COMPUTATION OF LYAPUNOV QUANTITIES

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Abstract
In the present work the general formulas for computation of the third Lyapunov quantity are obtained. A new method for computation of Lyapunov quantities, developed for Euclidian coordinates and in the time domain, is suggested.

Key words
Lyapunov quantity, symbolic computation, small-limit cycle.

1 Introduction

This work was motivated by studding the Lienard equation

$$\ddot{x} + F(x)\dot{x} + G(x) = 0.$$

Here F and G are sufficiently smooth functions.

It is well known that the Lienard equation has sufficiently simple mechanical interpretation. It describes a motion of material particle under two forces: the elastic recovering force $G(x)$ and the frictional force $F(x)$.

One of classical methods for the study of stability and instability of system in the critical case [Lyapunov, 1892] (when there exist two purely imaginary eigenvalues of the first approximation system) and the study of "small" cycles is the computing method for the study of Lyapunov quantities of system.

While for two-dimensional system the first and second Lyapunov quantities were computed in general form in the 40-50s of last century [Bautin & Leontovich 1976; Serebryakova, 1959], the third Lyapunov quantity was computed only in certain special cases (see, for example, [Lloyd & Pearson, 1997; Yu & Han, 2005; Lynch, 2005]).

In the present work the general formulas for computation of the third Lyapunov quantity are given. In this case new method for computation of Lyapunov quantities, developed for Euclidian coordinates and in the time domain, is suggested. The first steps in the development of this method were made in the works [Leonov, 2007; Kuznetsov & Leonov 2007, Leonov, 2008[1]] and some related results can be found in [Leonov, 2006]. Application of these results to computation of limit cycles can be found in the work [Leonov, Kuznetsov & Kudryashova, 2008].

The methods for the computation of Lyapunov quantities and their symbolic expressions for various special types of two-dimensional system can also be found in the work [Lloyd, 1988; Yu, 1998; Lynch, 2005].

2 Determination of approximate solution of two-dimensional system in the neighborhood of equilibrium

Consider a system of two autonomous differential equations

$$\frac{dx}{dt} = -y + f(x,y), \qquad \frac{dy}{dt} = x + g(x,y). \tag{1}$$

Here $x, y \in \mathbb{R}$ and the functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ have continuous partial derivatives of order $(n+1)$ in the open neighborhood U, of radius R_U, of the point $(x,y) = (0,0)$

$$f(\cdot,\cdot), g(\cdot,\cdot)\colon\ \mathbb{R}\times\mathbb{R}\to\mathbb{R} \quad \in \mathbb{C}^{(n+1)}(U). \tag{2}$$

Suppose, the expansion of the functions f, g begins with terms no lower than the second order and therefore we have

$$f(0,0) = g(0,0) = 0,$$
$$\frac{df}{dx}(0,0) = \frac{df}{dy}(0,0) = \frac{dg}{dx}(0,0) = \frac{dg}{dy}(0,0) = 0. \tag{3}$$

Below we use a smoothness of the functions f and g and follow the first Lyapunov method on finite time interval [Lefschetz, 1957; Cesari, 1959]. By assumption on smoothness (2) in the neighborhood U we have

$$\begin{aligned} f(x,y) &= \sum_{k+j=2}^{n} f_{kj}x^k y^j + o\big((|x|+|y|)^n\big) = \\ &= f_n(x,y) + o\big((|x|+|y|)^n\big), \\ g(x,y) &= \sum_{k+j=2}^{n} g_{kj}x^k y^j + o\big((|x|+|y|)^n\big) = \\ &= g_n(x,y) + o\big((|x|+|y|)^n\big). \end{aligned} \tag{4}$$

The existence condition of $(n+1)$-th partial derivatives with respect to x and y for f and g is used for the sake of simplicity of exposition and can be weakened.

Let $x(t,x(0),y(0)),\ y(t,x(0),y(0))$ be a solution of system (1) with the initial data

$$x(0)=0,\ y(0)=h. \tag{5}$$

Denote

$$x(t,h)=x(t,0,h),\ y(t,h)=y(t,0,h).$$

Further we will denote a time derivative by x' and $\dot{x}$.

Lemma 1. *A positive number $H \in (0,R_U)$ exists such that for all $h \in [0,H]$ the solution $\big(x(t,h),y(t,h)\big)$ is defined for $t \in [0,4\pi]$.*

The validity of this lemma follows from condition (3) and the existence of two purely imaginary eigenvalues of the matrix of linear approximation of system (1).

