LYAPUNOV-TYPE THEOREMS FOR ESTIMATION OF OUTPUT OF CONTROL SYSTEM WITH PERIODIC DIFFERENTIABLE NONLINEARITY

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Abstract

For multidimensional systems of indirect control with periodic differentiable nonlinearities the problem of cycle-slipping is investigated. By means of generalized Lyapunov periodic functions and Yakubovich-Kalman frequency theorem, new frequency-algebraic estimates for a number of slipped cycles are obtained.

Key words

Phase system, cycle-slipping, frequency-algebraic theorem, Lyapunov-type periodic function.

1 Introduction

In this paper the systems of indirect control with periodic differentiable nonlinearities and denumerable sets of equilibria are considered. Such control systems are often called phase systems.

Phase systems are widespread in mechanics, radio and electric engineering, communication. That is why the asymptotic behavior of phase systems is of great interest, and many published works are devoted to problems which arise here.

In most works the problem of gradient-like behavior is studied with the aim to obtain conditions which guarantee that every solution converges to a certain equilibrium. For details and bibliography one can turn to the papers [Leonov, 2006] and [Perkin,Smirnova, and Shepeljavyi, 2012].

This paper is devoted to the problem of cycle-slipping in gradient-like phase control systems. The problem was introduced in the book [Stoker, 1950], for math-

ematical pendulum which underwent the strength of the resistance of the medium proportional to the square of its angular speed. The problem was then extended to multidimensional phase systems in paper [Yershova, and Leonov, 1983].

The essence of cycle-slipping is as follows. Suppose the phase system is gradient-like. Let it have a Δ - periodic scalar input and let $\sigma(t)$ be its angular scalar output. They say that the output function has slipped $k \in \mathbf{N} \bigcup \{0\}$ cycles if there exists such a moment $\widehat{t} \geq 0$ that

$$|\sigma(\widehat{t}) - \sigma(0)| = k\Delta, \qquad (1)$$

but

$$|\sigma(t) - \sigma(0)| < (k + 1)\Delta \qquad (2)$$

for all $t \geq 0$. The problem which arises here is to establish certain estimates for a number of slipped cycles.

In the paper [Yershova, and Leonov, 1983] such estimates were obtained by means of Lyapunov-type functions and Yakubovich-Kalman frequency theorem. The estimates had the form of frequency-domain inequalities with varying parameters which ought to satisfy algebraic restrictions.

Ideas and theorems of [Yershova, and Leonov, 1983] were exploited then in a series of published works in order to investigate the cycle-slipping of various types of phase systems.

In [Yang, and Huang, 2007] the results of [Yershova, and Leonov, 1983] were joined with the LMI method.

In [Leonov, Reitmann, and Smirnova, 1992] and [Smirnova, Utina, and Shepeljavyi, 2006] the theorems proved in [Yershova, and Leonov, 1983] were extended to distributed phase systems and discrete phase systems correspondingly. At the same time in [Leonov, Reitmann, and Smirnova, 1992] differential properties of periodic nonlinearities were taken into consideration and new frequency-domain inequalities with more varying parameters were introduced.

In this paper frequency-domain inequalities intended for differentiable nonlinearities are used. At the same time generalized periodic Lyapunov functions introduced in [Perkin,Smirnova, and Shepeljavyi, 2012] are exploited here. These functions give the opportunity to establish for varying parameters new algebraic restrictions which are less limitative than the restrictions formulated in [Leonov, Reitmann, and Smirnova, 1992].

2 Frequency-algebraic estimates for cycle-slipping of multidimensional phase system

Consider an autonomous control system

$$
\left.\begin{array}{l}
\frac{dz}{dt} = Az + b\varphi(\sigma) \quad (z \in \mathbf{R}^m, \sigma \in \mathbf{R}), \\
\frac{d\sigma}{dt} = c^*z + \rho\varphi(\sigma).
\end{array}\right\} \quad (3)
$$

Here A — $m \times m$ - real matrix, b and c are real m - vectors, ρ is a number, $\varphi(\sigma)$ is a Δ - periodic function and symbol (*) is used for Hermitian conjugation.

