PHYSCON 2013, San Luis Potosí, México, 26–29th August, 2013

NONSMOOTH $\mathcal{L}_2$-GAIN ANALYSIS OF TWISTING CONTROLLER

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Abstract

The present work extends the $\mathcal{L}_2$-gain analysis towards sliding mode dynamic systems and it is tested on the twisting algorithm to illustrate that the resulting closed loop system is capable not only of rejecting matching uniformly bounded disturbances, but also of attenuating unbounded ones.

Key words

Hamilton-Jacobi inequality, discontinuous systems.

1 Introduction

Sliding mode control algorithms are well recognized for their useful robustness features against matching disturbances with *a-priori* known bounds on their magnitudes. Their capability of attenuating disturbances with unknown bounds on their magnitudes constitute the main topic of the present paper.

Most of the papers related to second-order sliding modes (SOSM) control assume that bounded and matched disturbances are only affecting the systems. However, from the practical point of view, unmatched and *a-priori* unknown of the upper bound can occurs. Recently, ([Benderradji *et al.*, 2012], [Estrada *et al.*, 2011], [Orlov *et al.*, 2011], and [Santiesteban *et al.*, 2010]) made analysis of SOSM control on the framework of nonsmooth Lyapunov functions where conditions are imposed on the parameters of the controller to guarantee finite-time stability and disturbance rejection but analysis under disturbances assuming unknown upper bound of the disturbance are not provided. By the other hand, Zhang et al. (2009) propose the $\mathcal{L}_2$-gain approach for analysis and synthesis of feedback controllers for discontinuous time-delay systems. Zhao and Wang (2013) consider the finite-time stability and finite-time boundedness problems for switched linear systems subject to $\mathcal{L}_2$ disturbances.

First, the $\mathcal{L}_2$-gain analysis is extended towards sliding mode dynamic systems and then is tested on a pre-selected sliding mode control algorithm, being the popular twisting controller. It is thus demonstrated that the twisting controller is capable of not only rejecting matching bounded disturbances but also of attenuating the ones of class $\mathcal{L}_2$.

The rest of the paper is outlined as follows. In Section 2 we introduce basic assumptions and definitions in autonomous nonsmooth systems. In Section 3, the $\mathcal{L}_2$-gain for the twisting algorithm is developed, and its effectiveness is then illustrated in Section 4 by some numerical simulations. Finally, Section 5 presents some conclusions.

2 Nonsmooth $\mathcal{L}_2$-Gain Analysis

The $\mathcal{L}_2$-gain analysis, presented here, is based on the game-theoretic approach from [Basar and Bernhard, 1995] and extends the results from [Isidori and Astolfi, 1992; Van Der Shaft, 1992], where investigations were confined to smooth autonomous systems, towards locally Lipschitz continuous autonomous systems.

2.1 Essential assumptions and definitions

The $\mathcal{L}_2$-gain analysis is developed for an autonomous system of the form

$$\dot{x} = \varphi(x) + \psi(x)w(t) \tag{1}$$

and is made with respect to the output

$$z = h(x). \tag{2}$$

Hereinafter, $x \in \mathbb{R}^n$ is the state vector, $t \in \mathbb{R}^+$ is the time variable, $w(t) \in \mathbb{R}^r$ is the unknown disturbance vector, $\varphi(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $h(x) : \mathbb{R}^n \to \mathbb{R}^p$ are vector functions, and $\psi(x) : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ is a matrix function. The following *assumptions* are imposed on the system.

(A1) The functions $\varphi(x)$, $\psi(x)$, and $h(x)$ are piecewise continuous locally Lipschitz continuous in x.

(A2) $\varphi(0) = 0$ and $h(0) = 0$ for almost all t.

Recall that the function $\varphi(x) : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise (locally Lipschitz) continuous iff $\mathbb{R}^n$ is partitioned into a finite number of domains $G_j \subset \mathbb{R}^n$, $j = 1, \ldots, N$, with disjoint interiors and boundaries ∂G_j of measure zero such that $\varphi(x)$ is (locally Lipschitz) continuous within each of these domains and for all $j = 1, \ldots, N$ it has a finite limit $\varphi^j(x)$ as the argument $x^* \in G_j$ approaches a boundary point $x \in \partial G_j$.

