INCOMPLETE NOISE-INDUCED SYNCHRONIZATION IN GINZBURG–LANDAU EQUATION

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Abstract

A new type of noise-induced synchronous behavior is described. This phenomenon, called *incomplete noise-induced synchronization*, arises for one-dimensional Ginzburg-Landau equations driven by common noise. The mechanisms resulting in the incomplete noise-induced synchronization in the spatially extended systems are revealed analytically. A very good agreement between the theoretical results and the numerically calculated data is shown.

Key words

spatial extended systems, chaotic synchronization, noise-induced synchronization

Noise-induced synchonization (Fahy and Hamann, 1992; Martian and Banavar, 1994) is an ubiquitous phenomenon in nonlinear science. It arises as the interplay between determined and random dynamics (Hramov *et al.*, 2006), with both the synchronization and noise influence being recently the subjects of considerable interest of scientific community. Indeed, on the one hand the synchronous behavior of nonlinear systems has attracted great attention of researchers for a long time (Pecora and Carroll, 1990; Hramov and Koronovskii, 2004). On the other hand discovering the fact that fluctuations can actually induce some degree of order in a large variety of nonlinear systems is one of the most surprising results of the last decades in the field of stochastic processes (Pikovsky and Kurths, 1997; Mangioni *et al.*, 1997; Zaikin *et al.*, 2000). Moreover, both these phenomena are relevant for physical, chemical, biological and other systems described in terms of nonlinear dynamics (see, e.g., (Shuai and Wong, 1998)).

Noise-induced synchronization (NIS) means that the random signal influencing two identical uncoupled dynamical chaotic systems $\mathbf{u}(t)$ and $\mathbf{v}(t)$ (starting from the different initial conditions $\mathbf{u}(t_0)$ and $\mathbf{v}(t_0)$, $\mathbf{u}(t_0) \neq \mathbf{v}(t_0)$) results in their synchronous (i.e., identical) behavior $\mathbf{u}(t) = \mathbf{v}(t)$ after transient finished.

Noise-induced synchronization can be detected by means of direct comparison of the states of two chaotic systems, $\mathbf{u}(t)$, and, $\mathbf{v}(t)$, being under the influence of noise. The other method of diagnostics of NIS is calculating the largest Lyapounov exponent (LE) of dynamical system that measures the stability of the motion. Indeed, in driven chaotic system the largest Lyapunov exponent may become negative, that results in synchronization: both systems forced by the same noise “forget” their initial conditions and evolve to identical state (Goldobin and Pikovsky, 2005). If the noise influence is vanishingly small the largest Lyapunov exponent is positive for such a system.

In all cases of the noise-induced synchronization being known hitherto the boundary of the noise-induced synchronization regime is associated with the point on the parameter axis where the largest Lyapunov exponent of the system under study crosses the zero value when its sign is changed from “plus” to “minus”. In this paper we report for the first time that the noise-induced synchronization regime of two spatially extended uncoupled identical systems driven by common noise may be preceded by a new type of behavior, when the largest Lyapunov exponent remains to be zero in a finite range of parameter values. This kind of behavior called *“incomplete noise induced synchronization”* (INIS) demonstrates the features of the synchronous motion of two uncoupled identical systems driven by common noise: although the states of the system differ from each other, moving one system along the second one someone can find such spatial shift that both systems start showing the identical behavior.

The system under study is represented by a pair of uncoupled complex Ginzburg–Landau equations (CGLEs) driven by common noise, whose equations may be written as

$$\begin{array}{l} u_t = u - (1 - i\beta)|u|^2 u + (1 + i\alpha)u_{xx} + D\zeta(x,t), \\ v_t = v - (1 - i\beta)|v|^2 v + (1 + i\alpha)v_{xx} + D\zeta(x,t), \end{array} \tag{1}$$

where $u(x,t)$, $v(x,t)$ are complex states of the considered systems, α and $\beta = 4$ are the control parameters, D defines the intensity of a noise term. We have used model noise with the asymmetrical probability distribution of the real and imagine parts of the random variable

$$p(\xi) = \begin{cases} 2\xi, \text{ if } 0 \le \xi \le 1, \\ 0, \ \ \text{otherwise} \end{cases} \tag{2}$$

on the unit interval $[0;1]$. The simulation of the random variable ζ with required probability distribution $p(\zeta)$ was carried out in the same way as it was described in (Sweet *et al.*, 2001) for the exponential stagger distribution. Equation (1) was solved with periodic boundary conditions with all numerical calculations being performed for a fixed system length $L = 40\pi$ and random initial conditions. To evaluate (1) the standard numerical scheme has been used (García-Ojalvo and Sancho, 1999), the value of the grid spacing is $\Delta x = L/1024$, the time step of the scheme $\Delta t = 2.0 \times 10^{-4}$.