We suppose that A is a Hurwitz matrix, the pairs (A, b) and (A, c) are controllable and observable respectively. We suppose also that $\varphi(\sigma)$ has simple zeros on $[0, \Delta)$ and is continuously differentiable. So

$$
\alpha_1 \le \frac{d\varphi}{d\sigma} \le \alpha_2 \quad (\sigma \in \mathbf{R}). \quad (4)
$$

with $\alpha_1\alpha_2 < 0$. Let us define the function

$$
\Phi(\sigma) = \sqrt{(1 - \alpha_1^{-1}\varphi'(\sigma))(1 - \alpha_2^{-1}\varphi'(\sigma))}.
$$

Let us for definiteness suppose that

$$
\int_0^\Delta \varphi(\sigma)d\sigma \le 0. \quad (5)
$$

First we shall present a number of auxiliary Lyapunov-type assertions. We shall need the functions

$$
r_j(k, \ae, x) = \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j \frac{x}{\ae k}}{\int_0^\Delta |\varphi(\sigma)|d\sigma} \quad (j = 1, 2),
$$

$$
r_{0j}(k, \ae, x) = \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j \frac{x}{\ae k}}{\int_0^\Delta \Phi(\sigma)|\varphi(\sigma)|d\sigma} \quad (j = 1, 2).
$$

Lemma 1. Suppose there exist such numbers $k \in \mathbf{N}$, $a \in [0, 1]$, $\ae \neq 0$, positive ε, δ, τ and such continuously differentiable $\sigma(t)$ and $W(t)$ that the following conditions are fulfilled:
1) $W(t) \ge 0$ for $t \ge 0$;
2)

$$
\frac{dW(t)}{dt} + \ae\varphi(\sigma(t))\frac{d\sigma(t)}{dt} + \varepsilon\left(\frac{d\sigma(t)}{dt}\right)^2 + \delta\varphi^2(\sigma(t))
$$
$$
+\tau\Phi^2(\sigma(t))\left(\frac{d\sigma(t)}{dt}\right)^2 \le 0, \forall t \ge 0;
$$

$$(6)$$

3) matrices $T_j(W(0))$ $(j = 1, 2)$, where $T_j(x) =$

$$
= \left\| \begin{array}{ccc}
\varepsilon & , & \frac{a\ae r_j(k, \ae, x)}{2} & , & 0 \\
\frac{a\ae r_j(k, \ae, x)}{2} & , & \delta & , & \frac{a_0\ae r_{0j}(k, \ae, x)}{2} \\
0 & , & \frac{a_0\ae r_{0j}(k, \ae, x)}{2} & , & \tau
\end{array} \right\|
$$

and $a_0 = 1 - a$, are positive definite. Then

$$
|\sigma(t) - \sigma(0)| < k\Delta, \quad t \ge 0. \quad (7)
$$

Lemma 1 is proved in [Perkin,Smirnova, Shepeljavyi, and Utina,2009]. We shall prove here a modification of lemma 1.

Consider the functions

$$
r_{1j}(k, \ae, \varepsilon, \tau, x) = \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j \frac{x}{\ae k}}{\int_0^\Delta |\varphi(\sigma)|\sqrt{\varepsilon + \tau\Phi^2(\sigma)}d\sigma} \quad (j = 1, 2).
$$

Lemma 2. Suppose there exist such numbers $k \in \mathbf{N}$, $\ae \neq 0$, ε, δ, $\tau > 0$ and such functions $\sigma(t) \in C^1(0, \infty)$ and $W(t) \in C^1(0, \infty)$ that the conditions 1) and 2) of Lemma 1 are fulfilled and

$$
4\delta > \ae^2(r_{1j}(k, \ae, \varepsilon, \tau, W(0))^2 \quad (j = 1, 2). \quad (8)
$$

Then the assertion (7) is true.

Proof. Let ε_0 be so small that

$$4\delta > æ^2(r_{1j}(k, æ, \varepsilon, \tau, W(0)+\varepsilon_0)^2 \quad (j = 1, 2). \quad (9)$$

Define the functions

$$Y_j(\sigma) = \varphi(\sigma) - r_{1j}|\varphi(\sigma)|P(\varepsilon, \tau, \sigma) \quad (j = 1, 2),$$

where $r_{1j} = r_{1j}(k, æ, \varepsilon, \tau, W(0) + \varepsilon_0), P(\varepsilon, \tau, \sigma) = \sqrt{\varepsilon + \tau\Phi^2(\sigma)}$, and consider two Lyapunov-type functions

$$V_j(t) = W(t) + æ \int_{\sigma(0)}^{\sigma(t)} Y_j(\sigma)d\sigma \quad (j = 1, 2).$$

Their derivatives are as follows

$$\frac{dV_j(t)}{dt} = \frac{dW(t)}{dt} + æY_j(\sigma(t))\dot{\sigma}(t). \quad (10)$$

From condition 1) of Lemma 1 we conclude that

$$\frac{dV_j}{dt} \leq -\varepsilon\dot{\sigma}^2 - \delta\varphi^2(\sigma) - \tau(\Phi(\sigma)\dot{\sigma})^2 - æ r_{1j}P(\varepsilon, \tau, \sigma)|\varphi(\sigma)|\dot{\sigma} \quad (j = 1, 2), \quad (11)$$

or

$$\frac{dV_j}{dt} \leq -(P(\varepsilon, \tau, \sigma)\dot{\sigma})^2 - \delta|\varphi(\sigma)|^2 - æ r_{1j}|\varphi(\sigma)|P(\varepsilon, \tau, \sigma)\dot{\sigma}. \quad (12)$$

