FEEDFORWARD CONTROL FOR TWO-LEVEL ATOM IN MODULATED OPTICAL FIELD

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Abstract

We propose a model of feedforward (open-loop) optical control of two-level atom in the linearized form. This model allows to express the general form of solution for the atomic level populations via the arbitrary shapes of the control signal. Then we make numerical investigations of different shapes for the optical control signal.

Key words

Feedforward control; Quantum optics; Two-level atomic system; Density matrix; Linearization.

1 Introduction

A wide spectrum of control methods can be discovered for the quantum systems. Among them feedforward (open-loop) approach seems to be the most natural, since an applied external field, can be easily designed as a time-dependent function. Here we will discuss the basic, but very important case of two-level atomic system controlled by modulated optical field. Our choice has been motivated by developed technique for practical design of external field in quantum optics.

Recently other authors studied the control of two-level atoms in the frame of open loop-ideology when the controlling field was known *a priori*. It allowed to get the different forms of atomic energy spectra, producing π- and $\pi/2$-pulses [Imoto 1996], including the observation of the geometric phase using stimulated photon echoes [Tian *et al.* 2004], taking special non-constant shapes of external field [Di. Piazza 2001] etc.

We propose a model of feedforward control for the density matrix in the linearized form. We use the "semiclassical approach" of the atom–field interaction, when a single quantum two-level atomic system (all other levels are neglected) is interacting with classical electromagnetic field. We use the standard notation following [Scully 2006], but in our model the optical field plays the role of a control signal $u(t)$ for open-loop

(feedforward) control scheme [Fradkov 1999]. A similar case for the probability amplitudes (without decay) is described in the frame of closed-loop scheme in [Saifullah, 2008]. The present model has a decay component, because it involves the effect of elastic collisions between atoms.

In Section 2 we present our dynamical model with atomic level population decay in generalized dimensionless form and then apply the linearized control procedure for different shapes of the optical control field $u(t)$. This model allows to express the general form of solution for the atomic level populations via the arbitrary shapes of the control signal. Then in Section 3 we make numerical investigations of different shapes for the signal u.

2 Feedforward optical control for two-level atom
2.1 Dynamical control model for two-level atom in classical optical field

We consider the quantum two-level atomic system in the classical optical field $E(t)$.
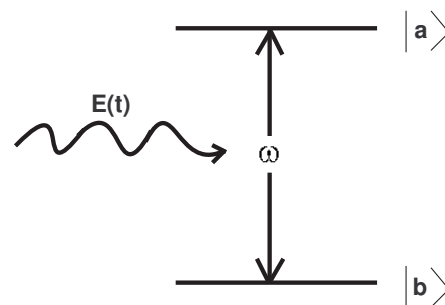

Figure 1. Interaction of a single two-level atom with an optical field.

Let $|a\rangle$ and $|b\rangle$ represent the upper and lower level states of the atom, i.e., they are eigenstates of the un-

perturbed part of the Hamiltonian $\hat{H}_0$ with the eigenvalues: $\hat{H}_0|a\rangle = \hbar\omega_a|a\rangle$ and $\hat{H}_0|b\rangle = \hbar\omega_b|b\rangle$.

The equations of motion for the density matrix elements are given by [Scully 2006]:

$$\dot{\rho}_{aa} = -\gamma_a\rho_{aa} + \frac{iE}{\hbar}\left(\wp_{ab}\rho_{ba}e^{i\omega t} - \wp_{ab}^*\rho_{ab}e^{-i\omega t}\right) \; ;$$

$$\dot{\rho}_{bb} = -\gamma_b\rho_{bb} - \frac{iE}{\hbar}\left(\wp_{ab}\rho_{ba}e^{i\omega t} - \wp_{ab}^*\rho_{ab}e^{-i\omega t}\right) \; ;$$

$$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - \frac{iE}{\hbar}\wp_{ab}(\rho_{aa} - \rho_{bb})e^{i\omega t} \; , \qquad (1)$$

where $\rho_{ba} = \rho_{ab}^*$; $\wp_{ab}$ is the matrix element of the electric dipole moment, γ_a and γ_b are the decay constants, $\gamma_{ab} = (\gamma_a + \gamma_b)/2 + \gamma_{ph}$, γ_{ph} is a decay rate including elastic collisions between atoms, and $\omega = \omega_a - \omega_b$ is the atomic transition frequency.