While Assumption 2 is made to ensure that the origin is an equilibrium point of the nominal (i.e., disturbance-free) system, Assumption 1 admits the underlying system to undergo discontinuities on the boundaries ∂G_j of measure zero, which is why the precise meaning of the differential equation (3) with a piecewise continuous right-hand side is defined in the sense of Filippov, throughout. Following [Filippov, 1988] we will give the following definition.

Definition 1. *Given the differential equation*

$$\dot{x} = \varphi(x), \tag{3}$$

let us introduce for each point $x \in \mathbb{R}^n$ the smallest convex closed set $\Phi(x)$ which contains all the limit points of $\varphi(x^*)$ as $x* \to x$, and $x^* \in \mathbb{R}^n \setminus (\cup_{j=1}^N \partial G_j)$. An absolutely continuous function x, is said to be a solution of (3) *if it satisfies the differential inclusion*

$$\dot{x} \in \Phi(x). \tag{4}$$

Now, we will give the following definition of finite $\mathcal{L}_2$-gain.

Definition 2. *Given a real number $\gamma > 0$, further referred to as a disturbance attenuation level, it is said that system (1) (locally) possesses $\mathcal{L}_2$-gain less than γ with respect to output (2) (or, simply, system (1), (2) (respectively, locally) possesses $\mathcal{L}_2$-gain less than γ) if the response z, resulting from w for initial state $x(t_0) = 0$, satisfies*

$$\int_{t_0}^{t_1} \|z(t)\|^2 \, dt < \gamma^2 \int_{t_0}^{t_1} \|w(t)\|^2 \, dt \tag{5}$$

for all $t_1 > t_0$ and all piecewise continuous functions $w(t)$ (locally around the origin).

Remark 1. *Respectively, system (1), (2) is said to have $\mathcal{L}_2$-gain less than γ, locally around the origin, if there exists a neighborhood U of the origin such that inequality (5) is satisfied for all $t_1 > t_0$ and all piecewise continuous functions $w(t)$ for which the state trajectory of the closed-loop system starting from the initial point $x(t_0) = 0$ remains in U for all $t \in [t_0, t_1]$.*

For later use, the following instrumental results, inspired from [Clarke, 1988], are involved.

Technical lemmas are now presented to be used in the subsequent nonsmooth $\mathcal{L}_2$-gain analysis. A standard notation

$$DV(x; \nu) = \lim_{\tau \to 0} \frac{V(x + \tau\nu) - V(x)}{\tau} \tag{6}$$

stands throughout for a Dini derivative (if any) of a scalar function $V(x)$, computed in the direction $\nu \in \mathbb{R}^n$ at $x \in \mathbb{R}^n$.

A vector $\zeta(\hat{x}) \in \mathbb{R}^n$ is a supergradient of a scalar function $f(x)$ at $\hat{x} \in \mathbb{R}^n$ if there exists some $\sigma(\hat{x}) > 0$ such that

$$f(x) \leq f(\hat{x}) + \zeta^T(\hat{x})(x - \hat{x}) + \sigma(\hat{x})\|x - \hat{x}\|^2 \tag{7}$$

for all x in some neighborhood $U(\hat{x})$ of $\hat{x}$.
The set of supergradients at x is denoted $\partial f(x)$, and is referred to as the *superdifferential*.

Lemma 1. *Let $x \in \mathbb{R}$ be an absolutely continuous function of time variable t and let $V(x)$ be a scalar locally Lipschitz function around $x \in \mathbb{R}$. Then the composite function $V(x)$ is absolutely continuous and its time derivative is given by*

$$\frac{d}{dt} V(x(t)) = DV(x(t), \dot{x}(t)) \tag{8}$$

almost everywhere. Furthermore,

$$DV(x(t), \dot{x}(t)) \leq \frac{\partial V}{\partial x} \dot{x}(t) \tag{9}$$

for almost all t and for all supergradients $\left(\frac{\partial V}{\partial x}\right)^T \in \partial V(x)$, if any.