It follows from (9) that

$$\frac{dV_j(t)}{dt} \leq 0 \quad (j = 1, 2), \quad (13)$$

and consequently for all $t > 0$

$$V_j(t) \leq V_j(0) = W(0) \quad (j = 1, 2). \quad (14)$$

Suppose now that for a certain $\bar{t}$ we have

$$\sigma(\bar{t}) = \sigma(0) + k\Delta. \quad (15)$$

Then

$$V_1(\bar{t}) = W(\bar{t}) + æk \int_0^\Delta Y_1(\sigma)d\sigma = W(\bar{t}) + W(0) + \varepsilon_0 \quad (16)$$

So $V_1(\bar{t}) > W(0)$ which contradicts (14).
If we suppose that

$$\sigma(\bar{t}) = \sigma(0) - k\Delta \quad (17)$$

we can use $V_2(t)$ and establish that

$$V_2(\bar{t}) = W(\bar{t}) - æk \int_0^\Delta Y_2(\sigma)d\sigma = = W(\bar{t}) + W(0) + \varepsilon_0 > W(0). \quad (18)$$

Lemma 2 is proved.

In order to use the two lemmas for estimation of solutions of system (3) we exploit certain approaches from monographs [Yakubovich, Leonov, and Gelig, 2004] and [Leonov, Reitmann, and Smirnova, 1992].
Let us introduce the function $\xi(t) = \frac{d}{dt}\varphi(\sigma(t))$ and the $(m + 1)$-vector-function

$$y(t) = \left\| \begin{array}{c} z(t) \\ \varphi(\sigma(t)) \end{array} \right\|.$$

We introduce also $(m + 1)$-vectors

$$L = \left\| \begin{array}{c} O \\ 1 \end{array} \right\|, \quad D = \left\| \begin{array}{c} c \\ \rho \end{array} \right\|$$

and the $(m + 1) \times (m + 1)$-matrix

$$Q = \left\| \begin{array}{cc} A & b \\ O & 0 \end{array} \right\|.$$

Functions $y(t)$ and $\xi(t)$ satisfy the system

$$\left. \begin{array}{l} \dfrac{dy(t)}{dt} = Qy(t) + L\xi(t), \\ \dfrac{d\sigma(t)}{dt} = D^*y(t) \end{array} \right\}. \quad (19)$$

The controlability of (A, b) implies the controlability of (Q, L) [Yakubovich, Leonov, and Gelig, 2004].
Consider the following quadratic form of $y \in \mathbf{R}^{m+1}$, $\xi \in \mathbf{R}$:

$$G(y, \xi) = 2y^*H(Qy + L\xi) + \varepsilon y^*DD^*y + æy^*LD^*y - \tau(D^*y - \alpha_1^{-1}\xi)(\alpha_2^{-1}\xi - D^*y) + \delta y^*LL^*y.$$

Here $H = H^*$ is an real $(m + 1) \times (m + 1)$-matrix and ε, $æ$, τ and δ are parameters.
Consider the function

$$K(p) = -\rho + c^*(A - pE_m)^{-1}b \quad (p \in \mathbf{C}),$$

where E_m is a unit $m \times m$-matrix, which is the transfer function of linear part of (3) from the input φ to the output $(-\dot{\sigma}(t))$. It is demonstrated in [Yakubovich,Leonov, and Gelig, 2004] by means of Yakubovich-Kalman theorem that if the frequency-domain inequality

$$Re\{æK(i\omega) - \tau(K(i\omega) + \alpha_1^{-1}i\omega)^*(K(i\omega) + \alpha_2^{-1}i\omega)\} \\ -\varepsilon|K(i\omega)|^2 - \delta \geq 0$$

(20)

is true for $\omega \geq 0$ then there exists such real matrix $H = H^*$ that

$$G(y,\xi) \leq 0, \quad \forall y \in \mathbf{R}^{m+1}, \quad \xi \in \mathbf{R}. \qquad (21)$$

Let $(z(t), \sigma(t))$ be a solution of (3) with the initial data $(z(0), \sigma(0))$. The corresponding solution $y(t)$ of (19) has the initial data
$$y(0) = \left\| \begin{array}{c} z(0) \\ \varphi(\sigma(0)) \end{array} \right\|.$$
Note that $y(t)$ is bounded (since A is Hurwitz matrix and $\varphi(\sigma)$ is a bounded function).

