Problems of Dynamic Optimization of Flow

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***Abstract*— The paper deals with mathematical models of motion of mechanical systems of sequentially joint solid bodies in a viscous medium. Control laws are found to move the considered systems from the initial state to a given one for optimum energy consumption.**

I. Introduction

Intelligent autonomous vehicles and robots intended for work in atypical environment has proved to form a great body of knowledge interesting from the viewpoint of challenging applications and being the source of new theoretical research. Particular emphasis is placed on mobile manipulation robots (just this term is preferred in E.P.Popov, A.F.Vereshchagin, S.L.Zenkevich [1], F.L.Chernous'ko, N.N.Bolotnik, and V.G.Gradetskii [2]) intended for work in a viscous medium. It is caused, for example, by a need in robots to inspect and assimilate water tanks, and to do various technological works in those places. Design of a special mobile manipulation robot (further we write, in shorthand, MMR) is a complicated problem. Working out control systems matching up the MMR destination is the principal step in solving this problem. The situation when one has to deal with rather limited energy supply of the MMR is natural and, sometimes, inevitable. Then, the following control problem is topical: to find the laws of the control forces and momentums behavior so as to move the MMR from the initial position to a given one for minimum energy consumption. Such a problem is close to the ones of dynamic optimization considered by F.L.Chernous'ko and other researches. So, the speech goes about a new set of problems being topical from the viewpoint of the theory of singular solutions of dynamic optimization problems [3]. The totality of the problems solved in the present paper can be used in both the applied theory of singular dynamic optimization problems and design of perspective samples of new machines.

II. Hydrodynamic Constraints

Hydrodynamic constraints listed below, being satisfied, give a possibility to analyze necessary conditions for optimality. It is assumed that an inertial system and, inside it, a right Cartesian coordinate system $Ox_1x_2x_3$ are chosen.

Let $\mathbf{v}(t,x)=\mathbf{v}(t,x_1,x_2,x_3)$ be the velocity vector of fluid particle at the point $M(x_1,x_2,x_3)$ at the instant t, and v_1, v_2, and v_3 be its projections in the coordinate axes. The first two constraints [4] are reduced to the following.

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L1: Fluid is incompressible.

With account of the equation of continuity, this constraint is equivalent to zero velocity of the volume strain $\operatorname{div}\mathbf{v}=0$.

L2: The generalized Newton hypothesis is fulfilled

$$P=-pE+\mu\Big(\frac{\partial\mathbf{v}}{\partial x}+\Big(\frac{\partial\mathbf{v}}{\partial x}\Big)^*\Big), \tag{1}$$

where P is the linear operator defined by the stress tensor, $p=p(t,x)$ denotes the scalar field of pressure, μ is the dynamic viscosity coefficient, E is the identity mapping, $\partial\mathbf{v}/\partial x$ is the Frechet derivative, and $(\partial\mathbf{v}/\partial x)^*$ is the conjugate operator.

Let a body of bounded size with sufficiently smooth boundary S [5] move in fluid. A fluid mechanics axiom and the constraint L1 imply that in the case of translational motion of the body the following equality is fulfilled at its surface

$$\Big(\frac{\partial\mathbf{v}}{\partial x}\Big)^*\mathbf{n}=0, \tag{2}$$

where $\mathbf{n}$ is the unit vector of the outward normal to the surface S at the point x.

The stress on an element dS of the body surface is calculated by the formula $\mathbf{p}_n=P\mathbf{n}$, where $\mathbf{n}$ is the unit vector of the outward normal to dS. This equality and (1) yield the formula for the principal vector of the forces acting from fluid upon the body surface (hydrodynamic forces)

$$\mathbf{R}=\iint\limits_S\Big(-pE+\mu\Big(\frac{\partial\mathbf{v}}{\partial x}+\Big(\frac{\partial\mathbf{v}}{\partial x}\Big)^*\Big)\mathbf{n}\,dS. \tag{3}$$

The formula for the principal momentum of hydrodynamic forces can be obtained similarly. According to (2), if the body moves translationally, then the formula (3) is reduced to

$$\mathbf{R}=\iint\limits_S\Big(-pE+\mu\frac{\partial\mathbf{v}}{\partial x}\Big)\mathbf{n}\,dS. \tag{4}$$

We need further the so-called moving coordinate system $O_cy_1y_2y_3$ with the body inertia center as the origin and the axes rigidly connected with the body.