Lemma 2. *Let a discontinuous system (3) possess a Lyapunov function $V(x)$ Lipschitz continuous. Then system (3) is stable. If in addition, the function $V(x)$ is a strict Lyapunov function (and radially unbounded) then system (3) is (globally) asymptotically stable.*

2.2 Hamilton–Jacobi inequality and their proximal solutions

System (1), (2) is subsequently analyzed under the hypothesis that

(H) There exists a locally Lipschitz continuous, positive definite, radially unbounded proximal solution of the Hamilton–Jacobi inequality

$$\frac{\partial V}{\partial x} \phi(x) + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x} \psi(x) \psi^T(x) \left(\frac{\partial V}{\partial x}\right)^T + h^T(x) h(x) \leq -v(x) \tag{10}$$

under some positive γ and some positive definite function $v(x)$.

A locally Lipschitz continuous function $V(x)$ is said to be a *proximal solution* of the partial differential inequality (10) iff its proximal superdifferential $\partial^P V(x)$ is everywhere non-empty and (10) holds with $V(x)$ for all $x \in \mathbb{R}^n$, $\phi(x) \in \Phi(x)$, and for all proximal supergradients $\frac{\partial V}{\partial x} \in \partial^P V(x)$. The interested reader may refer [Clarke, 1988] for the proximal superdifferential concept.

2.3 Global analysis

The following result presents sufficient conditions of the nonsmooth system (1), (2) to be internally asymptotically stable and to possess $\mathcal{L}_2$-gain less than γ.

Theorem 1. *Let Assumptions A1 and A2 be in force, and let Hypothesis H be satisfied (locally). Then the nominal system (3) is globally (locally) asymptotically stable whereas its disturbed version (1) (locally) possesses $\mathcal{L}_2$-gain less than γ with respect to output (2).*

Proof. It is clear that Lemma 1 is applicable to a proximal solution $V(x)$ of the Hamilton–Jacobi inequality (10) viewed on the solutions $x(t)$ of the disturbance-free system (3). Under the relations (8), (9), and (10), we have

$$\begin{aligned}\frac{d}{dt}V(x) = DV(x, \dot{x}) \leq \frac{\partial V}{\partial x}\dot{x} \\ = \frac{\partial V}{\partial x}\varphi(x) \leq -\nu(x).\end{aligned} \tag{11}$$

Taking into account that (11) holds almost everywhere, Hypothesis H thus ensures that $V(x)$ is a strict decrescent radially unbounded Lyapunov function of the nominal system (3). By Lemma 2, system (3) is globally (locally) asymptotically stable.

It remains to show that the disturbed system (1) (locally) possesses $\mathcal{L}_2$-gain less than γ with respect to output (2). For this purpose, let us introduce the multi-valued function

$$\begin{aligned}H(x,w) = \frac{\partial V(x)}{\partial x}[\phi(x) + \psi(x)w] \\ + h^T(x)h(x) - \gamma^2 w^T w\end{aligned} \tag{12}$$

where $\frac{\partial V}{\partial x} \in \partial^P V(x)$. Clearly, the multi-valued function (12) is quadratic in w. Then

$$\frac{\partial H(x,w)}{\partial w}\Big|_{w=\alpha(x)} = \frac{\partial V(x)}{\partial x}\psi(x) - 2\gamma^2\alpha^T(x) = 0 \tag{13}$$

for $\alpha(x) = \frac{1}{2\gamma^2}\psi^T(x)\left(\frac{\partial V(x)}{\partial x}\right)^T$ and $\frac{\partial V}{\partial x} \in \partial^P V(x)$. Expanding the quadratic function $H(x,w)$ in Taylor series, we derive that

$$H(x,w) = H(x,\alpha(x)) - \gamma^2\|w - \alpha(x)\|^2 \tag{14}$$

where $H(x,\alpha(x)) \leq -v(x)$ due to (10). Hence,

$$H(x,w) \leq -\gamma^2\|w - \alpha(x)\|^2 - v(x) \tag{15}$$

and employing (12) and (15) we arrive at

$$\begin{aligned}\frac{\partial V(x)}{\partial x}[\phi(x) + \psi(x)w] \\ \leq -\gamma^2\|w-\alpha(x)\|^2 - v(x) \\ - \|h(x)\|^2 + \gamma^2\|w\|^2.\end{aligned} \tag{16}$$