Let's denote $\wp_{ab} = |\wp_{ab}|e^{i\phi}$ and

$$\rho_+ \equiv \rho_{ba}e^{i(\omega t+\phi)} + \rho_{ab}e^{-i(\omega t+\phi)} \; ;$$

$$\rho_- \equiv i\left[\rho_{ba}e^{i(\omega t+\phi)} - \rho_{ab}e^{-i(\omega t+\phi)}\right] \; . \qquad (2)$$

Using (2) we can re-write the system (1) in the real form:

$$\dot{\rho}_{aa} = -\gamma_a\rho_{aa} + \frac{|\wp_{ab}|E}{\hbar}\cdot\rho_- \; ;$$

$$\dot{\rho}_{bb} = -\gamma_b\rho_{bb} - \frac{|\wp_{ab}|E}{\hbar}\cdot\rho_- \; ;$$

$$\dot{\rho}_+ = -\gamma_{ab}\rho_+ + \omega\rho_- \; ; \qquad (3)$$

$$\dot{\rho}_- = -\gamma_{ab}\rho_- - \omega\rho_+ - \frac{2|\wp_{ab}|E}{\hbar}\cdot(\rho_{aa} - \rho_{bb}) \; .$$

For further calculations we put $\gamma_a = \gamma_b \equiv \gamma$. Then

$$(\rho_{aa} + \rho_{bb})(t) = e^{-\gamma t}(\rho_{aa} + \rho_{bb})(0) \; . \qquad (4)$$

The first two equations of the system (3) can be combined together and we can put:

$$\rho_{aa}(t) - \rho_{bb}(t) \equiv e^{-\gamma t}x(t) \; ;$$

$$\rho_+(t) \equiv e^{-\gamma t}y(t) \; ; \qquad (5)$$

$$\rho_-(t) \equiv e^{-\gamma t}z(t) \; .$$

By substitution of (5) in (3) we can eliminate the decay γ-containing terms. Finally, rescaling the time by ω: $\tau = \omega t$, and denoting the dimensionless control signal by $u(t) \equiv 2|\wp_{ab}|E(t)/\hbar\omega$ and $\epsilon = \gamma_{ph}/\omega$, we get the simplified system

$$\dot{x} = u \cdot z \; ;$$

$$\dot{y} = -\epsilon \cdot y + z \; ; \qquad (6)$$

$$\dot{z} = -\epsilon \cdot z - y - u \cdot x \; .$$

Here the dot means the derivative with respect to the new dimensionless time τ. We remind that $x \in [-1, 1]$, since $(\rho_{aa} - \rho_{bb}) \in [-1, 1]$, and $(\rho_{aa} - \rho_{bb}) \to 0$ as $t \to \infty$.

2.2 Linearization of control

Let's suppose that we apply the linearized form of control:

$$x(\tau) = X_0(\tau) + u \cdot X_1(\tau) \; ;$$

$$y(\tau) = Y_0(\tau) + u \cdot Y_1(\tau) \; ; \qquad (7)$$

$$z(\tau) = Z_0(\tau) + u \cdot Z_1(\tau) \; .$$

We will skip all the terms of the order u^2 and elder. Then substituting (7) in (6), we split our system into two parts: the free (non-controlled) system:

$$\dot{X}_0 = 0 \; ;$$

$$\dot{Y}_0 = -\epsilon \cdot Y_0 + Z_0 \; ; \qquad (8)$$

$$\dot{Z}_0 = -\epsilon \cdot Z_0 - Y_0$$

and the controlled part:

$$\dot{u} \cdot X_1 + u \cdot \dot{X}_1 = u \cdot Z_0 \; ;$$

$$\dot{u} \cdot Y_1 + u \cdot \dot{Y}_1 = u \cdot Z_1 \; ; \qquad (9)$$

$$\dot{u} \cdot Z_1 + u \cdot \dot{Z}_1 = -u \cdot Y_1 - u \cdot X_0 \; .$$

In (9) we omitted the decay ϵ-terms, because the decay is supposed to be a slow process to compare with the control, i.e. ϵ and u are the small parameters of the same order, and the linearization deals only with their first orders. Then from the first equation of system (9) we get:

$$u(\tau)X_1(\tau) = \int_0^\tau u(t')Z_0(t')dt'$$

and from the first equation of system (7), we have

$$x(\tau) = X_0(\tau) + \int_0^\tau u(t')Z_0(t')dt' \; . \qquad (10)$$

Now we apply the initial conditions $X_0(0)$, $Y_0(0)$, $Z_0(0)$ to solve the system (8):

$$X_0(\tau) = X_0(0) \equiv x(0) \; ;$$

$$Y_0(\tau) = e^{-\epsilon\tau}\left[Y_0(0)\cos\tau + Z_0(0)\sin\tau\right] \; ; \quad (11)$$

$$Z_0(\tau) = e^{-\epsilon\tau}\left[Z_0(0)\cos\tau - Y_0(0)\sin\tau\right] \; .$$

If we denote the phase of ρ_{ab} by ϕ', then $\rho_+(0) = 2|\rho_{ab}|\cos(\phi' - \phi)$ and $\rho_-(0) = 2|\rho_{ab}|\sin(\phi' - \phi)$. We can put for the initial condition: $\phi' = \phi$, then $\rho_+(0) = 2|\rho_{ab}(0)| \equiv \delta$ and $\rho_-(0) = 0$. Let's demand $X_1(0) = Y_1(0) = Z_1(0) = 0$. Thus, $Y_0(0) = \delta$ and $Z_0(0) = 0$ are our initial conditions.

2.3 Control signal correction

If $X_0(0) = -1$ (that corresponds to the ground level of the atom as the initial condition), then from $-1 \leq x(\tau) \leq 1$ and (10) it follows:

$$0 \leq \int_0^\tau u(t')Z_0(t')dt' \leq 2. \qquad (12)$$

In other words this integral should be positive and bounded. We define first the arbitrary non-corrected control $u_0(\tau)$ and then put

$$\tilde{u}(\tau) \equiv |u_0(\tau)| \cdot \text{sign} Z_0(\tau) \,. \qquad (13)$$

Then the left inequality (12) will be satisfied automatically. The right part of (12) can be represented by Cauchy – Schwartz inequality:

$$\left| \int_0^\tau \tilde{u}(t')Z_0(t')dt' \right|^2 \leq \int_0^\tau \tilde{u}^2(t')dt' \cdot \int_0^\tau Z_0^2(t'')dt'' \,,$$

and then we demand:

$$\int_0^\tau \tilde{u}^2(t')dt' \cdot \int_0^\tau Z_0^2(t'')dt'' \leq 4 \,. \qquad (14)$$

Let's check the inequality (14):

$$\left| \int_0^\tau Z_0^2(t'')dt'' \right| = |Y_0(0)|^2 \left| \int_0^\tau e^{-2\epsilon t''} \sin^2 t'' dt'' \right| \leq$$

$$\leq \delta^2 \left| \int_0^\tau e^{-2\epsilon t''} dt'' \right| = \frac{\delta^2(1-e^{-2\epsilon\tau})}{2\epsilon} \,. \qquad (15)$$

Thus, from (14) and (15)

$$\int_0^\tau \tilde{u}^2(t')dt' \leq \frac{8\epsilon}{\delta^2(1-e^{-2\epsilon\tau})} \,. \qquad (16)$$

To satisfy (16) we also have to correct the signal u. Let's suppose that there are two functions: an initial arbitrary $u_0(\tau)$ and its corrected variant $u(\tau)$ that is bounded above by the condition (16). Of course, physically the external optical field should follow the signal u, and the initial u_0 is only a basic model to construct the behavior of the open-loop control field.

