AN APPROXIMATE ANALYTICAL SOLUTION DESCRIBING OSCILLATIONS OF A CONDUCTOR IN A MAGNETIC FIELD

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Abstract

An approximate analytical solution to a system exhibiting oscillations of a conductor in a magnetic field is sought by means of multiple scales. A comparison between the analytical solution and the numerical integration is made. The results show a nearness of frequency but discrepancy in amplitude. The foundation to implement an effective control strategy is established. This solution gives an indication of a starting point so that the trajectory may be steered towards a desired basin of attraction.

Key words

Electro-vibroimpact, multiple scales, approximate analytical method, target trajectory.

1 Introduction

An electro-vibroimpact system may be deployed underground to penetrate soil. To ensure the most efficient use of energy to achieve this, a robust feedback control system is required. However, the design of the transfer function of the feedback requires an understanding of system dynamics and characteristics. This document presents findings from an initial approximate analysis by means of multiple scales.

2 Multiple Scales Analysis

In order to implement an effective control strategy, an approximate analytical solution to the system is sought. In this paper, an approximate solution to a simplified system model is elucidated. To avoid the complexity of discontinuous functions present in the system model in [Ho, 2007], an attempt to analyse the system response for a single degree-of-freedom (without impact) metal bar vibrating inside a frictionless surface within the solenoid is made. In this case, the system governing equations can be described as follows:

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2(x-\delta) = \frac{1}{2}\frac{1}{m}\frac{\partial L}{\partial x}i^2 \tag{1}$$

$$\begin{aligned} &L\ddot{i} + \left[R + 2\frac{\partial L}{\partial x}\dot{x}\right]\dot{i} \\ &+ \left[\frac{1}{C} + \frac{\partial^2 L}{\partial x^2}\dot{x}^2 + \frac{\partial L}{\partial x}\ddot{x}\right] i = \Omega V_s \cos(\Omega t) \end{aligned} \tag{2}$$

where dot denotes differentiation with respect to time t, x is the displacement of the metal bar, ξ is the damping ratio, ω_n is the natural frequency of the metal bar and the spring, δ is the initial displacement of the metal bar, m is the mass of the metal bar, L is the inductance function in the RLC circuit, i is the current flow through the circuit, R and C are the resistance and the capacitance in the RLC circuit respectively, V_s is the externally supplied time dependent voltage and Ω is the frequency of the power supply.

The inductance function $L(x)$ can be adopted in a Gaussian form [Ho, 2007] to fit experimental data

$$L(x) = l_0 + l_a e^{-\sigma x^2} \tag{3}$$

The first and second derivative of $L(x)$ are obtained as

$$\frac{\partial L}{\partial x} = -2\sigma l_a x e^{-\sigma x^2} \tag{4}$$

$$\frac{\partial^2 L}{\partial x^2} = 4\sigma^2 l_a x^2 e^{-\sigma x^2} - 2\sigma l_a e^{-\sigma x^2} \tag{5}$$

where σ, l_0 and l_a are the parameters used in the Gaussian fit to experimental data.

To obtain an approximate analytical solution to the system described above, multiple scales analysis [Nayfeh, 1995] is performed. The solution that is valid near *x*=0 and accounting for nonlinear effects is sought. In this case, equations (3),(4) and (5) can be approximated by Taylor's series expanded about *x*=0 and only the first two terms are retained, giving rise to the following approximate expression:

$$\hat{L}(x) = l_0 + l_a - \sigma l_a x^2 + O(x^4) \tag{6}$$

$$\frac{\partial \hat{L}}{\partial x} = -2\sigma l_a x + 2\sigma^2 l_a x^3 + O(x^4) \tag{7}$$

$$\frac{\partial^2 \hat{L}}{\partial x^2} = -2\sigma l_a + 6\sigma^2 l_a x^2 + O(x^4) \tag{8}$$

