$ are rational numbers with odd numerators and denominators, $i = 1, \ldots, n$.*

Without loss of generality, suppose that $\mu_1 \leq \ldots \leq \mu_n$.

Assumption 2. *Let $L(t, 0) = 0$ for $t \geq 0$.*

Thus, the subsystems (4), (5) admit zero solutions. Our objective is to derive conditions under which the asymptotic stability of the zero solution of (4) and stability of the zero solution of (5) imply that the zero solution of (3) is stable with respect to all variables and asymptotically x-stable. It should be noted that for the case where the system (3) is delay-free and $F(x) = x$ such conditions are defined by Lyapunov–Malkin Theorem [Malkin, 1963] and its generalizations (see [Vorotnikov, 1998]). In [Halanay and Safta, 2020], Lyapunov–Malkin Theorem was extended to systems with constant delay. A counterpart of Lyapunov–Malkin Theorem for systems with continuous and bounded delay and power type functions $f_i(x_i)$ was obtained in [Aleksandrov, Aleksandrova, Zhabko and Chen, 2017]. In this contribution, the case of distributed delay is studied.

4 Stability Analysis

We impose some additional constraints on the right-hand side of (3).

Assumption 3. *The matrix $P + hQ$ is diagonally stable.*

Assumption 4. *Let $\mu_i > 1$, $i = 1, \ldots, n$.*

Remark 1. Assumption 4 means that the subsystem (4) is strongly nonlinear.

Remark 2. In [Aleksandrov, 2024], it was proved that if Assumptions 1, 3 and 4 are satisfied, then the zero solution of (4) is asymptotically stable.

Assumption 5. *The estimate*

$$\|D(t, x_t, y(t))\|_h \leq \beta_1(x_t, y(t))\Big(\|F(x(t))\|$$

$$+\int_{t-h}^{t} \|F(x(s))\| ds\Big)$$

holds for $t \geq 0$, $\|x_t\|_h < \rho$, $\|y(t)\| < \rho$, where $\beta_1(x_t, y(t)) \to 0$ as $\|x_t\|_h + \|y(t)\| \to 0$.

Assumption 6. *The estimate*

$$\|G(t, x_t, y(t))\|_h \leq \beta_2 \|x_t\|_h^{\nu}$$

holds for $t \geq 0$, $\|x_t\|_h < \rho$, $\|y(t)\| < \rho$, where $\beta_2 > 0$, $\nu > 0$.

Assumption 7. *The zero solution of (5) is stable, and for this subsystem there exists a continuously differentiable for $t \geq 0$, $\|y\| < \rho$ Lyapunov function $V_1(t, y)$ satisfying the conditions of the Lyapunov Stability Theorem, see [Lyapunov, 1992], and such that $\|\partial V_1(t, y)/\partial y\| \leq K$ for $t \geq 0$, $\|y\| < \rho$, where K is a positive constant.*

Theorem 1. *Let*

$$\nu > \mu_n - 1 \tag{6}$$

and Assumptions 1–7 be fulfilled. Then the zero solution of (3) is stable with respect to all variables and asymptotically x-stable.

Proof. In [Aleksandrov, 2024], it was shown that, under Assumptions 1, 3 and 4, a Lyapunov–Krasovskii functional for subsystem (4) can be constructed as follows:

$$V_2(x_t) = \sum_{i=1}^{n} \lambda_i \frac{x_i^{\mu_i+1}(t)}{\mu_i + 1}$$

$$+F^{\top}(x(t))\Lambda Q \int_{t-h}^{t} (s - t + h) F(x(s)) ds$$

$$+\int_{t-h}^{t} (\gamma + \beta(s - t + h)) \|F(x(s))\|^2 ds. \tag{7}$$

Here γ and β are positive tuning parameters, $\lambda_i > 0$ are entries of a matrix $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ for which the matrix $\Lambda(P + hQ) + (P + hQ)^{\top}\Lambda$ is negative definite.