By applying Lemma 1 and taking (16) into account, the time derivative of the solution $V(x)$ of the Hamilton–Jacobi inequality (10) on the trajectories of (1), (2) is estimated as follows

$$\begin{aligned}\frac{d}{dt}V(x) \leq -\gamma^2\|w-\alpha(x)\|^2 \\ -v(x) - \|z\|^2 + \gamma^2\|w\|^2.\end{aligned} \tag{17}$$

As a matter of fact, the latter inequality ensures that

$$\begin{aligned}\int_{t_0}^{t_1}(\gamma^2\|w(t)\|^2 - \|z(t)\|^2)dt \geq V(x(t_1)) - V(x(t_0)) \\ + \gamma^2\int_{t_0}^{t_1}[\|w(t) - \alpha(x(t))\|^2 + v(x(t))]dt > 0\end{aligned} \tag{18}$$

for any trajectory of (1), (2), initialized with $x(t_0) = 0$. Thus, inequality (5) is established thereby completing the proof of Theorem 1.

3 $\mathcal{L}_2$-Gain Analysis for Twisting Algorithm

In this Section, we will develop the $\mathcal{L}_2$-gain analysis of the twisting algorithm [Levant, 1993] driving the position $y(t) \in \mathbb{R}$ and velocity $\dot{y}(t) \in \mathbb{R}$ to the origin of a perturbed double integrator governed by

$$\ddot{y} = u + w \tag{19}$$

where $u(t) \in \mathbb{R}$ is the control input and $w(t) \in \mathbb{R}$ is the disturbance vector which is assumed to be unknown of class $\mathcal{L}_2$.

3.1 Twisting algorithm with proportional–derivative terms

Consider the twisting algorithm augmented with a proportional–derivative (PD) terms

$$u = -\alpha\,\mathrm{sgn}(y) - \beta\,\mathrm{sgn}(\dot{y}) - k_1 y - k_2\dot{y} \tag{20}$$

where the scalar constants k_1 and k_2 are positive, and α and β are chosen such that

$$\alpha > \beta > 0. \tag{21}$$

If the external disturbance would be uniformly bounded with an *a priori* known upper bound $M > 0$ such that

$$\sup_{t\geq 0} |w(t)| \leq M < \alpha - \beta \tag{22}$$

then the closed-loop system (20), (19) remains globally asymptotically stable regardless of whichever external disturbance affects the system [Orlov, 2005]. The aim is to demonstrate that the twisting algorithm not only rejects the uniformly bounded external disturbances subject to (22) but also attenuates external disturbances of class $\mathcal{L}_2$ with respect to the output

$$z = [y, \dot{y}]^T. \tag{23}$$

Setting $x_1 = y$ and $x_2 = \dot{y}$ and substituting (20) into (19) we have the state-space representation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha \operatorname{sgn}(x_1) - \beta \operatorname{sgn}(x_2) - k_1 x_1 - k_2 x_2 + w \end{aligned} \tag{24}$$

$$z = x = \begin{bmatrix} x_1 \ x_2 \end{bmatrix}^T. \tag{25}$$

According to the generalized representation (1)–(2) we have

$$\varphi(x) = \begin{bmatrix} x_2 \\ -\alpha \operatorname{sgn}(x_1) - \beta \operatorname{sgn}(x_2) - k_1 x_1 - k_2 x_2 \end{bmatrix}, \tag{26}$$

$$\psi(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{27}$$

In order to satisfy conditions of Theorem 1, let us introduce the following positive-definite function

$$V = \frac{1}{2}(k_1 + k_2)x_1^2 + x_1 x_2 + \frac{1}{2}x_2^2 + \alpha|x_1|. \tag{28}$$

with some $k_1, k_2 > 1$.