A perturbation parameter ϵ, is assigned to the following terms that are expected to be small

$$\begin{array}{l} \epsilon\lambda_0 = l_0 + l_a,\ \epsilon\lambda_1 = \frac{\sigma l_a}{m},\ \epsilon\lambda_2 = \frac{\sigma^2 l_a}{m} \\ \epsilon\lambda_3 = \sigma l_a,\ \epsilon\lambda_4 = \sigma^2 l_a,\ \epsilon\alpha = 2\xi\omega_n \end{array} \tag{9}$$

The system equations (1) and (2) can then be expressed as

$$\ddot{x} + \epsilon\alpha\dot{x} + \omega_n^2(x-\delta) = \frac{1}{2}(-2\epsilon\lambda_1 x + 2\epsilon\lambda_2 x^3)i^2 \tag{10}$$

$$\begin{array}{l} (\epsilon\lambda_0 - \epsilon\lambda_3 x)\ddot{i} + \left[R + (-4x\epsilon\lambda_1 + 4\epsilon\lambda_2 x^3)\dot{x}\right]\dot{i} \\ + \left[\frac{1}{C} + (-2\epsilon\lambda_3 + 6\epsilon\lambda_4 x^2)\dot{x}^2 \right. \\ \left. +(-2\epsilon\lambda_1 x + 2\epsilon\lambda_2 x^3)\ddot{x}\right] i = \Omega V_s \cos(\Omega t) \end{array} \tag{11}$$

To apply the method of multiple scales, different time scales are introduced according to

$$T_n = \epsilon^n t \qquad \text{for } n = 0, 1, 2, ... \tag{12}$$

In operator form,

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + ... = D_0 + \epsilon D_1 + ... \tag{13}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + ... \tag{14}$$

By setting the truncation at order ϵ^1, the assumed solution of (10) and (11) can be represented by

$$x(t,\epsilon) = x_0(T_0,T_1) + \epsilon x_1(T_0,T_1) + \epsilon^2 ... \tag{15}$$

$$i(t,\epsilon) = i_0(T_0,T_1) + \epsilon i_1(T_0,T_1) + \epsilon^2 ... \tag{16}$$

Substituting (13),(14),(15) and (16) into (10) and (11) and equating the coefficients of ϵ^0 and ϵ^1 to zero, the following is obtained

$$\epsilon^0 : \begin{array}{l} D_0^2 x_0 + \omega_n^2 x_0 = \omega_n^2 \delta \\ R(D_0 i_0) + \frac{i_0}{C} = \Omega V_s \cos(\Omega T_0) \end{array} \tag{17}$$

$$\epsilon^1 : \begin{array}{rl} D_0^2 x_1 + \omega_n^2 x_1 & = \lambda_2 i_0^2 x_0^3 - \lambda_1 i_0^2 x_0 \\ & \quad -\alpha(D_0 x_0) - 2(D_0 D_1 x_0) \\ R(D_0 i_1) + \frac{i_1}{C} & = -R(D_1 i_0) - \lambda_0 \left(D_0^2 i_0\right) \\ & \quad -4\lambda_2 (D_0 i_0)(D_0 x_0) x_0^3 \\ & \quad -2\lambda_2 \left(D_0^2 x_0\right) i_0 x_0^3 \\ & \quad +4\lambda_1 (D_0 i_0)(D_0 x_0) x_0 \\ & \quad +2\lambda_1 \left(D_0^2 x_0\right) i_0 x_0 \\ & \quad +2\lambda_3 i_0 (D_0 x_0)^2 \\ & \quad -6\lambda_4 i_0 x_0^2 (D_0 x_0)^2 \\ & \quad +\lambda_3 \left(D_0^2 i_0\right) x_0^2 \end{array} \tag{18}$$