To find the principal vector and momentum, one has to calculate on the body surface the pressure and the Frechet derivative of the fluid velocity vector. To do this, one has to solve a certain boundary-value problem for the vector-valued Navier–Stokes equation. This equation is written out below in the moving system $O_cy_1y_2y_3$ with axes parallel to the corresponding axes of the system $Ox_1x_2x_3$ (the body is assumed to move translationally). Let $\mathbf{V}$ be the velocity vector of the body, and $x_c(t)$ be the radius vector of its inertia center. In the moving coordinate system, denote the absolute velocity vector of fluid and the pressure as follows: $\hat{\mathbf{v}}(t,y) = \mathbf{v}(t, x_c(t)+y)$, $\hat{p}(t,y) = p(t, x_c(t)+y)$. Then the Navier–Stokes equation is of the form

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} = -\frac{\partial \hat{\mathbf{v}}}{\partial y}(\hat{\mathbf{v}} - \mathbf{V}) - \frac{1}{\rho}\Big(\frac{\partial \hat{p}}{\partial y}\Big)^{*} + \nu \operatorname{div} \frac{\partial \hat{\mathbf{v}}}{\partial y} + \mathbf{F}, \qquad (5)$$

where $\mathbf{F}$ is the strength of the gravity field, ρ is the fluid density, $\nu = \mu/\rho$ is the kinematic viscosity coefficient.

Now, the above-mentioned boundary-value problem is reduced to finding the solution of a system of partial differential equations, namely, equation (5) plus the equation of continuity $\operatorname{div} \hat{\mathbf{v}} = 0$. This solution must satisfy the sticking condition $\hat{\mathbf{v}}(t,y)\Big|_S = \mathbf{V}$ and the natural condition $\lim\limits_{y\to\infty} \hat{\mathbf{v}}(t,y) = 0$.

A flow is accepted to call established or stationary if the field of its absolute velocity vectors in the moving coordinate system does not change in time. Obviously, if the body moves translationally, the necessary condition for the flow to be stationary is $\mathbf{V} = \mathbf{V}_0 = \mathrm{const}$.

Suppose that the body has a symmetry axis. If the body moves in such a manner that this axis remains in a given plane (for example, in the plane Ox_1x_2), then, according to the statics theorems for an absolutely solid body, the totality of forces acting from fluid upon the body can be reduced to the resultant one called the hydrodynamic force [6]. As usual the point of intersection of the symmetry axis and the line of the hydrodynamic force action is referred to as center of pressure. The hydrodynamic force is resolved into components parallel to the velocity vector $\mathbf{V}$ of the body inertia center and perpendicular to $\mathbf{V}$. The first component $\mathbf{D}$ is known as the drag force, and the second one $\mathbf{D}^l$ is called the lift force.

Let $\mathbf{i}, \mathbf{j}$ be the unit vectors in the directions Ox_1 and Ox_2, respectively. We need further a mapping that puts a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ into correspondence to $\mathbf{a}^\perp = -a_2\mathbf{i} + a_1\mathbf{j}$. Let V be the magnitude of $\mathbf{V}$, D be that of the drag force, and D^l be that of the lift force. For needs of forthcoming references, it is convenient to formulate the following assertion as lemma.

Lemma 1: The drag and lift forces are calculated by the formulae

$$\begin{aligned} &\mathbf{D} = \operatorname{sgn}(\mathbf{V},\mathbf{D}) D V^{-1}\mathbf{V}, \\ &\mathbf{D}^l = \operatorname{sgn}(\mathbf{V},\mathbf{D}) s D^l V^{-1}\mathbf{V}^\perp, \\ &s = \operatorname{sgn}((\mathbf{V},\mathbf{e})(\mathbf{V},\mathbf{e}^\perp)), \end{aligned} \qquad (6)$$

where $\mathbf{e}$ is the directing vector of the body symmetry axis (see Fig. 1).