Theorem 1. Suppose there exist such $k \in \mathbf{N}$, $a \in [0, 1]$, positive ε, δ, τ and $æ \neq 0$ that the following conditions are fulfilled:

1) for all $\omega \geq 0$ the frequency-domain inequality (20) is true;

2) matrices $T_j (y^*(0)Hy(0) - I)$ $(j = 1, 2)$, where $I = \inf_{t \in R_+} y^*(t)Hy(t)$, are positive definite for a certain matrix $H = H^*$, satisfying (21).

Then for the solution of (3) with initial data $(z(0), \sigma(0))$ the estimate

$$|\sigma(t) - \sigma(0)| < \Delta k \qquad (22)$$

is true for all $t > 0$.

Proof. The proof is based on Lemma 1, where $\{z(t), \sigma(t)\}$ is the solution of system (3) with initial data $(z(0), \sigma(0))$. Let

$$W(t) = y^*(t)Hy(t) - I$$

where $y(t)$ is the solution of (19) with $y(0) = \left\| \begin{array}{c} z(0) \\ \varphi(\sigma(0)) \end{array} \right\|$. We have $W(t) \geq 0$ for $t \geq 0$ and

$$\frac{dW(t)}{dt} = 2y^*(t)H (Qy(t) + L\xi(t)). \qquad (23)$$

Condition 1) of the theorem guarantees that (21) is true, whence as $D^*y = \dot{\sigma}$ and $L^*y = \varphi(\sigma(t)$ it follows that

$$\frac{dW(t)}{dt} \leq -æ\varphi(\sigma(t))\dot{\sigma}(t) - \varepsilon\dot{\sigma}^2(t) - \delta\varphi^2(\sigma(t))- \\ -\tau\dot{\sigma}^2(t) \left(1 - \alpha_1^{-1}\varphi'(\sigma(t))\right)\left(1 - \alpha_2^{-1}\varphi'(\sigma(t))\right).$$

(24)

So condition 2) of Lemma 1 is fulfilled. Condition 3) of Lemma 1 coincides with condition 2) of the theorem. It follows from Lemma 1 that estimate (22) holds for all $t \geq 0$. Theorem 1 is proved.

Theorem 2. Suppose for certain parameters ε, δ, $\tau > 0$, $æ \neq 0$ and $k \in \mathbf{N}$ the following conditions are fulfilled:

1) for all $\omega \geq 0$ the frequency-domain inequality (20) is true;

2)

$$4\delta > æ^2(r_{1j}(k, æ, \varepsilon, \tau, y(0)^*Hy(0)-I))^2 \quad (j = 1, 2)$$

(25)

where I and H are defined in the text of Theorem 1. Then for the solution of (3) with the initial data $(z(0), \sigma(0))$ the estimate (22) is true for all $t \geq 0$.

The proof of Theorem 2 is based on Lemma 2. It is just alike the proof of Theorem 1 with the same functions $\sigma(t)$ and $W(t)$.

3 Sharpened estimates for cycle-slipping

In this section we continue to establish various frequency-algebraic estimates for cycle-slipping of system (3). We are going to handle the same tools as in the previous section. We shall use system (19), the quadratic form $G(y, \xi)$, the frequency-domain inequality (20). Our aim is to simplify algebraic conditions 2) of Theorems 1 and 2.

Let $z(t), \sigma(t)$ be a solution of (3) with the initial data $(z(0), \sigma(0))$. Let σ_0 be such a zero of $\varphi(\sigma)$ that

$$\sigma_0 - \Delta \leq \sigma(0) < \sigma_0. \qquad (26)$$

In this section we replace functions r_j, r_{0j}, r_{1j} $(j = 1, 2)$ by the functions

$$\nu_1(k, æ, x) = \frac{\int_{\sigma(0)}^{\sigma_0} \varphi(\sigma)d\sigma + k\int_0^{\Delta} \varphi(\sigma)d\sigma - æ^{-1}x}{\int_{\sigma(0)}^{\sigma_0} |\varphi(\sigma)|d\sigma + k\int_0^{\Delta} |\varphi(\sigma)|d\sigma},$$

$$\nu_2(k, æ, x) = \frac{\int_{\sigma_0 - \Delta}^{\sigma(0)} \varphi(\sigma)d\sigma + k\int_0^{\Delta} \varphi(\sigma)d\sigma + æ^{-1}x}{\int_{\sigma_0 - \Delta}^{\sigma(0)} |\varphi(\sigma)|d\sigma + k\int_0^{\Delta} |\varphi(\sigma)|d\sigma},$$