The solutions to (17) may be expressed as

$$x_0(T_0,T_1) = A_0(T_1)e^{i\omega_n T_0} + \bar{A}_0(T_1)e^{-i\omega_n T_0} + \delta \tag{19}$$

$$\begin{array}{l} i_0(T_0,T_1) = B_0(T_1)e^{-\frac{T_0}{CR}} \\ \qquad + \dfrac{C\Omega V_s\left[\cos(\Omega T_0) + CR\Omega\sin(\Omega T_0)\right]}{1 + C^2 R^2 \Omega^2} \end{array} \tag{20}$$

where A_0 and B_0 are unknown complex functions and $\bar{A}_0$ is the complex conjugate of A_0.To solve for $A_0(T_1)$, $\bar{A}_0(T_1)$ and $B_0(T_1)$, secular terms in (18) may be inspected. Substituting solution (19) and (20) into (18), expanding and collecting the terms involving $e^{i\omega_n T_0}$ gives

$$e^{i\omega_n T_0}\Big(\frac{C^4R^2A_0V_s^2\lambda_1\Omega^4}{2\left(C^2R^2\Omega^2+1\right)^2}+\frac{C^2A_0V_s^2\lambda_1\Omega^2}{2\left(C^2R^2\Omega^2+1\right)^2}$$
$$-\frac{3C^2\delta^2A_0V_s^2\lambda_2\Omega^2}{2\left(C^2R^2\Omega^2+1\right)^2}+i\alpha A_0\omega_n$$
$$-\frac{3C^4R^2\delta^2A_0V_s^2\lambda_2\Omega^4}{2\left(C^2R^2\Omega^2+1\right)^2}-\frac{3C^4R^2\Omega^4A_0^2V_s^2\lambda_2\bar{A}_0}{2\left(C^2R^2\Omega^2+1\right)^2}$$
$$-\frac{3C^2\Omega^2A_0^2V_s^2\lambda_2\bar{A}_0}{2\left(C^2R^2\Omega^2+1\right)^2}+e^{-\frac{2T_0}{CR}}A_0B_0^2\lambda_1$$
$$-3e^{-\frac{2T_0}{CR}}\delta^2A_0\lambda_2B_0^2-3e^{-\frac{2T_0}{CR}}A_0^2B_0^2\lambda_2\bar{A}_0$$
$$+e^{-2iT_0\Omega}(...)+e^{2iT_0\Omega}(...)+e^{-iT_0\Omega}(...)$$
$$+e^{iT_0\Omega}(...)+2i\omega_nA_0'\Big) \tag{21}$$

Terms on $e^{-i\omega_n T_0}$ also contribute to secular terms. However they are dependent on expression (21) since they are complex conjugates of each other. By further inspecting the terms on expression (21), it was found that not all terms in (21) will lead to secular terms. Terms that are associated with $e^{-\frac{2T_0}{CR}}$ will produce particular solutions that decay with time. Meanwhile, the case of nonresonant excitations is considered, for which Ω is away from ω_n and $2\omega_n$. Therefore terms that are associated with $e^{-2iT_0\Omega}$, $e^{2iT_0\Omega}$, $e^{-iT_0\Omega}$ and $e^{iT_0\Omega}$ will not produce secular terms.

Equating the coefficients of terms in (21) that will lead to secular terms to zero give

$$\frac{C^4R^2A_0V_s^2\lambda_1\Omega^4}{2\left(C^2R^2\Omega^2+1\right)^2}+\frac{C^2A_0V_s^2\lambda_1\Omega^2}{2\left(C^2R^2\Omega^2+1\right)^2}$$
$$-\frac{3C^2\delta^2A_0V_s^2\lambda_2\Omega^2}{2\left(C^2R^2\Omega^2+1\right)^2}-\frac{3C^4R^2\delta^2A_0V_s^2\lambda_2\Omega^4}{2\left(C^2R^2\Omega^2+1\right)^2}$$
$$-\frac{3C^4R^2\Omega^4A_0^2V_s^2\lambda_2\bar{A}_0}{2\left(C^2R^2\Omega^2+1\right)^2}-\frac{3C^2\Omega^2A_0^2V_s^2\lambda_2\bar{A}_0}{2\left(C^2R^2\Omega^2+1\right)^2}$$
$$+i\alpha A_0\omega_n+2i\omega_nA_0'=0 \tag{22}$$