Further we deal with mechanical systems of axially symmetric bodies (referred to as links). Let us introduce the following constraint.

L3: Systems move in a volume of fluid which is either very extended or is enclosed within rigid boundaries.

In the framework of the listed constraints, the coefficient C_D is a function of the body shape, Reynolds number and, probably, the angle of attack between the velocity vector of the body inertia center and the symmetry axis, i.e., $C_D = C_D(\text{shape}, \mathrm{Re}, \alpha)$ [7]. To determine the angle of attack (see Fig. 1), one can use the formula

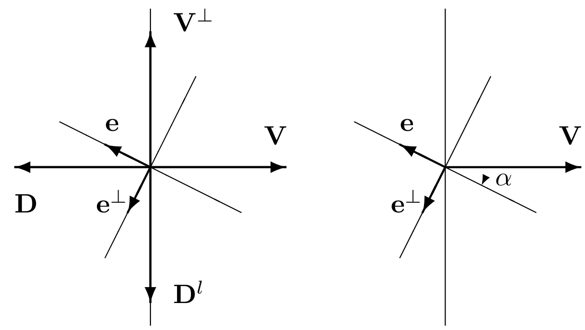

Fig. 1. Drag and lift forces and the angle of attack

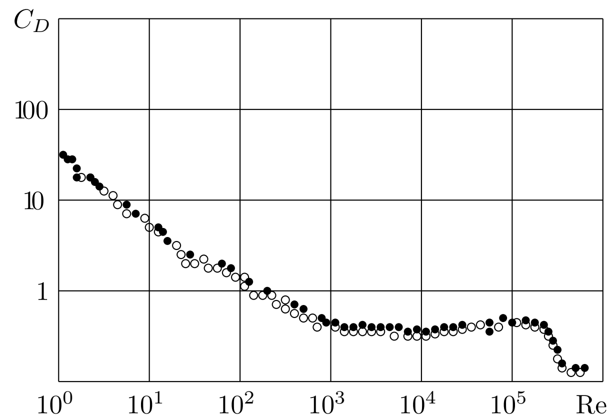

Fig. 2. Reynolds numbers

$$\alpha = -s \arccos |(\mathbf{e}, \mathbf{V}/V)|. \qquad (7)$$

The nonstationarity of the flow can be partially taken into account by means of introducing the apparent additional mass [7].

Hypothesis 1: The optimal displacement of the system produces quasistationary flow of the system links.

Hypothesis 2: The optimal displacements of the considered systems possess the following property: Reynolds numbers of each link provide that the drag and lift coefficients of the link are homogeneous functions of these numbers. See Fig. 2 for example.

III. The Mechanical System of Sequentially Joint Solid Bodies

A system is assumed to consist of axially symmetric links sequentially joint with the help of cylindrical hinges (see Fig. 3). The hinges are in the links inertia centers, their axes are mutually parallel, the links axes are in the same plane called the plane of the system [8]. The system hinges control momentums generated by internal forces operate so as the momentum $\mathbf{U}_k$, $k = 1, \dots 3$, acting on the kth link is accompanied by the momentum $-\mathbf{U}_k$ acting on the $(k-1)$th link. The system moves with the help of the force $\mathbf{F}$ acting in the plane. The mechanical system under consideration is placed in fluid. Therefore, besides the control and gravitational forces $m_k\mathbf{g}$ (m_k is the mass of the kth link) the links are subjected to the Archimedes and hydrodynamic ones. These last forces can be reduced to the forces $\mathbf{D}_k + \mathbf{D}_k^l$ applied to the links inertia centers and the momentums $\mathbf{M}_k$. The drag forces $\mathbf{D}_k$ and the lift ones $\mathbf{D}_k^l$ act in the plane, and the momentums $\mathbf{M}_k$ are perpendicular to it. These momentums are stipulated by the fact that the