$$\nu_{01}(k, æ, x) = \frac{\int_{\sigma(0)}^{\sigma_0} \varphi(\sigma)d\sigma + k\int_0^{\Delta} \varphi(\sigma)d\sigma - æ^{-1}x}{\int_{\sigma(0)}^{\sigma_0} \Phi(\sigma)|\varphi(\sigma)|d\sigma + k\int_0^{\Delta} \Phi(\sigma)|\varphi(\sigma)|d\sigma},$$

$$\nu_{02}(k,\text{æ},x) = \frac{\int\limits_{\sigma_0-\Delta}^{\sigma(0)} \varphi(\sigma)d\sigma + k\int\limits_{0}^{\Delta} \varphi(\sigma)d\sigma + \text{æ}^{-1}x}{\int\limits_{\sigma_0-\Delta}^{\sigma(0)} \Phi(\sigma)|\varphi(\sigma)|d\sigma + k\int\limits_{0}^{\Delta} \Phi(\sigma)|\varphi(\sigma)|d\sigma},$$

$$\nu_{11}(d,x) = \frac{\int\limits_{\sigma(0)}^{\sigma_0} \varphi(\sigma)d\sigma + k\int\limits_{0}^{\Delta} \varphi(\sigma)d\sigma - \text{æ}^{-1}x}{\int\limits_{\sigma(0)}^{\sigma_0} |\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma + k\int\limits_{0}^{\Delta} |\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma},$$

$$\nu_{12}(d,x) = \frac{\int\limits_{\sigma_0-\Delta}^{\sigma(0)} \varphi(\sigma)d\sigma + k\int\limits_{0}^{\Delta} \varphi(\sigma)d\sigma + \text{æ}^{-1}x}{\int\limits_{\sigma_0-\Delta}^{\sigma(0)} |\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma + k\int\limits_{0}^{\Delta} |\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma}$$

with $d = \{k,\text{æ},\varepsilon,\tau\}$, $P(\varepsilon,\tau,\sigma) = \sqrt{\varepsilon + \tau\Phi^2(\sigma)}$.

Lemma 3. Suppose there exist such numbers $k \in \mathbf{N}$, $a \in [0,1]$, ε, δ, $\tau > 0$, $\text{æ} \neq 0$ and such functions $\sigma(t), W(t) \in C^1[0,\infty)$ that the following conditions are fulfilled:
1) $W(t) \geq 0$ if $\varphi(\sigma(t)) = 0$;
2)

$$\frac{dW(t)}{dt} + \text{æ}\varphi(\sigma(t))\frac{d\sigma(t)}{dt} + \varepsilon\left(\frac{d\sigma(t)}{dt}\right)^2 +$$
$$+ \delta\varphi^2(\sigma(t)) + \tau\Phi^2(\sigma(t))\left(\frac{d\sigma(t)}{dt}\right)^2 \leq 0, \forall t \geq 0; \quad (27)$$

3) $\sigma(0) \in [\sigma_0 - \Delta, \sigma_0)$, where $\varphi(\sigma_0) = 0$;
4) matrices $U_j(W(0))$ $(j = 1,2)$, where $U_j(x) =$

$$= \left\| \begin{array}{ccc} \varepsilon & , \dfrac{a\text{æ}\nu_j(k,\text{æ},x)}{2} & , \quad 0 \\[2ex] \dfrac{a\text{æ}\nu_j(k,\text{æ},x)}{2} & , \quad \delta & , \dfrac{a_0\text{æ}\nu_{0j}(k,\text{æ},x)}{2} \\[2ex] 0 & , \dfrac{a_0\text{æ}\nu_{0j}(k,\text{æ},x)}{2} & , \quad \tau \end{array} \right\|$$

with $a + a_0 = 1$, are positive definite.
Then for all $t \geq 0$ the estimate

$$|\sigma(t) - \sigma(0)| < (k+1)\Delta \quad (28)$$

is true.

Proof. Let $\varepsilon_0 > 0$ be a small number, such that matrices $U_j(W(0) + \varepsilon_0)$ $(j = 1,2)$ are positive definite. Let us introduce functions

$$F_j(\sigma) = \varphi(\sigma) - \nu_j|\varphi(\sigma)|,$$

$$\Psi_j(\sigma) = \varphi(\sigma) - \nu_{0j}|\varphi(\sigma)|\Phi(\sigma),$$

with

$$\nu_j = \nu_j(k,\text{æ},W(0)+\varepsilon_0); \nu_{0j} = \nu_{0j}(k,\text{æ},W(0)+\varepsilon_0).$$

Consider Lyapunov-type functions

$$V_j(t) = W(t) + \text{æ}\left(a\int\limits_{\sigma(0)}^{\sigma(t)} F_j(\sigma)d\sigma + a_0\int\limits_{\sigma(0)}^{\sigma(t)} \Psi_j(\sigma)d\sigma \right).$$