Differentiating the functional (7) with respect to the system (3), we obtain (see [Aleksandrov, 2024]) that, under an appropriate choice of positive numbers of γ, β, ρ_1, the inequality

$$\dot{V}_2 \leq -\alpha_1 \|F(x(t)\|^2 - \beta \int_{t-h}^{t} \|F(x(s))\|^2 ds$$

$$+\alpha_2 \beta_1(x_t, y(t)) \|F(x(t)\| \left(\int_{t-h}^{t} \|F(x(s))\| ds\right.$$

$$\left.+\|F(x(t))\|\right) + (\alpha_3 + \beta_1(x_t, y(t)))\Big(\|F(x(t))\|$$

$$+\int_{t-h}^{t} \|F(x(s))\| ds\Big) \int_{t-h}^{t} \|F(x(s))\| ds \left\| \frac{\partial F(x(t))}{\partial x} \right\|$$

holds for

$$\|x_t\|_h + \|y(t)\| < \rho_1, \tag{8}$$

where $\alpha_1, \alpha_2, \alpha_3$ are positive constants.

If ρ_1 is sufficiently small, then, under the condition (8), the estimates

$$\alpha_4 \sum_{i=1}^{n} x_i^{\mu_i+1}(t) + \frac{1}{2}\gamma \int_{t-h}^{t} \|F(x(s))\|^2 ds \le V_2(x_t)$$

$$\le \alpha_5 \sum_{i=1}^{n} x_i^{\mu_i+1}(t) + 2(\gamma+\beta h) \int_{t-h}^{t} \|F(x(s))\|^2 ds, \tag{9}$$

$$\dot{V}_2 \le -\frac{1}{2}\alpha_1 \|F(x(t)\|^2 - \frac{1}{2}\beta \int_{t-h}^{t} \|F(x(s))\|^2 ds$$

are valid, where $\alpha_4 > 0$, $\alpha_5 > 0$. Hence, if the condition (8) holds for $t \in [t_0, \bar{t}]$, then

$$\dot{V}_2(x_t) \le -\alpha_6 V_2^{1+\frac{\mu_n-1}{\mu_n+1}}(x_t), \tag{10}$$

where $\alpha_6 = \text{const} > 0$.

Integrating the differential inequality (10) and taking into account the lower estimate in (9), we obtain

$$\|x(t)\|^{\mu_n+1} \le \alpha_7 \sum_{i=1}^{n} x_i^{\mu_i+1}(t) \le \alpha_8 V_2(x_{t_0}) \Big(1$$

$$+\alpha_9 V_2^{\frac{\mu_n-1}{\mu_n+1}}(x_{t_0})(t-t_0)\Big)^{-\frac{\mu_n+1}{\mu_n-1}} \tag{11}$$

for $t \in [t_0, \bar{t}]$, where $\alpha_7, \alpha_8, \alpha_9$ are positive constants.

Next, differentiating the function $V_1(t, y)$ with respect to the system (3), we have

$$\dot{V}_1(t, y(t)) \le \beta_2 K \|x_t\|_h^{\nu}.$$

Hence,

$$V_1(t, y(t)) \le V_1(t_0, y(t_0)) + \beta_2 K \int_{t_0}^{t} \|x_s\|_h^{\nu} ds$$

$$\le V_1(t_0, y(t_0)) + \omega_1 + \omega_2 V_2^{\frac{\nu}{\mu_n+1}}(x_{t_0}) \int_{t_0+h}^{t} \Big(1$$

$$+\alpha_9 V_2^{\frac{\mu_n-1}{\mu_n+1}}(x_{t_0})(s-t_0-h)\Big)^{-\frac{\nu}{\mu_n-1}} ds$$

$$\le V_1(t_0, y(t_0)) + \omega_1$$

$$+\omega_2 V_2^{\frac{\nu-\mu_n+1}{\mu_n+1}}(x_{t_0}) \int_0^{+\infty} (1+\alpha_9\tau)^{-\frac{\nu}{\mu_n-1}} d\tau \tag{12}$$