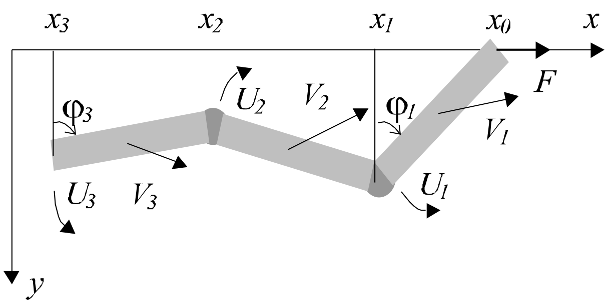

Fig. 3. The mechanical system of sequentially joined bodies

centers of pressure do not, in general, coincide with the links inertia centers. The described physical model being given, one can consider the equivalent plane mechanical systems of point masses sequentially joint by indefinitely thin absolutely solid rods. The above-listed forces are applied to those point masses. The differential equations of the system motion are obtained in the form of the 2nd kind Lagrange equations for the equivalent systems of point masses. The bibliography on robotics contains rather full exposition of various types of engines generating angular guiding momentums. The angular momentum, $\mathbf{U}_k$, which is, as a matter of fact, a momentum created by a couple of forces. This is, for example, the case when the momentum $\mathbf{U}_k$ is generated by an electric motor with stator placed in one link and the rotor rigidly connected with the next one.

Thus, formulas for the components of the generalized forces corresponding to the drag and lift forces acting on the links are obtained [9]. Moreover, the form of the generalized forces corresponding to the generalized angle coordinates is proven to be invariant with respect to the way of realization of the controlling momentums.

IV. Optimization

We indicated the three special features of the considered optimization problems. The third of them consists of calculating the energy consumption. To do this, one should determine a correct, from the viewpoint of the theory of distributions, method of multiplication of discontinuous velocities by impulse controls. However, another approach is chosen. Namely, with the use of schemes described in [10], [11], the original problem is reduced to some auxiliary of classical dynamic optimization. Such an approach corresponds to a strict mathematical formalization of the specified nonlinear operations with distributions. Reduction is based on the fact that the systems motion happens in the potential gravity field and a part of the work is spent to change the kinetic energy. Since the boundary conditions are given, the varied part of the work is the energy consumption needed to overcome drag. Calculation of the power corresponding to this consumption results in a certain auxiliary problem of minimization of this consumption subject to the constraints in the form of the kinematic constraints plus the equation for the energy consumption needed to overcome drag. The so-constructed auxiliary problem belongs formally to a number of problems of the classical calculus of variations. This is actually the case, if the system consists of bodies of smooth surface. If the structure of the system includes bodies of piecewise smooth surface (for example, cylindrical bodies), then in the space of generalized coordinates and velocities of the original problem there can appear manifolds in which the projection of the mentioned bodies onto the plane perpendicular to the velocity vector of their inertia centers and, consequently, the Hamiltonian become nondifferentiable.

The summarizing result of the carried out research can be formulated as follows. Along the optimal systems displacement the power is constant. If at all points of the optimal state path the power is differentiable with respect to the generalized velocities of the links, then its partial derivative with respect to the vehicle velocity also remains constant.

For more details, the following generalized coordinates of mechanical system are introduced: $q_0 = x_0$ is the horizontal coordinate of the point O_0, $q_k = \varphi_k$, $k = 1, 2, 3$ is the angle between a vertical line and the kth link. Then, the power of the hydrodynamic forces is of the form

$$\dot{Y} = \sigma^\top (C(q,\sigma)D + \hat{M}), \quad Y(\tau) = Y_\tau. \tag{8}$$

Here $D = (D_0, D_1, D_2, D_3)^\top$, D_0, D_1, D_2, D_3 are the magnitudes of the drag forces applied to the links inertia centers O_0, O_1, O_2, O_3 respectively, $\hat{M} = (0, M_1, M_2, M_3)^\top$, M_k is the momentum of the hydrodynamic forces acting upon the kth link, $k = 1, 2, 3$, $C(q,\sigma)$ is an 4×4 matrix,

$$\dot{q} = \sigma, \quad q(\tau) = q_\tau. \tag{9}$$

Problem 1: Solve the problem $Y(t_p) \to \min_\sigma$ subject to the dynamic constraints (8), (9) and the boundary conditions $q(t_p) = q_p$.

