Control of Dynamical Systems on a Final Interval of Time by Means of Stability Theory Methods

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The paper is devoted to the stability theory based methods of designing feedback control for dynamical systems under uncertainty.

A lot of approaches to designing control for dynamical systems with uncertain parameters are based on the stability theory and consist in constructing regimes that ensure the asymptotic stability of the desired motion (in particular, the terminal state) of the system. In contrast to these approaches, we are searching for the control laws that may be used to steer a system to a terminal state in a finite time.

In the present paper, two approaches to constructing feedback control algorithms are discussed. The first approach can be applied to linear systems, while the second one has been elaborated for Lagrangian mechanical systems. Both of these approaches are based on the Lyapunov direct method and enables one to steer the system to a given terminal state in a finite time under the assumption that the control variables are bounded and the system is subject to unknown perturbations. The peculiarity of the investigation is that the Lyapunov functions are defined implicitly in both cases.

The control algorithms under consideration employ linear feedbacks with the gains that are functions of the phase variables. The gains increase and tend to infinity as the trajectory approaches the terminal state; nevertheless, the control forces are bounded and meet the imposed constraint.

To compare the efficiency of the proposed control algorithms a computer simulation of the controlled motion of a double pendulum in a neighbourhood of the upper equilibrium state is presented.

Feedback Control for Linear Systems

Consider a linear system

$$z^{(n)} = u. \tag{1}$$

where $z \in R$, u is a scalar control function. A generic linear system that satisfies the controllability condition can be reduced to system (1).

A bounded feedback control $u(z)$ is proposed which steers system (1) from an arbitrary initial state to the phase space origin in a finite time. To this end, we rewrite system (1) in the form

$$\dot{x} = Ax + bu, \quad x \in R^n. \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Denote by $V(x)$ a scalar function which will be specified below and consider the diagonal matrices $D = \mathrm{diag}(V^{-n}, V^{-n+1}, \ldots, V^{-1})$ and $N = \mathrm{diag}(-n, -n+1, \ldots, -1)$.

Let us choose the constant vector $a \in R^n$ such that:

1° for $u = a^\top y$ and $\tilde{A}y = Ay + bu$, the system $\dot{y} = \tilde{A}y$ is asymptotically stable;

2° there exist positive definite symmetrical $n \times n$ matrix Q satisfying the linear matrix inequalities

$$Q\tilde{A} + \tilde{A}^\top Q < 0, \quad QN + NQ < 0.$$

Theorem 1. The equation

$$x^\top D^\top(x) Q D(x) x = 1 \tag{3}$$

uniquely defines continuously differentiable function $V(x) > 0$ with properties

$$\lim_{|x| \to 0} V(x) = 0, \qquad \lim_{|x| \to \infty} V(x) = \infty.$$

Theorem 2. The derivative of function $V(x)$ along the trajectory of system (2) with the control function $u(x) = a^\top D(x) x$ satisfies the inequality

$$\dot{V} \leq d < 0, \ d = \text{const}. \tag{4}$$

Corollary. The control function $u(x) = a^\top D(x) x$ is bounded and steers system (2) (and hence system (1)) from an arbitrary initial state to the phase space origin in a finite time.

Note 1. The proposed control is effective also for the system

$$z^{(n)} = u + v$$

if the disturbance v satisfies the inequality

$$|v| \leq \frac{q_{max} p_{min}}{2 q_{min}^{1/2}}$$

where q_{min} and q_{max} are the minimal and maximal eigenvalues of Q, and p_{min} is the minimal eigenvalue of the matrix $-(Q\tilde{A} + \tilde{A}^\top Q)$.

Note 2. A similar approach was proposed in [1] where, apart from inequality (4), the inequality

$$\dot{V}(x) \leq -\gamma V^\alpha(x), \ \gamma > 0, \ 0 < \alpha < 1,$$

for the derivative of the Lyapunov function V holds.

Feedback Control for Lagrange Mechanical Systems

Consider now a mechanical systems governed by Lagrange's equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q} - \frac{\partial T}{\partial q} = U + S. \tag{5}$$

Here $q, \dot q \in R^n$ are the vectors of the generalized coordinates and velocities, T is the kinetic energy

$$T(q, \dot q) = \frac{1}{2}\dot q^\top A(q)\dot q.$$

The vector S of unknown generalized forces (disturbances) and the vector U of generalized control forces are assumed to be bounded

$$|S| \le S_0, \quad |U| \le U_0, \quad S_0, U_0 > 0.$$

Suppose that, for all $q, z \in R^n$, the matrix A satisfies the inequality

$$mz^2 \le z^\top A(q) z \le M z^2, \quad m, M > 0.$$

A bounded feedback control is proposed which, under certain conditions, steers the system from an arbitrary initial state to the origin of the phase space in a finite time.

Following to [2, 3], the desired control is chosen in the form

$$U(q, \dot q) = -\alpha(q, \dot q)A(q)\dot q - \beta(q, \dot q)q \tag{6}$$

where

$$\begin{gathered} \beta = \frac{3U_0^2}{8V}, \quad \alpha = \frac{\sqrt{3}U_0}{2\sqrt{2MV}}, \\ V = \frac{1}{2}\dot q^\top A \dot q + \frac{1}{2}\beta q^2 + \frac{1}{2}\alpha q^\top A \dot q. \end{gathered} \tag{7}$$

Relations (7) define the functions $\alpha(q, \dot q)$, $\beta(q, \dot q)$, and $V(q, \dot q)$ implicitly.

The justification of the proposed control law is based on the Lyapunov direct method. The function V here serves as a Lyapunov function for system (5) and satisfies the inequality

$$\dot V(q, \dot q) \le -\frac{\delta}{3}V^{1/2}(q, \dot q), \quad \delta > 0.$$

The function V tends to zero as the trajectory approaches the terminal state. Since the function V appears in the denominators in relations (7), the feedback factors a and b tend to infinity as the trajectory approaches the origin. Nevertheless, control (6) does not go beyond the admissible boundaries.

A computer simulation

A computer simulation of the controlled motion of a double pendulum is presented. We utilize the proposed control algorithms for steering the pendulum to the upper equilibrium state.

The first approach is used for control of the pendulum by means of control torque applied to the first link. To this end, we construct the control law for the linearized equations of motion and apply the obtained control to the full nonlinear equations.

The second approach is utilized for the fully actuated system.

References

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