This implies [Hartman, 1964] the following

Lemma 2. *If smoothness condition (2) is satisfied, then*

$$x(\cdot,\cdot),y(\cdot,\cdot) \in \mathbb{C}^{(n+1)}\big([0,4\pi]\times[0,H]\big). \tag{6}$$

Further we will consider sufficiently small initial data $h \in [0,H]$ and a finite time interval $t \in [0,4\pi]$ and make use of a uniform boundedness of the solution $(x(t,h),y(t,h))$ and its mixed partial derivatives with respect to h and t up to the order $n+1$ inc in the considered set $[0,4\pi]\times[0,H]$.

Now we apply a well-known linearization procedure [Leonov & Kuznetsov, 2007; Leonov, 2008[2]].

From Lemma 2 it follows that for each fixed t the solution of system can be represented by the Taylor formula

$$\begin{aligned} x(t,h) &= h\frac{\partial x(t,\eta)}{\partial\eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_x(t,h)}, \\ &\quad 0 \le \theta_x(t,h) \le 1, \\ y(t,h) &= h\frac{\partial y(t,\eta)}{\partial\eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_y(t,h)}, \\ &\quad 0 \le \theta_y(t,h) \le 1. \end{aligned} \tag{7}$$

Note that by Lemma 2 and relation (7) the functions

$$\frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_x(t,h)}, \quad \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_y(t,h)}$$

and their time derivatives are smooth functions of t and have the order of smallness $o(h)$ uniformly with respect to t on a considered finite time interval $[0,4\pi]$.

Introduce the following denotations $\widetilde{x}_{h^k}(t) = \dfrac{\partial^k x(t,\eta)}{\partial^k\eta}\Big|_{\eta=0}$, $\widetilde{y}_{h^k}(t) = \dfrac{\partial^k y(t,\eta)}{\partial^k\eta}\Big|_{\eta=0}$.

We shall say that the sums

$$\begin{aligned} x_{h^m}(t,h) &= \sum_{k=1}^{m}\widetilde{x}_{h^k}(t)\frac{h^k}{k!} = \sum_{k=1}^{m}\frac{\partial^k x(t,\eta)}{\partial\eta^k}\Big|_{\eta=0}\frac{h^k}{k!}, \\ y_{h^m}(t,h) &= \sum_{k=1}^{m}\widetilde{y}_{h^k}(t)\frac{h^k}{k!} = \sum_{k=1}^{m}\frac{\partial^k y(t,\eta)}{\partial\eta^k}\Big|_{\eta=0}\frac{h^k}{k!} \end{aligned}$$

are the m-th approximation of solution of system with respect to h. Substitute representation (7) in system (1). Then, equating the coefficients of h^1 and taking into account (3), we obtain

$$\frac{d\widetilde{x}_{h^1}(t)}{dt} = -\widetilde{y}_{h^1}(t),\ \frac{d\widetilde{y}_{h^1}(t)}{dt} = \widetilde{x}_{h^1}(t). \tag{8}$$

Hence, by conditions on initial data (5) for the first approximation with respect to h of the solution $(x(t,h),y(t,h))$ we have

$$\begin{aligned} x_{h^1}(t,h) &= \widetilde{x}_{h^1}(t)h = -h\sin(t), \\ y_{h^1}(t,h) &= \widetilde{y}_{h^1}(t)h = h\cos(t). \end{aligned} \tag{9}$$

Similarly, to obtain the second approximation $(x_{h^2}(t,h),y_{h^2}(t,h))$, we substitute representation

$$\begin{aligned} x(t,h) &= x_{h^2}(t,h) + \frac{h^3}{3!}\frac{\partial^3 x(t,\eta)}{\partial\eta^3}\Big|_{\eta=h\theta_x(t,h)}, \\ y(t,h) &= y_{h^2}(t,h) + \frac{h^3}{3!}\frac{\partial^3 y(t,\eta)}{\partial\eta^3}\Big|_{\eta=h\theta_y(t,h)}. \end{aligned} \tag{10}$$

in formula (4) for $f(x,y)$ and $g(x,y)$. Note that in the expressions for f and g (denote their by $u^f_{h^2}$ and $u^g_{h^2}$, respectively) in virtue of (3) the coefficients of h^2 depend only on $\widetilde{x}_{h^1}(t)$ and $\widetilde{y}_{h^1}(t)$, i.e., by (9) they are known functions of time and are independent of the unknown functions $\widetilde{x}_{h^2}(t)$ and $\widetilde{y}_{h^2}(t)$. Thus, we have

$$\begin{aligned} & f\big(x_{h^2}(t,h)+o(h^2), y_{h^2}(t,h)+o(h^2)\big) = \\ &\quad = u^f_{h^2}(t)h^2 + o(h^2), \\ & g\big(x_{h^2}(t,h)+o(h^2), y_{h^2}(t,h)+o(h^2)\big) = \\ &\quad = u^g_{h^2}(t)h^2 + o(h^2). \end{aligned}$$