It is possible to verify that the Hamilton-Jacobi inequality (10) is satisfied with the positive definite function (28). For this purpose, denote $\mathcal{H}(x) = H(x, \alpha(x))$, i.e., the notation $\mathcal{H}(x)$ stands for the left-hand side of the Hamilton-Jacobi inequality (10). Then by inspection, one derives that

$$\begin{aligned} \mathcal{H}(x) &= -(k_1 - 1)x_1^2 - (k_2 - 2)x_2^2 - \alpha|x_1| - \beta|x_2| \\ &\quad - \beta x_1 \operatorname{sgn}(x_2) + \frac{1}{4\gamma^2}(x_1 + x_2)^2 \\ &\leq -(k_1 - 1)x_1^2 - (k_2 - 2)x_2^2 - (\alpha - \beta)|x_1| \\ &\quad - \beta|x_2| + \frac{1}{4\gamma^2}(x_1 + x_2)^2. \end{aligned} \tag{29}$$

Using the inequality

$$\frac{1}{2}(x_1 + x_2)^2 \leq x_1^2 + x_2^2, \tag{30}$$

it follows that

$$\begin{aligned} \mathcal{H}(x) \leq &- \left(k_1 - 1 - \frac{1}{2\gamma^2}\right) x_1^2 - \left(k_2 - 2 - \frac{1}{2\gamma^2}\right) x_2^2 \\ &- (\alpha - \beta)|x_1| - \beta|x_2| \\ \leq &- \left(k_1 - 1 - \frac{1}{2\gamma^2}\right) x_1^2 - \left(k_2 - 2 - \frac{1}{2\gamma^2}\right) x_2^2 \\ \leq &- \underbrace{\varepsilon\|x\|^2}_{v(x)} \end{aligned} \tag{31}$$

where

$$0 < \varepsilon \leq \min\left\{k_1 - 1 - \frac{1}{2\gamma^2}, k_2 - 2 - \frac{1}{2\gamma^2}\right\}. \tag{32}$$

The Hypothesis (H) is thus satisfied with the positive definite function (28), and

$$\begin{aligned} k_1 &> 1 + \frac{1}{2\gamma^2} \\ k_2 &> 2 + \frac{1}{2\gamma^2} \end{aligned}. \tag{33}$$

Summarizing, the following result is obtained.

Theorem 2. *Let the parameter gains be such that* (33) *is satisfied, and subordination* (21) *holds. Then the nominal system (3), (26) is globally asymptotically stable whereas its disturbed version (24) possesses* $\mathcal{L}_2$*-gain less than* γ *with respect to output (25) for any* $\gamma > 0$.

3.2 Twisting algorithm

In this subsection, we consider the twisting algorithm without PD part, that is

$$u = -\alpha \operatorname{sgn}(x_1) - \beta \operatorname{sgn}(x_2) \tag{34}$$

where α and β are subordinated according to (21).

The state-space representation of the closed-loop system (19), (34) in terms of the states $x_1 = y$ and $x_2 = \dot{y}$ is given by (1)–(2) where

$$\varphi(x) = \begin{bmatrix} x_2 \\ -\alpha \operatorname{sgn}(x_1) - \beta \operatorname{sgn}(x_2) \end{bmatrix}, \tag{35}$$

$$\psi(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{36}$$

Consider the following function

$$V = \alpha|x_1| + x_1x_2 + \frac{1}{2}x_2^2 \tag{37}$$

which is positive-definite for all $x \in \mathcal{D} = \{x \in \mathbb{R}^2 : |x_2| < \alpha\}$. Let us now verify that the Hamilton-Jacobi inequality (10), specified with (35) and (36), holds true with locally positive definite function (37). Indeed, for the left-hand side of (10), denoted by $\mathcal{H}(x)$, one derives

$$\begin{aligned}\mathcal{H}(x) &= x_1^2 + 2x_2^2 - \alpha|x_1| - \beta|x_2| - \beta x_1 \operatorname{sgn}(x_2) \\ &\quad + \frac{1}{4\gamma^2}(x_1 + x_2)^2 \\ &\le x_1^2 + 2x_2^2 - (\alpha - \beta)|x_1| - \beta|x_2| \\ &\quad + \frac{1}{4\gamma^2}(x_1 + x_2)^2. \end{aligned} \tag{38}$$