If the noise intensity is equal to zero ($D = 0$) and initial conditions $u(x,0)$ and $v(x,0)$ are not identical, both systems demonstrate the complex chaotic behavior (both in time and in space), with the system states being different, i.e., $u(x,t) \neq v(x,t)$ (Fig. 1,*a*). Alternatively, if the noise intensity D is large enough the states of both systems coincide with each other (Fig. 1,*b*), that is the evidence of the noise-induced synchronization.

To detect the presence of the noise–induced synchronization regime the averaged difference

$$\varepsilon = \frac{1}{TL} \int\limits_{\tau}^{\tau+T} \int\limits_{0}^{L} |u(x,t) - v(x,t)| \, dxdt, \tag{3}$$

between the spatio-temporal states of two CGLEs driven by common noise was calculated. The averaging process starts after a long-time transient with duration $\tau = 200$.

In the NIS regime the relation $\varepsilon = 0$ takes place, since in this case the difference between the states of two identical spatially extended systems (1) in every point of space tends to be zero. We have also calculated

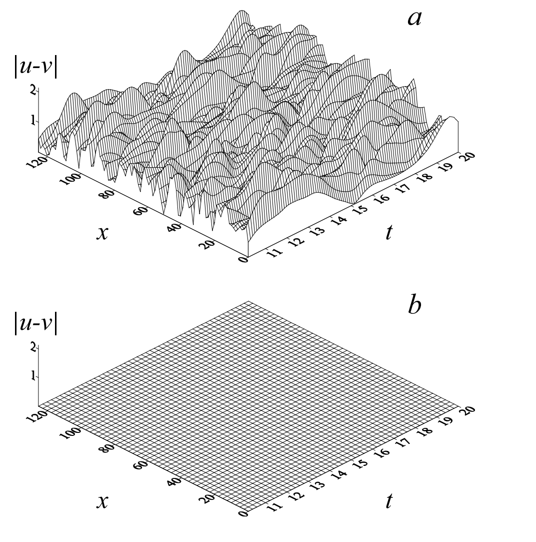


Figure 1. The evolution of the difference of the system states $|u(x,t) - v(x,t)|$ described by complex Ginzburg-Landau equations (1) (*a*) without noise and (*b*) with noise with the intensity $D = 3$. The control parameter values are $\alpha = 2, \beta = 4$

the largest Lyapunov exponent λ for one of the systems (1). As it was mentioned above in the NIS regime the largest Lyapunov exponent λ should be negative.

The dependencies of the largest Lyapunov exponent $\lambda(D)$ and the averaged difference $\varepsilon(D)$ on the noise intensity D are shown in Fig. 2 for two different values of the control parameter α. For the control parameter $\alpha = 1$ (curves 1 in Fig. 2,*a*,*b*) the value of the noise intensity D for which the largest Lyapunov exponent λ crosses the zero value and becomes negative coincides with the point where the averaged difference (3) starts being vanishingly small. So, in this case the noise–induced synchronization boundary is $D_{NIS} \approx 1.5$ and we deal with the occurrence of the noise-induced synchronization regime being typical and well-known.

Alternatively, the different scenario is observed in the same system (1) if the control parameter value $\alpha = 2$ is considered (see curves 2 in Fig. 2, *a*,*b*). For such a choice of α-parameter value the largest Lyapunov exponent becomes equal to zero for the large enough intensity of noise $D_{INIS} \approx 1.53$ whereas the averaged difference ε between the spatio-temporal states of two CGLEs driven by common noise exceeds the zero value sufficiently (Fig. 2, *a*,*b*). With further increase of the noise intensity D the value of ε becomes equal to zero and the largest Lyapunov exponent starts to be negative that is the evidence of the presence of the noise-induced synchronization regime.