Substitute (10) in system (1). Then for the determination of $\widetilde{x}_{h^2}(t)$ and $\widetilde{y}_{h^2}(t)$ we obtain

$$\begin{aligned} \frac{d\widetilde{x}_{h^2}(t)}{dt} &= -\widetilde{y}_{h^2}(t) + u^f_{h^2}(t), \\ \frac{d\widetilde{y}_{h^2}(t)}{dt} &= \widetilde{x}_{h^2}(t) + u^g_{h^2}(t). \end{aligned} \tag{11}$$

Lemma 3. *For the solutions of system*

$$\begin{aligned} \frac{d\widetilde{x}_{h^k}(t)}{dt} &= -\widetilde{y}_{h^k}(t) + u^f_{h^k}(t), \\ \frac{d\widetilde{y}_{h^k}(t)}{dt} &= \widetilde{x}_{h^k}(t) + u^g_{h^k}(t) \end{aligned} \tag{12}$$

with the initial data

$$\widetilde{x}_{h^k}(0) = 0, \quad \widetilde{y}_{h^k}(0) = 0 \tag{13}$$

we have

$$\begin{aligned} &\widetilde{x}_{h^k}(t) = u^g_{h^k}(0)\cos(t) + \\ &+\cos(t)\int_0^t \cos(\tau)\big((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau)\big)\,\mathrm{d}\tau + \\ &+\sin(t)\int_0^t \sin(\tau)\big((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau)\big)\,\mathrm{d}\tau - u^g_{h^k}(t), \\ &\widetilde{y}_{h^k}(t) = u^g_{h^k}(0)\sin(t) + \\ &+\sin(t)\int_0^t \cos(\tau)\big((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau)\big)\,\mathrm{d}\tau - \\ &-\cos(t)\int_0^t \sin(\tau)\big((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau)\big)\,\mathrm{d}\tau. \end{aligned} \tag{14}$$

The relations (14) are verified by direct differentiation.

Repeating this procedure for determination of the coefficients $\widetilde{x}_{h^k}$ and $\widetilde{y}_{h^k}$ of the functions $u^f_{h^k}(t)$ and $u^g_{h^k}(t)$, by formula (14) we obtain sequentially the approximations $(x_{h^k}(t,h), y_{h^k}(t,h))$ for $k = 1,...,n$. For $h \in [0,H]$ and $t \in [0, 4\pi]$ we have

$$\begin{aligned} x(t,h) &= x_{h^n}(t,h) + \frac{h^{n+1}}{(n+1)!}\frac{\partial^{n+1}x(t,\eta)}{\partial\eta^{n+1}}\Big|_{\eta=h\theta_x(t,h)} = \\ &= x_{h^n}(t,h) + o(h^n) = \sum_{k=1}^{n}\widetilde{x}_{h^k}(t)\frac{h^k}{k!} + o(h^n), \\ y(t,h) &= y_{h^n}(t,h) + \frac{h^{n+1}}{(n+1)!}\frac{\partial^{n+1}y(t,\eta)}{\partial\eta^{n+1}}\Big|_{\eta=h\theta_y(t,h)} = \\ &= y_{h^n}(t,h) + o(h^n) = \sum_{k=1}^{n}\widetilde{y}_{h^k}(t)\frac{h^k}{k!} + o(h^n), \\ &0 \le \theta_x(t,h) \le 1,\ 0 \le \theta_y(t,h) \le 1. \end{aligned} \tag{15}$$