Then using (30) and setting $\eta = \min\{\alpha - \beta, \beta\}$, it follows

$$\begin{aligned}\mathcal{H}(x) &\le x_1^2 + 2x_2^2 - \eta(|x_1| + |x_2|) + \frac{1}{2\gamma^2}x_1^2 + \frac{1}{2\gamma^2}x_2^2 \\ &\le -\eta\|x\|_1 + \left(2 + \frac{1}{2\gamma^2}\right)\|x\|_2^2 \\ &\le -\eta\|x\|_2 + \left(2 + \frac{1}{2\gamma^2}\right)\|x\|_2^2 \le v(x) \end{aligned} \tag{39}$$

The Hypothesis (H) is thus locally satisfied for all $x \in D_{\mathcal{A}}$ where

$$D_{\mathcal{A}} = \{x \in \mathbb{R}^2 : \|x\|_2 \le \frac{2\gamma^2}{4\gamma^2 + 1}\eta\} \tag{40}$$

Moreover, in the unperturbed case, the time derivative of V along the solution of the closed-loop system (19), (34)

$$\frac{d}{dt}V = x_2^2 - (\alpha - \beta)|x_1| - \beta|x_2| \tag{41}$$

remains negative-definite for any $\alpha > \beta$ and for all $x \in D = \{x \in \mathbb{R}^2 : |x_2| < \beta\} \subset \mathcal{D}$. Hence, semiglobal stability can be concluded for an arbitrarily large region of attraction by increasing the gains α and β.

Summarizing, the following result is obtained.

Theorem 3. *Let the parameter gains be such that subordination* (21) *holds. Then the nominal system (3), (35) is locally asymptotically stable for all $x \in D_{\mathcal{A}} \subset D \subset \mathbb{R}^2$, whereas its perturbed version* (1), (35), (36) *locally possesses $\mathcal{L}_2$-gain less than an arbitrary $\gamma > 0$ with respect to output $z = x$, locally within the region $D_{\mathcal{A}}$.*

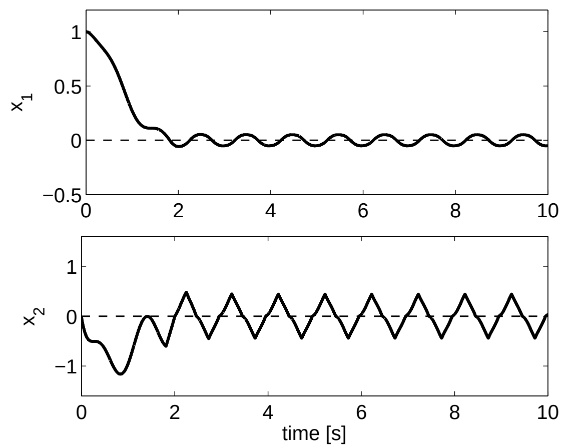

Figure 1. Responses of the perturbed double integrator (19) using the twisting algorithm with PD terms (20) and considering a sinusoidal perturbation.

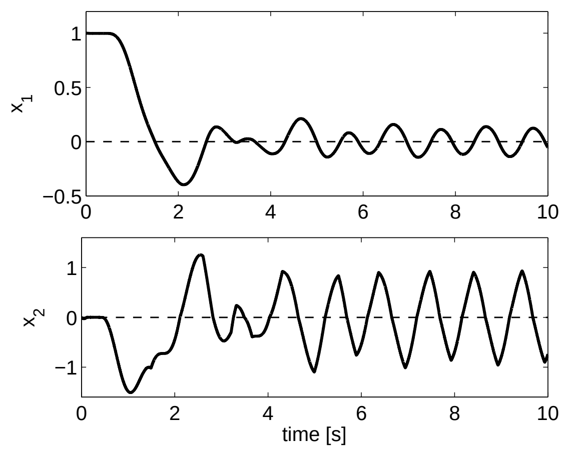

Figure 2. Responses of the perturbed double integrator (19) using the twisting algorithm (34) and considering a sinusoidal perturbation.