Then

$$\frac{dV_j}{dt} = \frac{dW}{dt} + \text{æ}\left(aF_j(\sigma(t)) + a_0\Psi_j(\sigma(t)) \right)\dot\sigma(t). \quad (29)$$

Condition 2) of Lemma 3 implies that

$$\frac{dV_j}{dt} \leq -\varepsilon\dot\sigma^2 - \tau\left(\Phi(\sigma)\dot\sigma\right)^2 - \text{æ}a\nu_j|\varphi(\sigma)|\dot\sigma - \\ - \text{æ}a_0\nu_{0j}|\varphi(\sigma)|\Phi(\sigma)\dot\sigma - \delta\varphi^2(\sigma). \quad (30)$$

By condition 4) of the Lemma we have that

$$\frac{dV_j}{dt} \leq 0 \quad (j = 1,2). \quad (31)$$

So

$$V_j(t) \leq V_j(0) = W(0). \quad (32)$$

Suppose that

$$\sigma(\bar{t}) = \sigma_0 + k\Delta. \quad (33)$$

for a certain $\bar{t}$. Then

$$V_1(\bar{t}) = W(\bar{t}) + \\ + \text{æ}\left(a\int\limits_{\sigma(0)}^{\sigma_0+k\Delta} F_1(\sigma)d\sigma + a_0\int\limits_{\sigma(0)}^{\sigma_0+k\Delta} \Psi_1(\sigma)d\sigma \right) = \\ = W(\bar{t}) + \text{æ}\left(a\int\limits_{\sigma(0)}^{\sigma_0} F_1(\sigma)d\sigma + a_0\int\limits_{\sigma(0)}^{\sigma_0} \Psi_1(\sigma)d\sigma + \\ + ak\int\limits_{0}^{\Delta} F_1(\sigma)d\sigma + a_0k\int\limits_{0}^{\Delta} \Psi_1(\sigma)d\sigma \right). \quad (34)$$

The following equalities are true

$$\int\limits_{\sigma(0)}^{\sigma_0} F_1(\sigma)d\sigma + k\int\limits_{0}^{\Delta} F_1(\sigma)d\sigma = \int\limits_{\sigma(0)}^{\sigma_0} \varphi(\sigma)d\sigma + \\ k\int\limits_{0}^{\Delta} \varphi(\sigma)d\sigma - \nu_1\left(\int\limits_{\sigma(0)}^{\sigma_0} |\varphi(\sigma)|d\sigma + k\int\limits_{0}^{\Delta} |\varphi(\sigma)|d\sigma\right) = \\ = \text{æ}^{-1}(W(0) + \varepsilon_0). \quad (35)$$

Similarly

$$\int_{\sigma(0)}^{\sigma_0} \Psi_1(\sigma)d\sigma + k\int_0^\Delta \Psi_1(\sigma)d\sigma = æ^{-1}(W(0) + \varepsilon_0).$$
(36)

It follows from (35) and (36) that

$$V_1(\bar{t}) = W(\bar{t}) + W(0) + \varepsilon_0.$$
(37)

Since $\varphi(\sigma(\bar{t})) = \varphi(\sigma_0 + k\Delta) = 0$ we conclude from condition 1) of the lemma that $W(\bar{t}) \geq 0$, and

$$V_1(\bar{t}) > W(0),$$
(38)

which contradicts with (32). So our assumption is wrong and for all t

$$\sigma(\bar{t}) < \sigma_0 + k\Delta.$$
(39)

Suppose now that

$$\sigma(\bar{t}) = \sigma_0 - (k+1)\Delta.$$
(40)

Then

$$V_2(\bar{t}) = W(\bar{t})-$$
$$-æ\left(a\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} F_2(\sigma)d\sigma + a\int_{\sigma_0-\Delta}^{\sigma(0)} F_2(\sigma)d\sigma+\right.$$
$$\left.+a_0\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} \Psi_2(\sigma))d\sigma + a_0\int_{\sigma_0-\Delta}^{\sigma(0)} \Psi_2(\sigma))d\sigma \right).$$
(41)

It is easy to see that

$$\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} F_2(\sigma)d\sigma + \int_{\sigma_0-\Delta}^{\sigma(0)} F_2(\sigma)d\sigma =$$
$$= k\int_0^\Delta F_2(\sigma)d\sigma + \int_{\sigma_0-\Delta}^{\sigma(0)} F_2(\sigma)d\sigma =$$
$$= -\frac{1}{æ}(W(0) + \varepsilon_0),$$
(42)

and

$$\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} \Psi_2(\sigma)d\sigma + \int_{\sigma_0-\Delta}^{\sigma(0)} \Psi_2(\sigma)d\sigma =$$
$$= -\frac{1}{æ}(W(0) + \varepsilon_0).$$
(43)