Here by Lemma 2 we have

$$\widetilde{x}_{h^k}(\cdot), \widetilde{y}_{h^k}(\cdot) \in \mathbb{C}^n([0,4\pi]), \quad k = 1,...,n \tag{16}$$

and the estimate $o(h^n)$ is uniform $\forall t \in [0, 4\pi]$. From (13) and by the choice of initial data in (8) we obtain

$$x_{h^k}(0,h) = x(0,h) = 0, \quad y_{h^k}(0,h) = y(0,h) = h, \\ k = 1,...,n.$$

3 The method for computation of Lyapunov quantities in time domain

For the initial datum $h \in (0, H]$, consider the time $T(h)$ of the first crossing of the solution $\big(x(t,h), y(t,h)\big)$ with the half-line $\{x = 0,\ y > 0\}$. Complete a definition (by continuity) of the function $T(h)$ in zero: $T(0) = 2\pi$. Since by (9) the first approximation solution crosses the half-line $\{x = 0, y > 0\}$ in a time 2π, then the crossing time can be represented as

$$T(h) = 2\pi + \Delta T(h),$$

where $\Delta T(h) = O(h)$. We shall say that $\Delta T(h)$ is a residual of crossing time.

By definition of $T(h)$ we have

$$x(T(h), h) = 0. \tag{17}$$

Since by (6), $x(\cdot,\cdot)$ has continuous partial derivatives with respect to either arguments up to the order n inc and $\dot{x}(t,h) = \cos(t)h + o(h))$, by the theorem on implicit function [Zorich, 2002] the function $T(\cdot)$ is n times differentiable. It is possible to show (for example, considering the function $z(t,h) = x(t,h)/h$ and completing its definition in zero by the function $x_{h^1}(t)$ or making use of special theorems of mathematical analysis) that $T(h)$ is also differentiable n times in zero. By the Taylor formula we have

$$T(h) = 2\pi + \sum_{k=1}^{n}\widetilde{T}_k h^k + o(h^n), \tag{18}$$

where $\widetilde{T}_k = \dfrac{1}{k!}\dfrac{d^kT(h)}{dh^k}$. We shall say that the sum

$$\Delta T_k(h) = \sum_{j=1}^{k}\widetilde{T}_j h^j \tag{19}$$

is the k-th approximation of the residual of time $T(h)$ of the crossing of the solution $\big(x(t,h), y(t,h)\big)$ with the half-line $\{x = 0,\ y > 0\}$. Substituting relation (18) for $t = T(h)$ in the right-hand side of the first equation (15) and denoting the coefficient of h^k by $\widetilde{x}_k$, we obtain the series $x(T(h), h)$ in terms of powers of h:

$$x(T(h), h) = \sum_{k=1}^{n}\widetilde{x}_k h^k + o(h^n). \tag{20}$$

In order to express the coefficients $\widetilde{x}_k$ by the coefficients $\widetilde{T}_k$ of the expansion of residual of crossing time we assume that in (15) $t = 2\pi + \tau$:

$$x(2\pi + \tau, h) = \sum_{k=1}^{n}\widetilde{x}_{h^k}(2\pi + \tau)\frac{h^k}{k!} + o(h^n). \tag{21}$$

By smoothness condition (16) we have

$$\widetilde{x}_{h^k}(2\pi+\tau)=\widetilde{x}_{h^k}(2\pi)+\sum_{m=1}^{n}\widetilde{x}_{h^k}^{(m)}(2\pi)\frac{\tau^m}{m!}+o(\tau^n),\quad k=1,...,n.$$

Substitute this representation in (21) for the solution $x(2\pi+\tau,h)$ for $\tau = \Delta T(h)$, and bring together the coefficients of the same exponents h. Since $(\Delta T(h))^n = O(h^n)$, taking into account (18) for $T(h)$, by (17) we obtain

$$\begin{aligned}
&h: 0=\widetilde{x}_1=\widetilde{x}_{h^1}(2\pi),\\
&h^2: 0=\widetilde{x}_2=\widetilde{x}_{h^2}(2\pi)+\widetilde{x}'_{h^1}(2\pi)\widetilde{T}_1,\\
&\dots\\
&h^n: 0=\widetilde{x}_n=\dots
\end{aligned}$$

From the above we sequentially find $\widetilde{T}_j$. The coefficients $T_{k=1,...,n-1}$ can be determined sequentially since the expression for $\widetilde{x}_k$ can involve only the coefficients $T_{m<k}$ and the factor $\widetilde{x}'_{h^1}(2\pi)$ multiplying T_{k-1} is equal to -1.