Let the links be of smooth surface.

Problem 2: Find the initial velocities

$$\sigma_i(\tau + 0) = \sigma_i^0(q_\tau, \tau), \quad i = 0, \ldots, n$$

that provide the boundary conditions $q(t_p) = q_p$ for the system

$$\dot{q} = \sigma, \quad \frac{d}{dt}\Big(\frac{\partial \dot{Y}}{\partial \sigma}\Big) = \frac{\partial \dot{Y}}{\partial q}. \tag{10}$$

The standard Euler–Lagrange procedure results in the assertion.

Theorem 1: Let the hydrodynamic constraints and Hypothesis be fulfilled. Then

– Problems 1 and 2 are equivalent;

– in the interval (τ, t_p), the optimal state path of the system belongs to the manifold described by the equation

$$\sigma = \sigma^0(q,t), \quad \sigma^0 = (\sigma_0^0, \sigma_1^0, \sigma_2^0, \sigma_3^0)^\top\,; \tag{11}$$

– on the manifold (11), the following relations hold: $\dfrac{\partial \dot{Y}}{\partial v} = \text{const}$ and $\dot{Y} - \dfrac{\partial \dot{Y}}{\partial \sigma}\sigma = \text{const}\,.$

Remark: Following to the accepted terminology, the manifolds of type (11) are called singular [3], [12].

Corollary: The Euler theorem on homogeneous functions makes the third statement of Theorem 1 be of the form

$$\dot{Y} = \text{const}, \quad \frac{\partial \dot{Y}}{\partial v} = \text{const}\,.$$

Thus, the optimal control histories are shown to have two-impulse structure. The aim of the initial impulse is to move the systems state to the singular surface, then travel along the singular surface until a state is reached from which the terminal impulse will take the state to the specified position. As a result, the original problem is reduced to a boundary-value one. Such a problem can be numerically solved by the so-called method of shooting.

In the circumstances of indeterminate fluid fluctuations, the equations of the systems motion contain the control actions jointly with additive disturbances. The problem is to construct an optimal feedback control law ensuring the following requirement: in the case when the disturbances vanish, the law provides the optimal completion of the control process with respect to the attained position. For the law specified, we make use of a positioning procedure of impulse correction to move the systems state to the singular surface (11). As the time between two sequential corrections decreases, the state is more and more frequently moves to the surface (11). As a result, the control process is supplemented by an effect like sliding along the singular surface. One of the questions naturally arising here is whether the state path of the system tends to the so-called "ideal sliding" [14] as the frequency of correction increases. Such sliding is described by the original perturbed system in which the controls make the singular surface (11) be an integral manifold. Since this system coincides with the system of the optimal displacements (9), (10), the affirmative answer to the raised question would allow one to conclude that the procedure of impulse correction, when the frequency grows infinitely, ensures the optimal behaviour of the system for all perturbations $\delta U_i(t)$ $(i = 1, \ldots, n)$, $\delta F(t)$, $0 < t < t_p$ and, in particular, solves the problem of optimal feedback control. This is really the case, since the equations of the systems motion and the singular surface (11) satisfy the conditions of Theorem 2.2 [14].

To produce the 2nd kind Lagrange equations to describe the systems motion, one has to estimate the kinetic energy of the system. It consists of the kinetic energy of its vehicle and links. By the Kenig formula, the kinetic energy of the kth link is equal to

$$\frac{1}{2}\, m_k V_k^2 + \frac{1}{2}\, J_{ck}\omega_k^2, \tag{12}$$

where J_{ck} is the inertia momentum of the kth link. Then, the kinetic energy can be presented as follows: $2K = \dot{q}^\top A(q)\dot{q}$. Here $q = (q_0, q_1, q_2, q_3)^\top$, $A(q) = (a_{ij})$ is the 4×4 symmetric matrix with the elements

$$a_{00} = \sum_{k=0}^{3} m_k, \quad a_{0j} = \sum_{k=j}^{3} m_k\, l_j \cos\varphi_j,$$
$$a_{ij} = J_{ci}\delta_{ij} + \sum_{k=j}^{3} m_k\, l_j\, l_i \cos(\varphi_j - \varphi_i),\ (1 \leqslant i \leqslant j \leqslant 3),$$

where δ_{ij} is the Kronecker symbol.