for $t \in [t_0, \bar{t}]$. Here

$$\omega_1 = \beta_2 K h \max\left\{ \|x_{t_0}\|_h^{\nu};\ (\alpha_8 V_2(x_{t_0}))^{\frac{\nu}{\mu_n+1}} \right\},$$

$$\omega_2 = \beta_2 K \alpha_8^{\frac{\nu}{\mu_n+1}}.$$

Thus, if the condition (6) holds, $t_0 \ge 0$ and values of $\|x_{t_0}\|_\tau$ and $\|y(t_0)\|$ are sufficiently small, then, for the corresponding solution $(x^\top(t), y^\top(t))^\top$ of (3) the inequalities (8), (11), (12) are satisfied for all $t \ge t_0$. This completes the proof. □

Next, consider the case where the system (3) is of the form

$$\begin{aligned} \dot{x}(t) &= PF(x(t)) + Q \textstyle\int_{t-h}^{t} F(x(s)) ds \\ &\quad + D(t, x_t, y(t)), \\ \dot{y}(t) &= L(t, y(t)) + AF(x(t)) + B \textstyle\int_{t-h}^{t} F(x(s)) ds. \end{aligned} \tag{13}$$

Here A, B are constant matrices and the remaining notation is the same as for (3).

Similarly to the proof of Theorem 1 it can be verified that the following theorem is valid

Theorem 2. *Let*

$$2\mu_1 > \mu_n - 1 \tag{14}$$

and Assumptions 1–5, 7 be fulfilled. Then the zero solution of (13) is stable with respect to all variables and asymptotically x-stable.

Let us show that, under an additional constraint on matrices P and Q, the condition (14) in Theorem 2 can be eliminated.

Assumption 8. *For any positive rational number r with odd numerator and denominator, there exists a positive definite matrix $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ such that the function $\tilde{F}_r^\top(x)\Lambda(P+hQ)F(x)$ is negative definite, where $\tilde{F}_r(x) = (f_1^r(x_1), \ldots, f_n^r(x_n))^\top$.*

Remark 3. It is known [Aleksandrov, 2021] that Assumption 8 is satisfied if the matrix $P + hQ$ is Metzler and Hurwitz.

Theorem 3. *Let Assumptions 1, 2, 4, 5, 8 be fulfilled. Then the zero solution of (13) is stable with respect to all variables and asymptotically x-stable.*

Proof. For an arbitrary chosen positive rational number r with odd numerator and denominator, construct a Lyapunov–Krasovskii functional for (13) by the formula

$$V_3(x_t) = \sum_{i=1}^{n} \lambda_i \frac{x_i^{r\mu_i+1}(t)}{r\mu_i+1}$$

$$+\tilde{F}_r^\top(x(t))\Lambda Q\int_{t-h}^{t}(s-t+h)F(x(s))ds$$

$$+\int_{t-h}^{t}(\gamma+\beta(s-t+h))\|F(x(s))\|^{r+1}ds.$$

Here γ and β are positive tuning parameters, λ_i are entries of a matrix $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ with the properties specified in Assumption 8.

In this case, for an appropriate choice of γ, β, ρ_1, we obtain that if the condition (8) holds, then

$$\dot{V}_3(x_t) \le -\tilde{\alpha}V_3^{1+\frac{\mu_n-1}{r\mu_n+1}}(x_t),$$

where $\tilde{\alpha} = \mathrm{const} > 0$.

Using similar arguments as in the proof of Theorem 1, it is easy to verify that, under the condition

$$\mu_1(r+1) > \mu_n - 1, \qquad (15)$$

the zero solution of (13) is stable with respect to all variables and asymptotically x-stable. Thus, for any set of powers $\mu_1, \ldots, \mu_n$, one can take sufficiently large value of r to ensure the fulfillment of (15). The proof is completed. □

5 Persidskii Type Subsystems with Nontrivial Linear Approximations

The results of the previous section are obtained for the case where the subsystem (4) is strongly nonlinear (see Assumption 4). However, in numerous applications, models with nontrivial linear approximations are used [Fridman, 2014; Kazkurewicz and Bhaya, 1999; Lur'e, 1957; Vorotnikov, 1998]. Therefore, in this section, we will study two scenarios for which the Persidskii type subsystem contains both linear and strongly nonlinear terms.