We apply a similar procedure to determine the coefficients $\widetilde{y}_k$ of the expansion

$$y(T(h),h)=\sum_{k=1}^{n}\widetilde{y}_k h^k+o(h^n).$$

Substitute the representation

$$\begin{aligned}
&\widetilde{y}_{h^k}(2\pi+\Delta T(h))=\\
&\quad=\widetilde{y}_{h^k}(2\pi)+\sum_{m=1}^{n}\widetilde{y}_{h^k}^{(m)}(2\pi)\frac{\Delta T(h)^m}{m!}+o(h^n),\\
&\quad k=1,...,n
\end{aligned}$$

in the expression

$$y(2\pi+\Delta T(h),h)=\sum_{k=1}^{n}\widetilde{y}_{h^k}(2\pi+\tau)\frac{h^k}{k!}+o(h^n).$$

Equating the coefficients of the same exponents h, we obtain the following relations

$$\begin{aligned}
&h: \ \widetilde{y}_1=\widetilde{y}_{h^1}(2\pi),\\
&h^2: \widetilde{y}_2=\widetilde{y}_{h^2}(2\pi)+\widetilde{y}'_{h^1}(2\pi)\widetilde{T}_1,\\
&\dots\\
&h^n: \widetilde{y}_n=\dots
\end{aligned}$$

for the sequential determination of $\widetilde{y}_{i=1,...,n}$, where $\widetilde{y}_{h^{k=1,..,n}}(\cdot)$ and $\widetilde{T}_{k=1,..,n-1}$ are the above-obtained quantities.

Thus, for $n = 2m+1$ under the condition $f(\cdot,\cdot),g(\cdot,\cdot)\in\mathbb{C}^{(2m+2)}(U)$ we sequentially obtained the approximations of the solution $\big(x(t,h),y(t,h)\big)$ at time $t = T(h)$, where $T(h)$ is the time of the first crossing with the half-line $\{x=0,y>0\}$, accurate to $o(h^{2m+1})$ and the approximation of the time $T(h)$ itself accurate to $o(h^{2m})$. If in this case $\widetilde{y}_k = 0$ for $k=2,..,2m$, then $\widetilde{y}_{2m+1}$ is called the m-th Lyapunov quantity L_m. Note, that, according to the Lyapunov theorem, the first nonzero coefficient of the expansion $\widetilde{y}_i$ has always an odd number and for sufficiently small initial data h the sign of $\widetilde{y}_i$ (of the Lyapunov quantity) designates a qualitative behavior (winding, unwinding) of the trajectory $\big(x(t,h),y(t,h)\big)$ on plane [Lyapunov, 1892].

3.1 Computation of the first, second, and third Lyapunov quantities in general form

Consider a complete system in the case of expansion of the right-hand side up to the seventh order

$$\begin{aligned}
&\dot{x}=-y++f_{20}x^2+f_{11}xy+f_{02}y^2+\\
&+f_{30}x^3+f_{21}x^2y+f_{12}xy^2+f_{03}y^3+\\
&+f_{40}x^4+f_{31}x^3y+f_{22}x^2y^2+f_{13}xy^3+f_{04}y^4+\\
&+f_{50}x^5+f_{41}x^4y+f_{32}x^3y^2+f_{23}x^2y^3+f_{14}xy^4+f_{05}y^5+\\
&+f_{60}x^6+f_{51}x^5y+f_{42}x^4y^2+f_{33}x^3y^3+f_{24}x^2y^4\\
&+f_{15}xy^5+f_{06}y^6+\\
&+f_{70}x^7+f_{61}x^6y+f_{52}x^5y^2+f_{43}x^4y^3+f_{34}x^3y^4+\\
&+f_{25}x^2y^5+f_{16}xy^6+f_{07}y^7,\\
&\dot{y}=x+g_{20}x^2+g_{11}xy+g_{02}y^2+\\
&+g_{30}x^3+g_{21}x^2y+g_{12}xy^2+g_{03}y^3+\\
&+g_{40}x^4+g_{31}x^3y+g_{22}x^2y^2+g_{13}xy^3+g_{04}y^4+\\
&+g_{50}x^5+g_{41}x^4y+g_{32}x^3y^2+g_{23}x^2y^3+g_{14}xy^4+g_{05}y^5\\
&+g_{60}x^6+g_{51}x^5y+g_{42}x^4y^2+g_{33}x^3y^3+g_{24}x^2y^4+\\
&+g_{15}xy^5+g_{06}y^6+\\
&+g_{70}x^7+g_{61}x^6y+g_{52}x^5y^2+g_{43}x^4y^3+g_{34}x^3y^4+\\
&+g_{25}x^2y^5+g_{16}xy^6+g_{07}y^7.
\end{aligned} \tag{22}$$

For the first Lyapunov quantity we have [Bautin, 1952]

$$\begin{aligned}
&\mathrm{L}_1=\frac{\pi}{4}(g_{21}+f_{12}+3f_{30}+3g_{03}+f_{20}f_{11}+\\
&+f_{02}f_{11}-g_{11}g_{20}+2g_{02}f_{02}-2f_{20}g_{20}-g_{02}g_{11}).
\end{aligned}$$

Note that since $\widetilde{T}_1 = 0$, the residual of crossing time does not influence L_1.