As a result we obtain that

$$V_2(\bar{t}) = W(\bar{t}) + W(0) + \varepsilon_0 > W(0),$$
(44)

which contradicts with (32). So for all $t \geq 0$

$$\sigma_0 - (k+1)\Delta < \sigma(t).$$
(45)

From (39) and (45) we conclude that for all $t \geq 0$

$$\sigma_0-\sigma(0)-(k+1)\Delta < \sigma(t)-\sigma(0) < \sigma_0-\sigma(0)+k\Delta,$$
(46)
whence

$$-(k+1)\Delta < \sigma(t) - \sigma(0) < (k+1)\Delta.$$
(47)

Thus Lemma 3 is proved.

Lemma 4. Suppose there exist such numbers $k \in \mathbf{N}$, ε, δ, $\tau > 0$, $æ \neq 0$ and such functions $\sigma(t) \in C^1(0,\infty)$ and $W(t) \in C^1(0,\infty)$ that the conditions 1), 2) and 3) of Lemma 3 are fulfilled and

$$4\delta > æ^2(\nu_{1j}(k, æ, \varepsilon, \tau, W(0))^2 \quad (j = 1, 2).$$
(48)

Then the estimate (28) holds for all $t \geq 0$.

Proof. The proof is alike that of Lemma 3. It is true that

$$4\delta > æ^2(\nu_{1j}(k, æ, \varepsilon, \tau, W(0) + \varepsilon_0))^2$$
(49)

for a positive ε_0, small enough. We introduce functions

$$Z_j(\sigma) = \varphi(\sigma) - \nu_{1j}|\varphi(\sigma)|P(\varepsilon, \tau, \sigma) \quad (j = 1, 2),$$

where $\nu_{1j} = \nu_{1j}(k, æ, \varepsilon, \tau, W(0) + \varepsilon_0)$ and consider Lyapunov-type functions

$$V_j(t) = W(t) + æ\int_{\sigma(0)}^{\sigma(t)} Z_j(\sigma)d\sigma.$$

Their derivatives are as follows

$$\frac{dV_j(t)}{dt} = \frac{dW(t)}{dt} + æZ_j(\sigma(t))\dot{\sigma}(t),$$
(50)

and in virtue of condition 2) of Lemma 3 we have

$$\frac{dV_j}{dt} \leq -\varepsilon\dot{\sigma}^2 - \delta\varphi^2(\sigma) - \tau(\Phi(\sigma)\dot{\sigma})^2-$$
$$-æ\nu_{1j}P(\varepsilon, \tau, \sigma)|\varphi(\sigma)|\dot{\sigma} =$$
$$= -(P(\varepsilon, \tau, \sigma)\dot{\sigma})^2 - \delta|\varphi(\sigma)|^2-$$
$$-æ\nu_{1j}|\varphi(\sigma)|P(\varepsilon, \tau, \sigma)\dot{\sigma}.$$
(51)

It follows from (49) that

$$\frac{dV_j(t)}{dt} \le 0 \quad (j = 1, 2), \qquad (52)$$

and consequently for all $t \ge 0$

$$V_j(t) \le W(0) \quad (j = 1, 2). \qquad (53)$$

Suppose that

$$\sigma(\bar{t}) = \sigma_0 + k\Delta. \qquad (54)$$

Then

$$V_1(\bar{t}) = W(\bar{t}) + \text{æ} \int\limits_{\sigma(0)}^{\sigma_0 + k\Delta} Z_1(\sigma)d\sigma =$$

$$W(\bar{t}) + \text{æ}(\int\limits_{\sigma(0)}^{\sigma_0} \varphi(\sigma)d\sigma + k\int\limits_0^\Delta \varphi(\sigma)d\sigma) -$$

$$-\text{æ}\nu_{11}(\int\limits_{\sigma(0)}^{\sigma_0} |\varphi(\sigma)|P(\varepsilon, \tau, \sigma)d\sigma +$$

$$k\int\limits_0^\Delta |\varphi(\sigma)|P(\varepsilon, \tau, \sigma)d\sigma) = W(\bar{t}) + W(0) + \varepsilon_0$$

$$(55)$$

So

$$V_1(\bar{t}) > W(0). \qquad (56)$$

Similarly, if

$$\sigma(\bar{t}) = \sigma_0 - (k+1)\Delta \qquad (57)$$

then

$$V_2(\bar{t}) > W(0). \qquad (58)$$

Both (56) and (58) contradict with (53). Consequently

$$\sigma_0 - (k+1)\Delta < \sigma(t) < \sigma_0 + k\Delta \qquad (59)$$

for all $t > 0$. It follows that estimate (28) holds for all $t > 0$.
Lemma 4 is proved.