In order to solve (22), A_0 and $\bar{A}_0$ are assumed to be in the polar form

$$A_0(T_1)=\frac{1}{2}a(T_1)e^{i\beta(T_1)},\bar{A}_0(T_1)=\frac{1}{2}a(T_1)e^{-i\beta(T_1)}. \tag{23}$$

Substituting (23) into (22) and separating the result into real and imaginary parts, we obtain

$$\frac{1}{2}\alpha\omega_n a+\omega_n a'=0 \tag{24}$$

$$\frac{aC^2\Omega^2V_s^2\left(4\lambda_1-3\left(a^2+4\delta^2\right)\lambda_2\right)}{16\left(C^2R^2\Omega^2+1\right)}-a\omega_n\beta'=0. \tag{25}$$

The solution to (24) is

$$a(T_1)=c_1e^{-\frac{\alpha T_1}{2}} \tag{26}$$

where c_1 is the integration constant that can be determined through initial conditions. Substituting (26) into (25) and the solution for $\beta(T_1)$ is obtained:

$$\beta(T_1)=c_2+\frac{C^2\Omega^2V_s^2\left(\frac{3e^{-\alpha T_1}\lambda_2c_1^2}{\alpha}+4T_1\left(\lambda_1-3\delta^2\lambda_2\right)\right)}{16\left(C^2R^2\Omega^2+1\right)\omega_n} \tag{27}$$

where c_2 is another integration constant.

Therefore the first order approximation to the solution of (10) is

$$x=\delta+c_1e^{-\frac{\alpha T_1}{2}}\cos(\beta+\omega_nT_0)+O(\epsilon). \tag{28}$$

Repeating the same procedure to solve for $B_0(T_1)$ (i.e. to set up the conditions for secular terms in the current equation in (18) to solve for $B_0(T_1)$). The solution is

$$B_0(T_1)=c_3e^{-\frac{8\left(\lambda_3\delta^2+\lambda_0\right)T_1+u(T_1)}{8C^2R^3}} \tag{29}$$

where $u(T_1)$ is

$$u(T_1)=\frac{1}{\alpha}e^{-2\alpha T_1}\Big[3C^2R^2\left(\lambda_2-\lambda_4\right)\omega_n^2c_1^4$$
$$+4e^{\alpha T_1}\left(2C^2R^2\left(3\left(\lambda_2-\lambda_4\right)\delta^2-\lambda_1+\lambda_3\right)\omega_n^2-\lambda_3\right)c_1^2\Big] \tag{30}$$

and c_3 is the integration constant.

The first order approximation to the solution of (11) is

$$i=B_0(T_1)e^{-\frac{T_0}{CR}}+\frac{C\Omega V_s\left[\cos(\Omega T_0)+CR\Omega\sin(\Omega T_0)\right]}{1+C^2R^2\Omega^2}+O(\epsilon). \tag{31}$$

Finally the time scales, T_0 and T_1, are replaced and the parameters λ_0, λ_1, λ_2, λ_3 and λ_4, by using the relationship in (9) and (12) to obtain the final form of the displacement function $x(t)$ and current function $i(t)$:

$$x(t) = \delta + e^{-\xi\omega_n t} c_1 \cos\left(c_2 + \omega_n t + g(t)\right) + O(\epsilon) \tag{32}$$

$$\begin{aligned} i(t) = {} & \frac{C\Omega V_s(\cos(t\Omega) + CR\Omega\sin(t\Omega))}{C^2R^2\Omega^2+1} \\ & + c_3 e^{\frac{h(t)-16t\left(CR^2+l_0\right)}{16C^2R^3}} + O(\epsilon) \end{aligned} \tag{33}$$

where $g(t)$ and $h(t)$ are

$$\begin{aligned} g(t) = {} & V_s^2 C^2 \sigma \Omega^2 l_a e^{-2t\xi\omega_n}\Big[3\sigma c_1^2 \\ & -8e^{2t\xi\omega_n} t\xi\left(3\delta^2\sigma - 1\right)\omega_n\Big] \Big/ \\ & \Big[32m\xi\left(C^2R^2\Omega^2+1\right)\omega_n^2\Big] \end{aligned} \tag{34}$$