To compute the second Lyapunov quantity, we obtain the coefficients $\widetilde{T}_2$ and $\widetilde{T}_3$ of the expansion of residual of crossing time.

$\widetilde{T}_2 = \frac{\pi}{12}(-9g_{30}+4f_{20}^2+9f_{03}-3g_{12}+10g_{20}^2+10f_{02}^2+4g_{02}^2+g_{11}^2+f_{11}^2+3f_{21}--5f_{20}g_{11}-f_{11}g_{20}-5f_{11}g_{02}+10g_{02}g_{20}-f_{02}g_{11}+10f_{20}f_{02})$,

$\widetilde{T}_3 = -\frac{\pi}{18}(2f_{20}+f_{02}+g_{11})(-9g_{30}+4f_{20}^2+9f_{03}-3g_{12}+10g_{20}^2+10f_{02}^2+4g_{02}^2+g_{11}^2+f_{11}^2+3f_{21}-5f_{20}g_{11}-f_{11}g_{20}-5f_{11}g_{02}+10g_{02}g_{20}-f_{02}g_{11}+10f_{20}f_{02})$.

We obtain the coefficient g_{03} from the condition $\mathrm{L}_1 = 0$:

$$g_{03} = -\frac{1}{3}(g_{21}+f_{12}+3f_{30}+f_{20}f_{11}+f_{02}f_{11}-g_{11}g_{20}+ 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11}),$$