In other words, there is the finite interval of the noise intensity values $(D_{INIS}; D_{NIS})$ for which the noise-induced synchronization is not observed, and the largest Lyapunov exponent λ is equal to zero. To prove this fact we have calculated the largest Lyapunov exponent of the complex Ginzburg-Landau equation for different values of the spatial grid spacing. We obtain

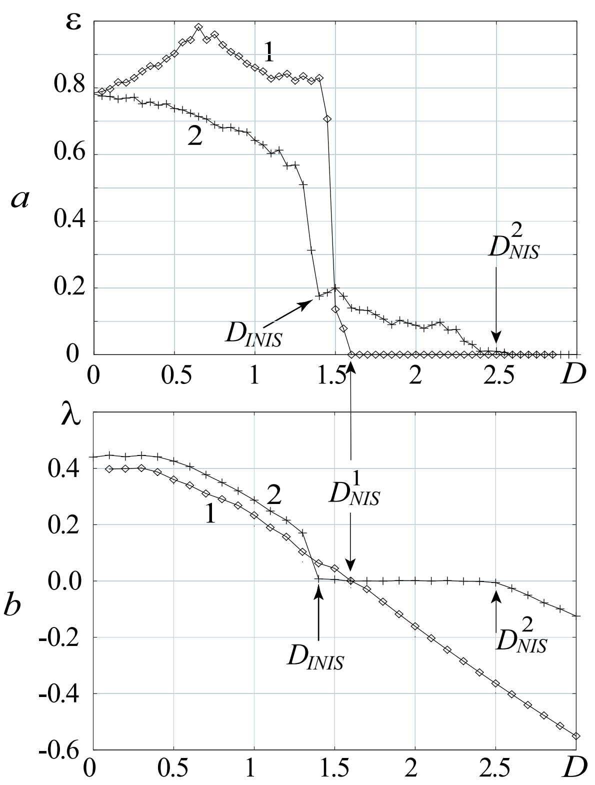


Figure 2. The dependencies of (*a*) the averaged difference (3) and (*b*) the largest Lyapounov exponent of the CGLE on the noise intensity D for the different values of the control parameter α. Curves 1 correspond to the case of $\alpha = 1$, curves 2 were calculated for $\alpha = 2$. The values of noise intensity corresponding to the onset of noise–induced synchronization are shown by arrows with labels D^1_{NIS} and D^2_{NIS} for the curves 1 and 2, respectively. The boundary of the incomplete noise-induced synchronization is also shown by arrow marked as D_{INIS}

that the largest Lyapunov exponent calculations with the different values of the spatial grid step give the similar results. Based on these calculations we come to conclusion that the largest Lyapunov exponent is actually equal to zero in the finite range of the noise intensity.

Despite the fact that the noise-induced synchronization is not observed in the region where $\lambda = 0$, this range of the noise intensities corresponds to the behavior showing the features of synchronous dynamics. The manifestation of synchronism may be observed if one of the complex media described by the Ginzburg-Landau equation starts to be shifted slowly along the second one with the spatial shift δ. In other words, if one uses the shifted state of one of the system $v = v(x + \delta, t)$ in Eq. (1) the averaged difference ε changes depending on this shift δ. This movement of one of the systems supposed to be very slow for the transient to be completed. In this case such a spatial shift δ_0 may be found that both Ginzburg-Landau equations start to behave identically, with the largest Lyapunov exponent being equal to zero. Therefore, we have called this regime *"incomplete noise-induced synchronization"* (INIS).

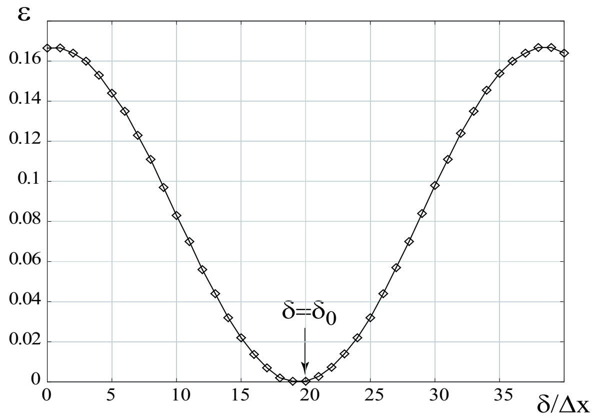
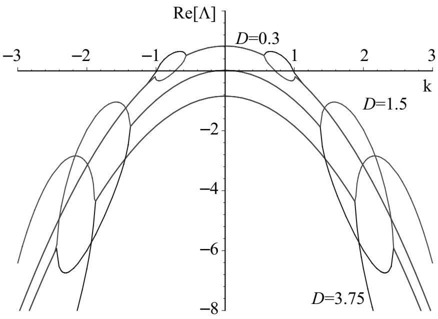