4 Numerical results

We run numerical simulations in *Simulink* in order to corroborate that external harmonic disturbance $w(t) = 3\sin(2\pi t)$, affecting the double integrator (19), is attenuated by the twisting algorithm with proportional-derivative part (20) and by the twisting algorithm (34). The controller gains $\beta < \alpha < M = 3$ are chosen not to exceed the magnitude of the harmonic disturbance applied. The initial conditions are chosen at $x_1(0) = 1$ and $x_2(0) = 0$ for both simulations. Figure 1 shows the response of the closed-loop system (19), (20) with gains $k_1 = k_2 = 5$, $\alpha = 2$, and $\beta = 1$. Figure 2 shows the response of the closed-loop system (19), (34) specified with $\alpha = 2$ and $\beta = 1$.

Figure 3 shows the responses of the twisting algorithm considering (20) and (34) affected by the following disturbance,

$$w(t) = \frac{5}{t^{\frac{1}{3}}}\cos 5\pi t, \tag{42}$$

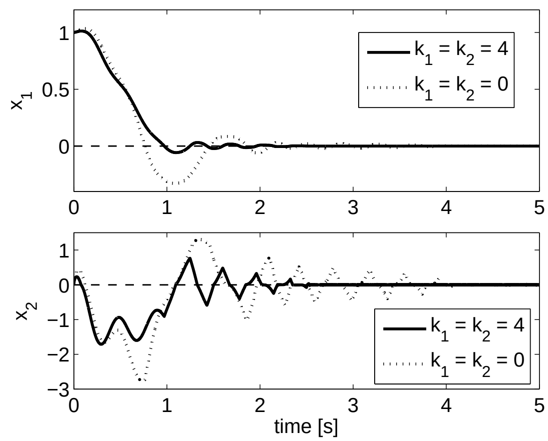

Figure 3. Responses of the perturbed system (19) considering the disturbance given by (42).

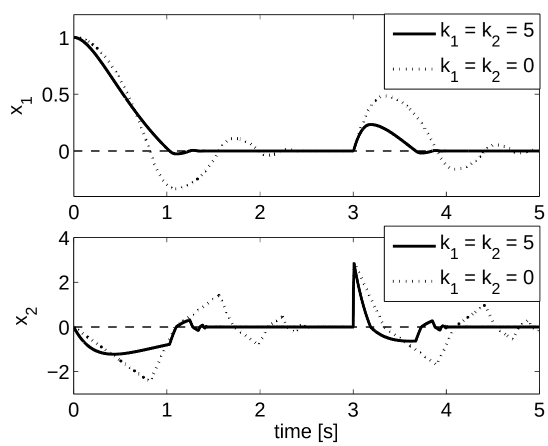

Figure 4. Responses of the perturbed system (19) considering the disturbance given by (43).

and figure 4 shows the responses of the double integrator considering the following disturbance,

$$w(t) = \begin{cases} 0 & 0 \le t < 3 \\ 300 & 3 \le t < 3.01 \\ 0 & t \ge 3.01 \end{cases}. \qquad (43)$$

It is concluded from these figures that as predicted by the theory, the disturbance is not rejected but only attenuated, that is, the system response is no longer rejected but it remains bounded.

5 Conclusions

$\mathcal{L}_2$-gain analysis, developed for the twisting controller in a perturbed double integrator, has clearly shown its applicability to sliding mode dynamic systems and the capability of the popular controller to attenuate unbounded disturbances.

6 Acknowledgment

Y. Orlov and L. T. Aguilar gratefully acknowledge the financial support from CONACYT (Consejo Nacional de Ciencia y Tecnología) under Grants 165958 and 127575.

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