Now let's define

$$\Delta(\tau) \equiv \int_0^\tau \tilde{u}^2(t')dt' - \frac{8\epsilon}{\delta^2(1-e^{-2\epsilon\tau})} \qquad (17)$$

and

$$u(\tau) = \begin{cases} \tilde{u}(\tau) & , \ \Delta(\tau) < 0 \,; \\ B(\tau) & , \ \Delta(\tau) \geq 0 \,, \end{cases} \qquad (18)$$

where a positive function $B(\tau)$ is defined from the equation:

$$\int_0^\tau B^2(t')dt' = \frac{8\epsilon}{\delta^2(1-e^{-2\epsilon\tau})} \,, \qquad (19)$$

or

$$B^2(\tau) = \left| \frac{d}{d\tau} \frac{8\epsilon}{\delta^2(1-e^{-2\epsilon\tau})} \right| = \frac{16\epsilon^2}{\delta^2} \cdot \frac{e^{-2\epsilon\tau}}{(1-e^{-2\epsilon\tau})^2} \,.$$

Thus,

$$B(\tau) = \frac{4\epsilon}{\delta} \cdot \frac{e^{-\epsilon\tau}}{1-e^{-2\epsilon\tau}} = \frac{2\epsilon}{\delta \cdot \cosh(\epsilon\tau)} \,. \qquad (20)$$

For small time intervals $\tau << 1/\epsilon$ we have: $\delta^2\tau$ in RHS (15), and $B(\tau) \simeq 2/(\delta \cdot \tau)$.

Finally by the corrections (13) and (18) we have:

$$x(\tau) = -1 + \delta \cdot \int_0^\tau dt' \, e^{-\epsilon t'} |\sin t'| \cdot |u(t')| \,. \quad (21)$$

Eq.(21) solves the problem of open-loop control in linearized form. Now defining the control signal $u(\tau)$ we restore by (21) the shape of the difference $\rho_{aa}(t) - \rho_{bb}(t)$. Their sum (4) is known, thus, we can find $\rho_{aa}(t)$ and $\rho_{bb}(t)$ separately.

3 Numerical simulation of different shapes for the control signal

Now we can apply the general solution of Eq.(21) to study the influence of control optical field u on the behavior of the system (6).

In the case of an ideal open-loop control the behavior of $x(t)$ is the saturation of the population at the ground level, in other words, $x(t) \rightarrow 1$ as $t \rightarrow \infty$. Sure, not every control will satisfy this condition.

On. Figs. 2–6 we plot the different shapes of the initial u_0 (Figs.A) and corrected u (Figs.B) control signals: constant, ramp, step, sine wave, and repeating sequence stair. To compare their efficiency we check also the corresponding time derivatives dx/dt (Figs.D).

We can see from the plots that the ramp control in our case is definitely more effective. The speed of the saturation for $x(t)$ is faster for the signals on Figs. 3, 6.

4 Conclusion

Finally we can conclude that our model for the open-loop control has several important features:

1. It can be easily extended for the case of multi-level atomic systems by adding the correspondent components in the density matrix;

2. For the two-level system it can be re-formulated in general form if we propose the linear approximation of control;

3. It can be an origin of studying the behavior of controlled non-linear systems in quantum optics.
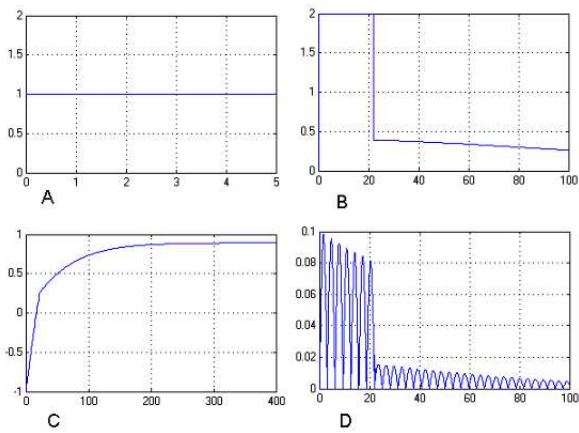
Figure 2. Constant control signal: (A) The initial signal $u_0(t)$; (B) The corrected signal $u(t)$; (C) $x(t)$; (D) The derivative dx/dt.
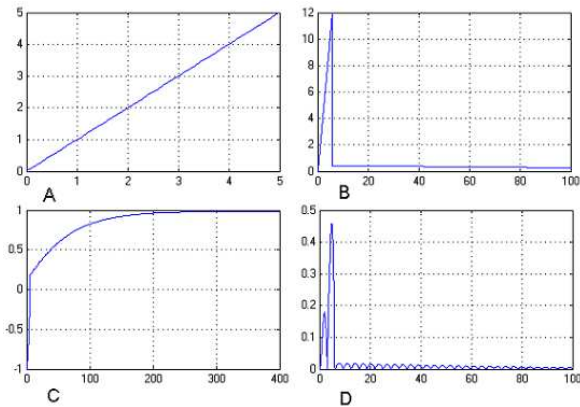