Figure 3. The dependence of the difference ε between the states of the media $u(x, t)$ and $v(x, t)$ described by the complex Ginzburg-Landau equations (1) on the space shift δ for the control parameters $\alpha = 2, \beta = 4, D = 2$

This statement is illustrated in Fig. 3 where the dependence of the difference ε (3) on the space shift δ is shown. One can see that there is the value δ_0 of the shift δ for which the averaged difference ε becomes equal to zero. Therefore, for this space shift δ_0 both systems demonstrate identical behavior and the noise-induced synchronization is observed. This shift δ_0 depends on the initial conditions. For the other values of the spatial shift δ the system states (both in space and time) are different, but the largest Lyapunov exponent is always equal to zero for the considered set of the control parameter values.

We study the mechanisms resulting in the occurrence of the incomplete noise-induced synchronization regime. In work (Hramov *et al.*, 2006) it has been shown, that for dynamical systems with small number of degrees of freedom the mechanisms of arising of noise-induced synchronization and generalized synchronization are equivalent. The mechanism of the generalized synchronization occurrence can be considered with the help of the modified system approach as it was done in Ref. (Hramov and Koronovskii, 2005) for the chaotical systems with small number of degrees of freedom and in Ref. (Hramov *et al.*, 2005) for the spatially extended system. It is possible to assume, that the mechanism of the incomplete noise-induced synchronization arising may be also explained in the same way. Therefore, following Ref. (Hramov *et al.*, 2006; Hramov and Koronovskii, 2005; Hramov *et al.*, 2005) we consider the dynamics of the modified spatially extended system with the additional term determined by the mean value of noise.

The modified Ginzburg-Landau equation with the additional term, determined by the noise with the mean



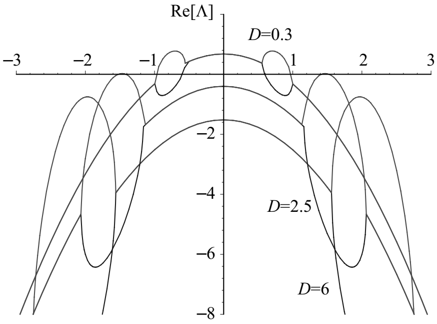


Figure 4. The dependencies of the real part of the eigenvalues Λ on the wave number k for the different values of D-parameter when the control parameter α has been fixed as (*a*) $\alpha = 1$ and (*b*) $\alpha = 2$

value $\langle D\zeta\rangle$ can be written as

$$\frac{\partial u_m}{\partial t} = u_m-(1-i\beta)|u_m|^2u_m+(1+i\alpha)\frac{\partial^2 u_m}{\partial x^2}+\langle D\zeta\rangle. \tag{4}$$

For the selected kind of noise with the probability distribution (2) $\langle D\zeta\rangle = 2D/3$.

Equation (4) is forced CGLE, widely studied and well documented in the literature (see, e.g. (Coullet and Emilsson, 1992; Glendinning and Proctor, 1993; Chate *et al.*, 1999)). It is well-known, that the different types of the spatio-temporal patterns may be observed depending on the domain of the control parameter values. If the value D is large enough, the homogeneous stationary state $u_0 = u_0(x,t) = \mathrm{const}$ is observed in the system (4). In this case the largest Lyapunov exponent is negative, with the stationary state regime in the system (4) corresponding to the noise-induced synchronization in the system (1). With decrease of the noise intensity D the stationary state u_0 loses its stability that corresponds to the boundary of the noise-induced synchronization of the initial Ginsburg-Landau equations (1) driven by noise.

At the same time the loss of the stability of the homogeneous stationary state occurs in the different ways depending on the control parameter values of the modified Ginzburg-Landau equation (4).