5.1 Lur'e type subsystem

First, assume that with the aid of the decomposition $x(t) = (\eta^\top(t), \zeta^\top(t))^\top$ the subsystem (4) is rewritten as follows:

$$\begin{aligned}\dot{\eta}(t) &= P_1\Psi(\eta(t)) + Q_1\textstyle\int_{t-h}^{t}\Psi(\eta(s))ds + P_2\zeta(t),\\ \dot{\zeta}(t) &= P_3\Psi(\eta(t)) + Q_2\textstyle\int_{t-h}^{t}\Psi(\eta(s))ds + P_4\zeta(t).\end{aligned} \qquad (16)$$

Here $\eta(t) \in \mathbb{R}^{n_1}$, $\zeta(t) \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, $P_1, P_2, P_3, P_4, Q_1, Q_2$ are constant matrices, $\Psi(\eta) = (\eta_1^{\mu_1}, \ldots, \eta_{n_1}^{\mu_{n_1}})^\top$, μ_j are rational numbers with odd numerators and denominators, $j = 1, \ldots, n_1$, $1 < \mu_1 \le \ldots \le \mu_{n_1}$. Thus, linear terms in (16) are delay free, whereas strongly nonlinear terms may be delay dependent. It is worth noticing that (16) can be considered as Lur'e indirect control system with distributed delay in the feedback law [Fridman, 2014; Kazkurewicz and Bhaya, 1999; Lur'e, 1957].

Assumption 9. *The matrix P_4 is Hurwitz.*

Assumption 10. *The matrix*

$$\tilde{P} = P_1 + hQ_1 - P_2P_4^{-1}(P_3 + hQ_2)$$

is diagonally stable.

Assumption 11. *The estimate*

$$\|D(t, x_t, y(t))\|_h \le \beta_1(x_t, y(t))\Big(\|\Psi(\eta(t))\|$$

$$+\int_{t-h}^{t}\|\Psi(\eta(s))\|ds + \|\zeta(t)\|\Big)$$

holds for $t \ge 0$, $\|x_t\|_h < \rho$, $\|y(t)\| < \rho$, where $\beta_1(x_t, y(t)) \to 0$ as $\|x_t\|_h + \|y(t)\| \to 0$.

Theorem 4. *Let*

$$\nu > \mu_{n_1} - 1$$

and Assumptions 2, 6, 7, 9–11 be fulfilled. Then the zero solution of (3) is stable with respect to all variables and asymptotically x-stable.

Proof. In [Aleksandrov, 2024], it was proposed to construct a Lyapunov–Krasovskii functional for (16) in the form

$$V_4(\eta_t, \zeta(t)) = \sum_{j=1}^{n_1}\lambda_j\frac{\eta_j^{\mu_j+1}(t)}{\mu_j+1} + \varepsilon\zeta^\top(t)\Delta\zeta(t)$$

$$+\Psi^\top(\eta(t))\Lambda Q_1\int_{t-h}^{t}(s-t+h)\Psi(\eta(s))ds$$

$$-\Psi^\top(\eta(t))\Lambda P_2P_4^{-1}Q_2\int_{t-h}^{t}(s-t+h)\Psi(\eta(s))ds$$

$$+\int_{t-h}^{t}(\gamma+\beta(s-t+h))\|\Psi(\eta(s))\|^2\,ds$$

$$-\Psi^\top(\eta(t))\Lambda P_2P_4^{-1}\zeta(t).$$

Here $\beta, \gamma, \varepsilon$ are positive tuning parameters, Δ is a symmetric positive definite matrix for which the matrix $\Delta P_4 + P_4^\top\Delta$ is negative definite, $\lambda_i > 0$ are entries of a matrix $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ for which the matrix $\Lambda\tilde{P} + \tilde{P}^\top\Lambda$ is negative definite.