Figure 5. Sine wave control signal: (A) The initial signal $u_0(t)$; (B) The corrected signal $u(t)$; (C) $x(t)$; (D) The derivative dx/dt.

Figure 3. Ramp control signal: (A) The initial signal $u_0(t)$; (B) The corrected signal $u(t)$; (C) $x(t)$; (D) The derivative dx/dt.
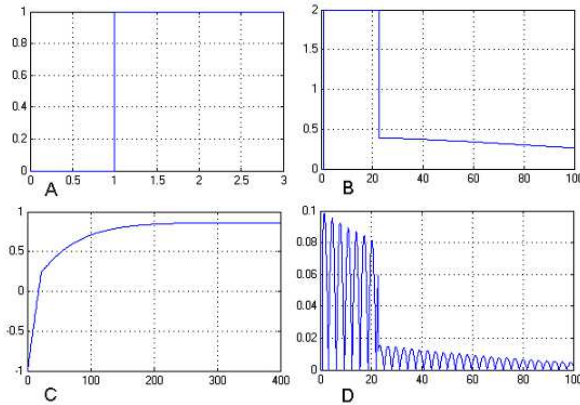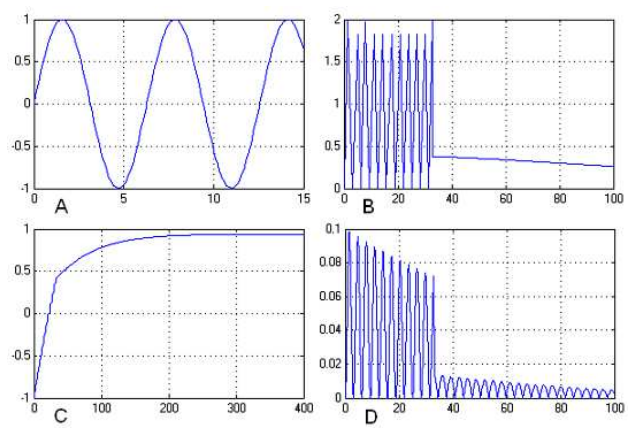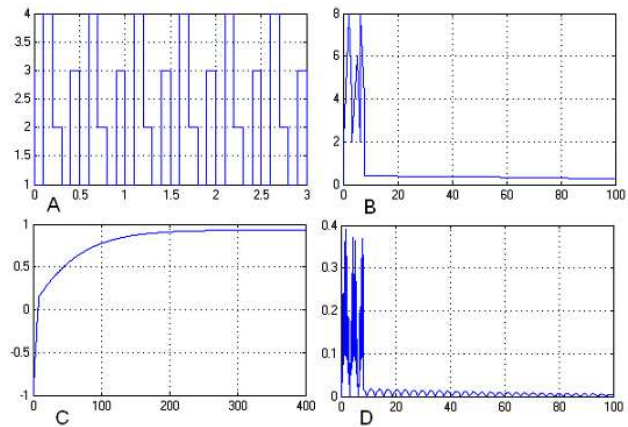

Figure 6. Repeating sequence stair control signal: (A) The initial signal $u_0(t)$; (B) The corrected signal $u(t)$; (C) $x(t)$; (D) The derivative dx/dt.

Figure 4. Step control signal: (A) The initial signal $u_0(t)$; (B) The corrected signal $u(t)$; (C) $x(t)$; (D) The derivative dx/dt.

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