Consider now a solution $(z(t), \sigma(t))$ of system (3) with initial data $(z(0), \sigma(0))$ and corresponding solution $y(t)$ of system (19) with $y(0) = \left\| \begin{matrix} z(0) \\ \varphi(\sigma(0)) \end{matrix} \right\|$.

Theorem 3. Suppose there exist such numbers ε, δ, $\tau > 0$, $k \in \mathbf{N}$, $a \in [0, 1]$ and $\text{æ} \ne 0$ that the following conditions are fulfilled:

1) for all $\omega \ge 0$ the inequality (20) is true;
2) matrices $U_j(y^*(0)Hy(0))$ $(j = 1, 2)$ are positive definite for a certain matrix $H = H^*$ satisfying (21). Then for solution of (3) with initial data $(z(0), \sigma(0))$ the estimate

$$|\sigma(t) - \sigma(0)| < (k+1)\Delta. \qquad (60)$$

is true for all $t \ge 0$.

Proof. The proof is based on Lemma 3. Let $\{z(t), \sigma(t)\}$ with initial data $(z(0), \sigma(0))$ be a solution of (3) and $W(t) = y^*(t)Hy(t)$, where $y(t)$ is a solution of (19) with $y(0) = \left\| \begin{matrix} z(0) \\ \varphi(\sigma(0)) \end{matrix} \right\|$. Repeating the argument of the proof of Theorem 1 we can demonstrate that conditions 1) and 2) of the theorem provide that conditions 2) and 4) of Lemma 3 are fulfiled. It remains only to show that condition 1) of Lemma 3 is fulfilled as well. Let

$$H = \left\| \begin{matrix} H_0 & h \\ h^* & \alpha \end{matrix} \right\|,$$

where $H_0 = H_0^*$, is an $m \times m$-matrix, h is an m-vector, α is a number. It follows from (21) that

$$G(y, 0) \le 0, \quad y \in \mathbf{R}^{m+1}. \qquad (61)$$

So for $y \in \mathbf{R}^{m+1}$ it is true that

$$2y^*HQy + \text{æ}y^*LDy + (\varepsilon + \tau)|y^*D|^2 + \delta|y^*L|^2 \le 0. \qquad (62)$$

Let $\bar{y}^* = (z^*, 0)$, $z \in \mathbf{R}^m$. Then $\bar{y}^*L = 0$, $D^*\bar{y} = c^*z$, $Q\bar{y} = \left\| \begin{matrix} Az \\ 0 \end{matrix} \right\|$, $\bar{y}^*HQ\bar{y} = z^*H_0Az$. So for $z \in \mathbf{R}^m$ we have

$$2z^*H_0Az + (\varepsilon + \tau)|c^*z|^2 \le 0. \qquad (63)$$

Then since (A, c) is observable and A is a Hurwitz matrix it follows from (63) that H_0 is positive definite [Yakubovich, 1973]. For all $\tilde{\tau}$ such that $\varphi(\sigma(\tilde{\tau})) = 0$ we have

$$W(\tilde{\tau}) = y^*(\tilde{\tau})Hy(\tilde{\tau}) = z^*(\tilde{\tau})H_0z(\tilde{\tau}) \ge 0. \qquad (64)$$

Theorem 3 is proved.

Theorem 4. Suppose there exist such numbers ε, δ, $\tau > 0$, $k \in \mathbf{N}$, and $\text{æ} \ne 0$, that the following conditions are fulfilled:

1) for all $\omega \ge 0$ the inequality (20) is true;
2)

$$4\delta > \text{æ}^2(\nu_{1j}(k, \text{æ}, \varepsilon, \tau, y(0)^*Hy(0))^2 \quad (j = 1, 2), \qquad (65)$$

where $H = H^*$ satisfies (21).

Then for solution of (3) with initial data $(z(0), \sigma(0))$ the estimate (60) is true for all $t \geq 0$.

The proof of Theorem 4 is based on Lemma 4. It is alike the proof of Theorem 3. We exploit the same function $W(t)$. Then conditions 1) and 2) of the theorem provide that conditions 2) and 4) of Lemma 4 are fulfilled. Condition 1) of Lemma 4 coincides with condition 1) of Lemma 3. The fact the condition is true is proved in Theorem 3.

4   Conclusion

The paper is devoted to the problem of cycle-slipping for multidimensional systems of indirect control with periodic nonlinear input (phase control systems). The case of differentiable nonlinearities is considered. The problem is investigated with the help of Lyapunov direct method and Yakubovich-Kalman frequency-domain theorem. In the paper various types of periodic Lyapunov functions are used. As a result a series of multiparametric frequency-algebraic estimates for the number of slipped cycles of the output of the system is established.

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