It can be easily verified that there exist such values of $\gamma, \beta, \varepsilon, \rho_1$ that, if the condition (8) holds, then the derivative of $V_4(\eta_t, \zeta(t))$ with respect to the system (3) satisfies the estimate

$$\dot{V}_4(x_t) \le -\hat{\alpha}V_4^{1+\frac{\mu_{n_1}-1}{\mu_{n_1}+1}}(x_t),$$

where $\hat{\alpha} = \mathrm{const} > 0$. The subsequent proof is similar

5.2 Positive Persidskii type subsystem

Next, consider the case where the subsystem (4) is positive. Let Assumption 1 be satisfied.

Assumption 12. *The matrix P is Metzler and the matrix B is nonnegative.*

Remark 4. Assumption 12 is the necessary and sufficient condition for positivity of (4), see [Aleksandrov, 2018].

Assumption 13. *The matrix $P + hQ$ is Hurwitz.*

Remark 5. It is known [Aleksandrov, 2018], that, under Assumptions 12 and 13, the zero solution of (4) is asymptotically stable.

Assumption 14. *Let $1 \leq \mu_1 \ldots \leq \mu_n$ and $\mu_n > 1$.*

Remark 6. Unlike to the previous subsection, here both linear and strongly nonlinear terms in (4) may be delay dependent.

Theorem 5. *Let the inequality (6) and Assumptions 1, 2, 5–7, 12–14 be fulfilled. Then the zero solution of (3) is stable with respect to all variables and asymptotically x-stable.*

Proof. Applying the approach developed in [Aleksandrov, 2018], construct a Lyapunov–Krasovskii functional as follows:

$$V_5(x_t) = \sum_{i=1}^{n} \lambda_i \frac{x_i^{\mu_i+1}(t)}{\mu_i + 1}$$

$$+ \int_{-h}^{0} \int_{t+\theta}^{t} F^\top(x(u)) \Gamma F(x(u)) du d\theta$$

$$+ \gamma \int_{-h}^{0} \int_{t+\theta}^{t} (u - \theta - t) \|F(x(u))\|^2 du d\theta,$$

where $\gamma, \lambda_1, \ldots, \lambda_n$ are positive coefficients, Γ is a diagonal positive definite matrix.

Using results of [Aleksandrov, 2018], it is easy to verify that there exist matrix Γ and positive numbers $\rho_1, \gamma, \lambda_1, \ldots, \lambda_n$ such that, under the condition (8), functional $V_5(x_t)$ and its derivative with respect to the system (3) satisfy the estimates

$$\sum_{i=1}^{n} \lambda_i \frac{x_i^{\mu_i+1}(t)}{\mu_i + 1} \leq V_5(x_t) \leq \sum_{i=1}^{n} \lambda_i \frac{x_i^{\mu_i+1}(t)}{\mu_i + 1}$$

$$+ c_1 \int_{-h}^{0} \int_{t+\theta}^{t} \|F(x(u))\|^2 du d\theta,$$

$$\dot{V}_5 \leq -c_2 \|F(x(t))\|^2 - \gamma \int_{-h}^{0} \int_{t+\theta}^{t} \|F(x(u))\|^2 du d\theta,$$

where c_1 and c_2 are positive coefficients. Hence, we arrive at the differential inequality

$$\dot{V}_5(x_t) \leq -\bar{\alpha} V_5^{1+\frac{\mu_n - 1}{\mu_n + 1}}(x_t),$$

where $\bar{\alpha} = \text{const} > 0$. The remaining part of the proof is similar to that of Theorem 1. □

6 Examples

Consider some examples of the application of the developed theory.

6.1 The damping the angular motions of a rigid body

Let a rigid body rotate around its mass center O with the angular velocity $\omega(t) = (\omega_1(t), \omega_2(t), \omega_3(t))^\top$.