Indeed, the homogeneous stationary state u_0 can be obtained numerically from equation

$$u_0 - (1-i\beta)|u_0|^2u_0 + 2D/3 = 0, \tag{5}$$

To analyze the stability of Eq. (5) we have to consider the linearization of the modified Ginzburg-Landau equation in the vicinity of the stationary solution u_0. Let $\tilde{u} = \tilde{u}_r + i\tilde{u}_i$ be a small perturbation of the homogenous stationary state $u_0 = u_r + iu_i$, i.e., $u_m = u_0 + \tilde{u}$. Having linearized equation (4) and assuming that $\tilde{u}_r(x,t) = \hat{u}_r(k)\exp(\Lambda t + ikx)$, $\tilde{u}_i(x,t) = \hat{u}_i(k)\exp(\Lambda t + ikx)$ we obtain the dispersion relation $\Lambda(k)$ determining the stability of the homogenous stationary state u_0. The homogenous stationary state u_0 is stable if the condition $\mathrm{Re}\,\Lambda(k) < 0$, $\forall k$ is satisfied.

The evolution of $\mathrm{Re}\,\Lambda(k)$ with the decrease of D-value for $\alpha = 1$ and $\alpha = 2$ is shown in Fig. 4, *a* and Fig. 4, *b*, respectively. One can see, that for $\alpha = 1$ the homogenous stationary state u_0 loses its stability when $D \approx 1.5$. In this case the spatial perturbation with the wave number $k = 0$ starts growing exponentially. As a result, the stationary state u_0 becomes unstable, the spatio-temporal chaos taking place in system (4). The largest Lyapunov exponent becoming positive both in the modified (4) and original (1) Ginzburg-Landau equation, the noise-induced synchronization regime in Eq. (1) is destroyed.

For the value of the control parameter $\alpha = 2$ the homogenous stationary state u_0 loses its stability for $D \approx 2.5$ and the spatial mode with the wave number $k = \pm 0.5$ becomes unstable in contrast to the case of $\alpha = 1$ considered before (see Fig. 4, *b*). Therefore, for $\alpha = 2$ the periodic spatial state $u_k(x) = u_k(x+l)$ (where l is close to $2\pi/k$ due to periodical boundary conditions) being stationary in time replaces the homogenous state u_0 in the modified Ginzburg-Landau equation. Obviously, for such stationary states the largest Lyapunov exponent is equal to zero. Evidently, in the initial Ginzburg-Landau equation driven by noise, $D\zeta(x,t)$, with the mean value $\langle D\zeta\rangle$ the stationary in time and periodical in space structure $u_k(x)$ is perturbed by the fluctuations. Therefore, the spatio-temporal dynamics of $u_k(x)$ looks like aperiodic motion, with the largest Lyapunov exponent being also equal to zero. Since two identical media, $u(x,t)$, and, $v(x,t)$, driven by common noise start with different initial conditions $u(x,0)$ and $v(x,0)$ the periodical in space structures do not coincide with each other, i.e., $u_k(x) \neq v_k(x)$, but there is such a shift in space δ_0 depending on the initial conditions $u(x,0)$ and $v(x,0)$ that $u_k(x) = v_k(x+\delta_0)$. Therefore, for $D_{INIS} < D < D_{NIS}$ Ginzburg-Landau equations (1) driven by common noise are characterized by zero largest Lyapunov exponent and their states are not identical. If one of the systems is shifted along the second one with a certain shift δ_0 that, depending on initial conditions, satisfies the requirement $u_k(x) = v_k(x+\delta_0)$, the identical behavior of both considered systems is observed.

We come to conclusion that the occurrence of the incomplete noise-induced synchronization regime is determined by the mean value of noise, whereas the variation of it practically does not play the role.

So, we have reported for the first time a new type of noise-induced synchronous behavior occurring in the spatially extended systems. Such a type of incomplete noise-induced synchronization differs remarkably from all the other types of synchronous behavior known so far. It may be observed in a certain range of the noise intensity values, where the largest Lyapunov exponent is equal to zero and the states of two identical spatially extended systems driven by common noise are different, although there is an indication of the synchronism: if one of the systems is shifted along the second one on the certain shift the identical behavior of the considered systems is observed. The theoretical equations allowing to explain the mechanism resulting in such a type of behavior have also been given, and they are in perfect agreement with the numerically obtained data. Since the noise influence may result in the pattern formation we suppose that incomplete noise-induced synchronization can be also observed for noise with the zero mean value, with the other types of the spatio-temporal patterns (e.g., traveling waves) being observed.

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