$$\begin{aligned} h(t) = {} & \Big\{e^{-4t\xi\omega_n} l_a\big[C^2(m-1)R^2\sigma\left(3\sigma c_1^2\right. \\ & \left.+8e^{2t\xi\omega_n}\left(3\delta^2\sigma-1\right)\right)\omega_n^2 c_1^2 \\ & +4e^{2t\xi\omega_n} m\sigma c_1^2 \\ & -16e^{4t\xi\omega_n} mt\xi\left(\sigma\delta^2+1\right)\omega_n\big]\Big\} \Big/ \\ & (m\xi\omega_n). \end{aligned} \tag{35}$$

Initial conditions can be used to solve for the integration constants c_1, c_2 and c_3. These conditions are initial displacement, δ, whereas initial velocity and initial current are both equal to zero. Differentiating equation (32) to get the velocity function $v(t)$ and substituting the initial conditions into equations (32), (33) and $v(t)$, the following are obtained

$$c_1 = \pm 2\sqrt{\frac{2}{3}}\sqrt{\frac{\xi\omega_n\left(-3\delta^2\sigma^2 + \sigma + \frac{4m(C^2R^2\Omega^2+1)\omega_n^2}{C^2\Omega^2 l_a V_s^2}\right)}{\sigma^2}} \tag{36}$$

$$\begin{aligned} c_2 = {} & (2k+1)\tfrac{\pi}{2} - \omega_n + \tfrac{C^2\sigma\left(3\delta^2\sigma-1\right)\Omega^2 l_a V_s^2}{4m(C^2R^2\Omega^2+1)\omega_n}, \\ & k \text{ is any integer} \end{aligned} \tag{37}$$

$$c_3 = -\frac{Ce^{-\frac{2pq}{3C^4mR^3\sigma^2\Omega^4 l_a V_s^4}}\Omega V_s}{C^2R^2\Omega^2+1} \tag{38}$$

where p and q are

$$p = C^2\sigma\left(1-3\delta^2\sigma\right)\Omega^2 l_a V_s^2 + 4m\left(C^2R^2\Omega^2+1\right)\omega_n^2 \tag{39}$$

$$\begin{aligned} q = {} & \sigma\Omega^2 l_a V_s^2\Big[m - 2C^2(m-1) \\ & R^2\left(3\delta^2\sigma-1\right)\omega_n^2\left(\xi\omega_n-1\right)\Big] \\ & +8(m-1)mR^2\xi\left(C^2R^2\Omega^2+1\right)\omega_n^5. \end{aligned} \tag{40}$$

3 Discussion and Conclusion

Numerical integration was performed to verify the approximate analytical solution obtained from multiple scales analysis. The integration was carried out by using the Dynamics software [Yorke, 1998]. The following parameters were used in the integration: l_0 = 0.16568, l_a = 0.41199, σ = 857.31, m = 1, ξ = 0.00353, ω_n = 14.142, δ = 0.001, C = 0.000032, R = 27.5, V_s = 14.142 and Ω = 314.159.

A comparison between numerical predictions in Figure 1 and analytical descriptions in Figure 2 shows a nearness of frequency, but discrepancy in amplitude.

Meanwhile, a close co-relation in both amplitude and frequency was observed for the current response obtained from analytical solutions and numerical predictions as shown in Figure 3 and Figure 4. However the analytical solution failed to predict the transient component captured in the numerical integration.

In conclusion, this non-distortion of the frequency component of the displacement, together with the close co-relation of the current response, gives an indication of the relevance of this solution, which may then be used as a platform to implement an effective control strategy.