The attitude motion of the body under a control torque M is modeled by the dynamical Euler equations (see [Rumyantsev and Oziraner, 1987])

$$J\,\dot{\omega}(t) + \omega(t) \times (J\omega(t)) = M. \qquad (17)$$

Here $J = \operatorname{diag}\{J_1, J_2, J_3\}$ is the inertia tensor of the body (we assume that axes of the coordinate system coincide with the principal central axes of inertia of the body).

Let $M = (M_1, M_2, M_3)^\top$. Here

$$M_i = p_i \omega_i^{\mu_i}(t) + q_i \int_{t-h}^{t} \omega_i^{\mu_i}(s) ds, \quad i = 1, 2,$$

$M_3 = 0$, p_i and q_i are constant coefficients, μ_1 and μ_2 are rational numbers with odd numerators and denominators, $1 < \mu_1 \leq \mu_2$, $h = \text{const} > 0$.

Then the system (17) takes the form

$$\begin{aligned} J_1\dot{\omega}_1(t) &= (J_2 - J_3)\omega_2(t)\omega_3(t) \\ &\quad + p_1\omega_1^{\mu_1}(t) + q_1 \textstyle\int_{t-h}^{t} \omega_1^{\mu_1}(s) ds, \\ J_2\dot{\omega}_2(t) &= (J_3 - J_1)\omega_1(t)\omega_3(t) \\ &\quad + p_2\omega_2^{\mu_2}(t) + q_2 \textstyle\int_{t-h}^{t} \omega_2^{\mu_2}(s) ds, \\ J_3\dot{\omega}_3(t) &= (J_1 - J_2)\omega_1(t)\omega_2(t). \end{aligned} \qquad (18)$$

Consider the case where $J_1 > J_3$, $J_2 > J_3$. Choose a Lyapunov–Krasovskii functional for (18) as follows:

$$V_6(\omega_{1t}, \omega_{2t}) = J_1(J_1 - J_3)\omega_1^2(t) + J_2(J_2 - J_3)\omega_2^2(t)$$

$$+ \int_{t-h}^{t} (\gamma + \beta(s - t + h)) \left(\omega_1^{\mu_1+1}(s) + \omega_2^{\mu_2+1}(s)\right) ds$$

$$+2q_1(J_1 - J_3)\omega_1(t)\int_{t-h}^{t}(s-t+h)\omega_1^{\mu_1}(s)ds$$

$$+2q_2(J_2 - J_3)\omega_2(t)\int_{t-h}^{t}(s-t+h)\omega_2^{\mu_2}(s)ds, \quad (19)$$

where γ and β are positive parameters.

It is worth mentioning that Assumption 5 is not satisfied for the system (18). However, using the functional (19) and arguments similar to those in the proof of Theorem 1, it can be verified that if $p_i + hq_i < 0$, $i = 1, 2$, and $\mu_2 < 3$, then the zero solution of (18) is stable with respect to all variables and asymptotically stable with respect to ω_1, ω_2.

6.2 Interaction of two mechanical systems

Let the system be given

$$\ddot{\eta}(t) + B\dot{\eta}(t) + P\Psi(\eta(t)) + Q\int_{t-h}^{t}\Psi(\eta(s))ds = D(\eta_t, \xi(t), \dot{\xi}(t)), \quad (20)$$
$$\ddot{\xi}(t) + C\xi(t) = G(\eta_t, \xi(t), \dot{\xi}(t)).$$

Here $\eta(t) \in \mathbb{R}^l$, $\xi(t) \in \mathbb{R}^m$ are vectors of generalized coordinates, B, P, Q are constant matrices, C is a constant symmetric and positive definite matrix, $h = \text{const} > 0$, $\Psi(w) = (\eta_1^{\mu_1}, \dots, \eta_l^{\mu_l})^\top$, μ_j are rational numbers with odd numerators and denominators, $j = 1, \dots, l$, $1 < \mu_1 \le \dots \le \mu_l$, functionals $D(t, \eta_t, \xi(t), \dot{\xi}(t))$ and $G(t, \eta_t, \xi(t), \dot{\xi}(t))$ are continuous for $t \ge 0$, $\|\eta_t\|_h < \rho$, $\|\xi(t)\| < \rho$, $\|\dot{\xi}(t)\| < \rho$, $0 < \rho \le +\infty$.