We now begin to calculate the generalized forces. We need the following notation: $\mathbf{V}_k$ is the velocity vector of the inertia center of the kth link, and V_k is its length. The following formulae hold:

$$\mathbf{D}_k = -D_k \frac{1}{V_k}\mathbf{V}_k, \quad \mathbf{D}_k^l = -s_k D_k^l \frac{1}{V_k}\mathbf{V}_k^\perp,$$
$$s_k = \operatorname{sgn}((\mathbf{V}, \mathbf{e}_k)(\mathbf{V}, \mathbf{e}_k^\perp)),$$

where $\mathbf{e}_k = -\sin\varphi_k\,\mathbf{i} + \cos\varphi_k\,\mathbf{j}$ is the direction vector of the kth link axis.

Let $Q_q = (Q_{x_0}, Q_{\varphi_1}, Q_{\varphi_2}, Q_{\varphi_3})^\top$ be the generalized forces column-matrix corresponding to the generalized coordinates x_0, φ_1, φ_2, φ_3. With the help of the above-written formulae for the components of the hydrodynamic forces, one can obtain the expression

$$Q_q = B(q)P - C(q,\dot{q})D + E(q,\dot{q})D^l + M + NU.$$

Here we denote the column $(k_0 m_0, k_1 m_1, k_2 m_2, k_3 m_3)^\top g$ by P, where k_i are the correction coefficients to take into account the Archimedes forces

$$D = (D_0, D_1, D_2, D_3)^\top, \quad D^l = (D_0^l, D_1^l, D_2^l, D_3^l)^\top,$$
$$M = (M_0, M_1, M_2, M_3)^\top, \quad U = (F, U_1, U_2, U_3)^\top.$$

The matrices in the formula for the generalized forces column-matrix are defined as follows: $B(q) = (b_{ij})$ is the 4×4 matrix with the elements

$$b_{0j} = 0, \quad (0 \leqslant j \leqslant 3),$$
$$b_{ij} = 0, \quad (0 \leqslant j \leqslant i-1,\ 1 \leqslant i \leqslant 3),$$
$$b_{ij} = l_i \sin\varphi_i \quad (1 \leqslant i \leqslant j \leqslant 3),$$

$C(q,\dot{q}) = (c_{ij})$ is the 4×4 matrix with the elements

$$c_{00} = \operatorname{sgn} v,$$
$$c_{0j} = \frac{1}{V_j}\Big(v - \sum_{k=1}^{j} l_k\omega_k \cos\varphi_k\Big), \quad (1 \leqslant j \leqslant 3),$$
$$c_{ij} = 0, \quad (0 \leqslant j \leqslant i-1,\ 1 \leqslant i \leqslant 3),$$
$$c_{ij} = \frac{l_i}{V_j}\Big(-v\cos\varphi_i + \sum_{k=1}^{j} l_k\omega_k \cos(\varphi_i - \varphi_k)\Big),$$
$$(1 \leqslant i \leqslant j \leqslant 3),$$

$E(q,\dot{q}) = (e_{ij})$ is the 4×4 matrix with the elements

$$e_{00} = 0, \quad e_{0j} = \frac{s_j}{V_j}\sum_{k=1}^{j} l_k\omega_k \sin\varphi_k, \quad (1 \leqslant j \leqslant 3),$$
$$e_{ij} = 0, \quad (0 \leqslant j \leqslant i-1,\ 1 \leqslant i \leqslant 3),$$
$$e_{ij} = s_j\,\frac{l_i}{V_j}\Big(v\sin\varphi_i - \sum_{k=1}^{j} l_k\omega_k \sin(\varphi_i - \varphi_k)\Big),$$
$$(1 \leqslant i \leqslant j \leqslant 3),$$

and $N=(\nu_{ij})$ is the 4×4 triangular matrix with the elements

$$\nu_{ii}=1,\quad (1\leqslant i\leqslant 3),$$
$$\nu_{01}=0,\quad \nu_{i,i+1}=-1,\quad (1\leqslant i\leqslant 2),$$
$$\nu_{ij}=0,\quad (i+2\leqslant j\leqslant 3,\,0\leqslant i\leqslant 1).$$