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(a)

(b)

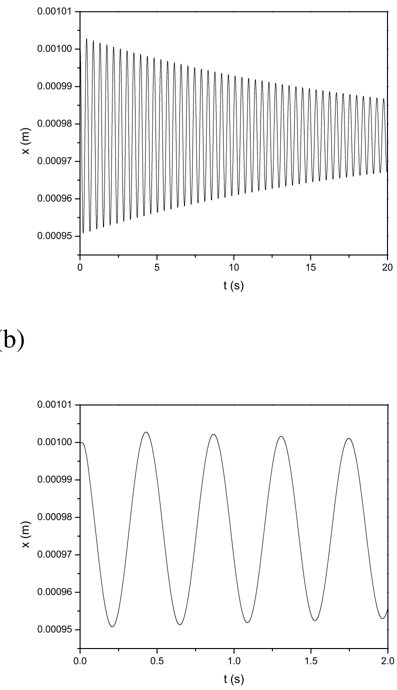

Figure 1. Displacement response obtained from numerical integration, for (a) the first 20 seconds and (b) the first 2 seconds.

(a)

(b)

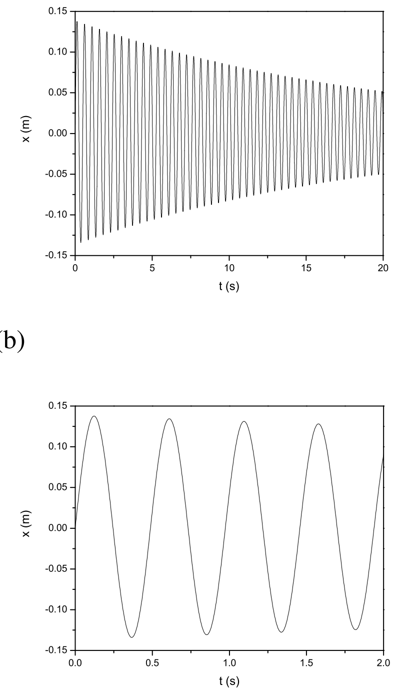

Figure 2. Displacement response obtained from multiple scales analysis, for (a) the first 20 seconds and (b) the first 2 seconds.

(a)

(b)

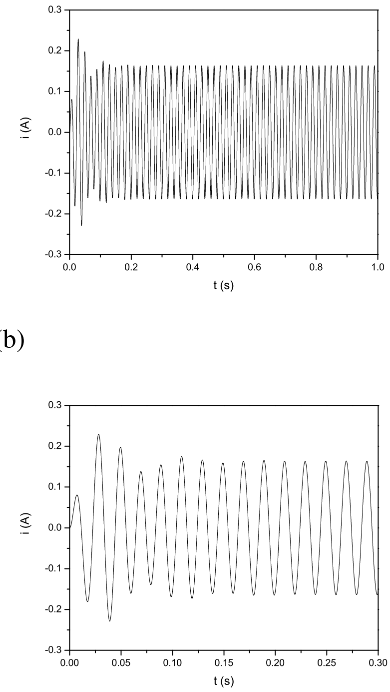

Figure 3. Current response obtained from numerical integration, for (a) the first second and (b) the first 0.3 seconds.

(a)

(b)

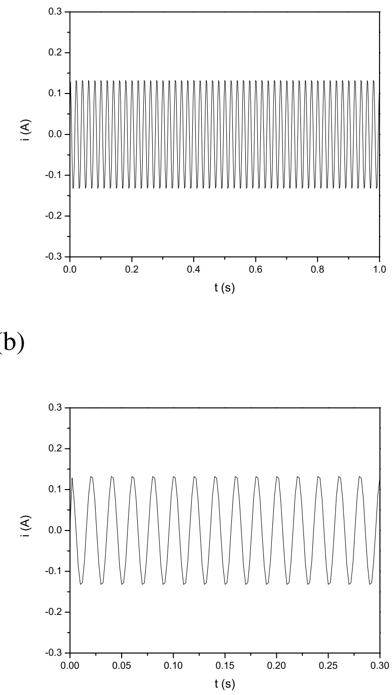

Figure 4. Current response obtained from multiple scales analysis, for (a) the first second and (b) the first 0.3 seconds.