The system (20) models the interaction of the linear potential mechanical subsystem

$$\ddot{\xi}(t) + C\xi(t) = 0 \quad (21)$$

and the mechanical subsystem with linear velocity forces and strongly nonlinear positional ones

$$\ddot{\eta}(t) + B\dot{\eta}(t) + P\Psi(\eta(t)) + Q\int_{t-h}^{t}\Psi(\eta(s))ds = 0, \quad (22)$$

the functionals $D(t, \eta_t, \xi(t), \dot{\xi}(t))$, $G(t, \eta_t, \xi(t), \dot{\xi}(t))$ describe couplings between the subsystems whereas the term $Q\int_{t-h}^{t}\Psi(\eta(s))ds$ can be interpreted as integral part of a PID-controller [Formal'sky, 1997].

It is well known [Lyapunov, 1992], that the equilibrium position $\xi = \dot{\xi} = 0$ of (21) is stable, and this subsystem admits the Lyapunov function $V_7(\xi, \dot{\xi}) = \|\dot{\xi}\|^2 + \xi^\top C\xi$.

Rewrite (22) in the form

$$\dot{\eta}(t) = \zeta(t),$$
$$\dot{\zeta}(t) = -B\zeta(t) - P\Psi(\eta(t)) - Q\int_{t-h}^{t}\Psi(\eta(s))ds. \quad (23)$$

It is worth noticing that (23) is a special case of (16).

Assume that the following conditions are valid:
(i) the matrix $-B$ is Hurwitz;
(ii) the matrix $-B^{-1}(P + hQ)$ is diagonally stable;
(iii) the estimate

$$\|D(t, \eta_t, \xi(t), \dot{\xi}(t))\|_h \le \beta_1(\eta_t, \xi(t), \dot{\xi}(t))\Big(\|\Psi(\eta(t))\| + \int_{t-h}^{t}\|\Psi(\eta(s))\|ds + \|\dot{\eta}(t)\|\Big)$$

holds for $t \ge 0$, $\|\eta_t\|_h < \rho$, $\|\xi(t)\| < \rho$, $\|\dot{\xi}(t)\| < \rho$, where $\beta_1(\eta_t, \xi(t), \dot{\xi}(t)) \to 0$ as $\|\eta_t\|_h + \|\xi(t)\| + \|\dot{\xi}(t)\| \to 0$;
(iv) the estimate

$$\|G(t, x_t, y(t))\|_h \le \beta_2\|\eta_t\|_h^\nu$$

holds for $t \ge 0$, $\|\eta_t\|_h < \rho$, $\|\xi(t)\| < \rho$, $\|\dot{\xi}(t)\| < \rho$, where $\beta_2 > 0$, $\nu > 0$.

Applying Theorem 4, we obtain that if $\nu > \mu_l - 1$, then the equilibrium position $\eta = \dot{\eta} = 0$, $\xi = \dot{\xi} = 0$ of (20) is stable with respect to all variables and asymptotically stable with respect to $\eta, \dot{\eta}$.

7 Conclusion

In the present paper, new sufficient conditions of the asymptotic stability with respect to a part of variables are derived for a class of nonlinear systems with distributed delay. Our analysis was based on the using the differential inequalities method, the Lyapunov direct method and special constructions of Lyapunov–Krasovskii functionals. It should be noted that, with the aid of the proposed functionals, not only stability conditions but also estimates for solutions of the investigated systems were obtained. An interesting direction for further research is an extension of the developed approach to systems with unbounded delay.

Acknowledgements

The research was supported by the grant of Russian Science Foundation, N 24-21-00091, https://rscf.ru/en/project/24-21-00091/.

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