Above information is sufficient to present the systens motion in the form

$$A(q)D_t^2q+\sum_{j=0}^{n}\dot q_j\Big(\frac{\partial A_j}{\partial q}-\frac{1}{2}\Big(\frac{\partial A_j}{\partial q}\Big)^{\top}\Big)\dot q=$$
$$=B(q)P-C(q,\dot q)D+E(q,\dot q)D^l+M+NU, \qquad (13)$$

where A_j is the jth column of the matrix $A(q)$, $\partial A_j/\partial q$ is the matrix of the Frechet derivative, i.e., the matrix with the ith row of the form $\partial a_{ij}/\partial x$, $\partial a_{ij}/\partial\varphi_1$, $\partial a_{ij}/\partial\varphi_2$, $\partial a_{ij}/\partial\varphi_3$, $(0\leqslant i,j\leqslant 3)$, D_t is the distributional differentiation operator. The use of such an operator corresponds to expecting impulse components in the optimal control.

The expression for the power of the control forces and momentums is of the form

$$\dot W=\dot q^{\top}NU. \qquad (14)$$

In (14), the difficulty to multiply impulse actions by the system simultaneously jumping generalized velocities was circumvented above by physical analysis of the done work. However, there exists another possibility to endow the expression (14) with strict sense.

Theorem 2: The work of control actions on the system can be written out in the form

$$W=K+\Pi+Y+\mathrm{const},\quad \dot Y=\dot q^{\top}(CD+\hat M), \qquad (15)$$

where Π is the potential energy that is equal to

$$\Pi=\sum_{i=1}^{3}\Big(\sum_{j=1}^{3}k_jm_jg\Big)l_i\cos\varphi_i.$$

Proof: Indeed, to justify this assertion, one should eliminate the control variables between the Lagrange equations (13) and substitute the resultant expression into (14). Then, in the obtained relation

$$D_tW=\Big(D_t\big(\frac{\partial K}{\partial\dot q}\big)\Big)\dot q-\frac{\partial K}{\partial q}\dot q+$$
$$+\big(C(q,\dot q)D+\hat M-E(q,\dot q)D^l-B(q)P\big)^{\top}\dot q,$$

one should make use of the definitions

$$\frac{\partial K}{\partial\dot q}D_t^2q=D_tK-\frac{\partial K}{\partial q}\dot q,$$
$$\Big(D_t\frac{\partial K}{\partial\dot q}\Big)\dot q+\frac{\partial K}{\partial\dot q}D_t^2q=D_t\Big(\frac{\partial K}{\partial\dot q}\dot q\Big),$$

which corresponds to the approach developed in [11] for the problem of multiplication of distributions. Finally, applying the Euler theorem on homogeneous functions, we see that $\partial K/\partial\dot q\,\dot q=2K$. The last enables one to write out the expression for the work as follows: $W=K+\Pi-\Pi(0-0)+Y+\mathrm{const}$, where Y is a solution of (15), if only the identity $(v,\omega_1,\omega_2,\omega_3)\,E(q,\dot q)=0$ is taken into account. It should be stressed that the above-written formulae of multiplication have the classical sense in the case of ordinary control functions. ■

V. CONCLUSIONS

In conclusion, let us discuss the physical sense of the above-described dynamic optimization problems. A piece of the body state history in which the power of each control action is nonnegative is naturally to call accelerating. The problems considered here consist, in essence, in minimizing the difference between the energy (in the technical sense) consumption needed to accelerate the body and, then, to damp it, the time and distance of the displacement being given. Hence, if the interval $[0,t_p]$ of the optimal control process is accelerating, then the speech goes about solving the problem of the given displacement for minimum (in the technical sense) energy consumption. Obviously, hypothesis on the quasistationarity of the optimal flow provides this if the terminal control impulse is accelerating.

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