and the expression for the second Lyapunov quantity

$$\mathrm{L}_2 = -\frac{\pi}{72}(-66f_{20}g_{04}-3f_{11}g_{30}f_{20}-24g_{20}g_{02}g_{21}+ 12f_{30}g_{11}f_{02} + 4f_{11}f_{20}^2g_{11} - 12f_{11}f_{21}f_{20} + 2g_{20}g_{11}^3-9g_{11}g_{02}g_{12}-12f_{20}f_{11}f_{03}-12g_{11}g_{02}f_{03}+ 3g_{20}f_{12}f_{11} + 9g_{21}g_{30} - 6f_{02}f_{11}g_{12} + 9g_{20}g_{11}g_{02}^2 + 30f_{20}g_{02}g_{12}+30g_{02}f_{21}f_{20}-60g_{04}f_{02}+g_{11}^2f_{11}f_{20}- 5f_{11}f_{20}^3 - 21f_{20}f_{13} - 3f_{11}^3f_{20} - 9g_{02}g_{21}f_{11} + 7g_{11}g_{21}f_{02}-5f_{11}g_{11}f_{02}^2+5f_{02}^2f_{11}f_{20}-3g_{11}g_{20}f_{21}+ 6g_{02}f_{20}f_{11}^2 + 9g_{21}f_{03} - 3f_{30}f_{11}^2 + 15f_{11}f_{40} - 21g_{11}g_{30}g_{02}-6g_{11}f_{03}f_{11}+f_{11}f_{02}g_{11}^2-18g_{20}f_{03}f_{20}- 42g_{20}g_{02}f_{30} - 6g_{11}g_{12}g_{20} - 30f_{02}^2g_{20}f_{20} + 3f_{11}^2g_{02}f_{02} + 60f_{40}g_{20} + 9g_{11}g_{40} + 24f_{20}g_{20}f_{21} - 9g_{11}g_{20}f_{03} - 10g_{11}f_{20}^2g_{02} + 18g_{02}g_{12}f_{02} - 6g_{11}f_{11}g_{30} - 24f_{20}f_{03}g_{02} - 30f_{03}f_{02}g_{02} - 24g_{11}g_{20}g_{30} - 12f_{11}f_{30}g_{02} - 3g_{12}f_{11}f_{20} + f_{12}g_{11}^2 - 9f_{21}f_{30} + 27f_{30}g_{30} + 3f_{30}g_{11}^2 + 15f_{30}g_{02}^2 - 9f_{02}f_{31} - 28g_{20}f_{02}f_{20}^2 - 2g_{11}^2g_{21} - 3f_{22}f_{11} - 14f_{12}f_{20}^2 - 6f_{12}g_{12} + 27g_{13}g_{02} - 3f_{02}f_{11}^3 + 7f_{20}g_{21}g_{11} + 3g_{20}^2f_{11}f_{20} - 10g_{02}g_{11}^2f_{20} - 10f_{02}f_{12}f_{20} - 12g_{20}f_{30}f_{11} + 6f_{02}f_{21}g_{20} + 18f_{02}f_{11}g_{02}^2+3f_{12}g_{02}f_{11}+6g_{20}g_{02}f_{12}+18g_{02}^2f_{12}+ 9g_{13}g_{20} - 3f_{12}f_{11}^2 - 45g_{20}^2f_{30} - 15f_{13}f_{02} + 30f_{20}g_{20}^3 - 18g_{02}f_{04} + 18f_{20}g_{40} - 21g_{20}f_{20}^2g_{11} + 2g_{02}g_{11}^3 + 3f_{02}g_{20}f_{11}^2 + 20f_{02}f_{20}^2g_{02} - 9g_{02}^2g_{21} - 9g_{21}g_{20}f_{11} - 9f_{04}f_{11} + 6f_{22}g_{20} + 45f_{30}f_{02}^2 + 15g_{11}g_{20}^3 - 15g_{11}g_{04} + 12f_{02}g_{02}f_{21} - 5f_{12}g_{11}f_{20} + 18g_{12}g_{20}f_{20}-5f_{12}g_{11}f_{02}+20f_{02}g_{21}f_{20}+21g_{02}g_{31}- 30g_{20}g_{02}^2f_{20} + 6g_{20}g_{12}f_{02} + 12f_{22}g_{02} + 3f_{21}g_{21} + 18f_{20}^3g_{02} + 24g_{11}g_{20}^2g_{02} + 18f_{20}g_{02}g_{20}^2 + 6f_{11}g_{31} - 6g_{22}f_{02}+15g_{31}g_{20}+3g_{22}g_{11}-12g_{22}f_{20}-9f_{30}g_{12}- 18f_{20}g_{02}^3-24f_{20}g_{02}g_{30}+15f_{20}f_{11}g_{02}^2-7g_{20}f_{02}g_{11}^2+ 6g_{20}f_{02}f_{11}g_{02} - 6g_{11}f_{13} - 28f_{02}g_{11}g_{20}f_{20} - 12g_{11}f_{20}g_{02}f_{02} + 9g_{02}g_{11}f_{11}g_{20} - 9f_{02}f_{21}f_{11} - f_{11}g_{11}f_{02}f_{20}-15f_{30}f_{20}^2+10f_{02}^2f_{20}g_{02}-8g_{11}^2f_{02}g_{02}+ 42f_{20}f_{30}f_{02} - 15g_{11}g_{20}f_{02}^2 + 6f_{20}g_{20}f_{11}g_{02} - 6f_{21}f_{12}+6g_{20}f_{11}^2f_{20}+66f_{40}g_{02}+27f_{30}f_{03}-45g_{05}- 9g_{23} + 15g_{21}f_{02}^2 - 27f_{20}f_{31} - 9g_{41} + 3g_{21}g_{12} + 9g_{11}g_{20}^2f_{11} - 15f_{11}f_{02}f_{03} - 45f_{50} + 12f_{30}g_{11}f_{20} + 10g_{20}f_{20}^3-48f_{20}g_{20}g_{30}-10g_{02}f_{02}^2g_{11}-9g_{11}^2g_{20}f_{20}+ 13g_{21}f_{20}^2 - 9f_{14} - 9f_{32} - 15g_{21}g_{20}^2).$$

For the first time this result was apparently obtained by Serebryakova in 1959 [Serebryakova, 1959].

To compute the third Lyapunov quantity, we obtain first $\widetilde{T}_4$:

$$\widetilde{T}_4 = \frac{\pi}{1152}(784f_{20}^4 + 1540g_{20}^4 + 49g_{11}^4 - 352g_{21}g_{20}g_{11} + 48g_{21}^2 - 336f_{40}g_{11} + 2616f_{20}^2f_{03} + 480g_{22}g_{02} + 200g_{20}^2f_{02}^2 + 700f_{11}g_{20}^3 + 270f_{03}g_{30} - 154f_{20}g_{11}^3 + 1728g_{40}g_{02} + 424g_{11}^2g_{02}^2 - 48f_{22}g_{11} + 54f_{03}f_{21} + 453f_{02}^2g_{11}^2 - 2184f_{20}^2g_{30} + 400g_{02}^4 + 864f_{30}^2 + 1540f_{02}^4 + 556g_{20}^2f_{02}g_{11} + 945g_{30}^2 + 240f_{12}^2+768f_{20}f_{40}+2352f_{20}f_{02}g_{02}^2-320g_{11}g_{21}g_{02}- 180g_{20}^2f_{20}g_{11} - 1134f_{11}g_{30}g_{20} + 5172f_{20}f_{03}f_{02} + 513f_{03}^2 + 153g_{12}^2 + f_{11}^4 - 762f_{11}f_{03}g_{02} - 1800f_{20}^2g_{20}f_{11} - 708f_{20}g_{20}g_{11}g_{02} - 888g_{12}g_{02}^2 - 84f_{21}g_{20}^2 + 1040f_{02}^2g_{02}^2 + 432g_{40}f_{11} + 4692f_{20}^2f_{02}^2 + 648f_{21}g_{02}^2 + 1180f_{02}^3g_{11} - 96g_{21}f_{02}g_{02} - 198f_{11}g_{30}g_{02} - 480f_{12}g_{20}f_{02} - 1500g_{12}g_{20}g_{02} + 228f_{20}^2f_{02}g_{11} + 912f_{02}f_{12}f_{11} - 1944g_{30}g_{02}^2 + 672f_{02}f_{40} - 48g_{22}f_{11} + 402f_{03}g_{11}^2 + 2772g_{20}^2g_{02}^2 - 63f_{20}^2g_{11}^2 + 150f_{21}g_{11}^2 + 444f_{21}f_{02}^2 + 3300f_{03}f_{02}^2 + 384f_{22}f_{02}+432g_{11}f_{04}-712f_{11}g_{02}^3+880f_{20}f_{12}f_{11}- 18f_{11}f_{03}g_{20} + 1992f_{20}f_{02}^2g_{11} + 162f_{20}f_{21}g_{11} - 2080f_{20}f_{12}g_{02} - 150f_{11}f_{02}g_{11}g_{02} - 828f_{11}g_{20}g_{02}^2 - 96f_{11}f_{30}g_{11} + 1392f_{11}f_{30}f_{02} + 980g_{11}^2g_{20}g_{02} + 64f_{11}f_{20}g_{21} - 64f_{12}f_{11}g_{11} - 6f_{11}f_{21}g_{20} - 1812g_{30}f_{20}f_{02} + 112g_{11}f_{12}g_{02} - 2744f_{11}f_{20}^2g_{02} - 1280f_{20}g_{20}f_{12} + 680f_{02}^2g_{20}g_{02} + 80f_{12}g_{20}g_{11} - 4164f_{20}f_{02}f_{11}g_{02}-1212f_{20}f_{02}g_{12}-408f_{20}g_{11}g_{02}^2+ 102f_{11}g_{20}g_{12} - 3552f_{20}f_{30}g_{20} + 22f_{11}^2f_{20}g_{11} - 1128f_{20}^2g_{12} - 48g_{20}f_{13} + 696f_{11}^2f_{20}^2 + 672g_{04}g_{20} + 2336f_{20}^2g_{02}^2 - 162f_{21}g_{12} + 96g_{31}g_{11} + 50f_{11}^2g_{11}^2 + 1128f_{03}g_{02}^2 - 3780g_{30}g_{20}^2 + 21f_{11}^2g_{20}^2 + 480f_{20}f_{22} - 432g_{02}f_{31} + 2016g_{40}g_{20} + 3080g_{20}^3g_{02} + 66f_{11}^2f_{03} - 48f_{02}g_{31} + 768g_{04}g_{02} + 1728f_{04}f_{20} - 54f_{11}^2g_{12} + 3984g_{20}^2f_{20}^2 - 2f_{11}^3g_{20} - 198f_{03}g_{12} - 432f_{20}g_{13} + 630g_{30}g_{12} - 46f_{11}f_{20}g_{11}g_{02} + 10f_{11}g_{20}f_{20}g_{11} + 30f_{20}g_{12}g_{11} + 318f_{20}f_{02}g_{11}^2 + 168f_{11}g_{20}^2g_{02} - 4788g_{30}g_{20}g_{02} - 2400f_{02}f_{30}g_{02} + 1734f_{20}f_{03}g_{11} + 4168f_{02}f_{20}g_{20}g_{02}-62f_{02}f_{11}g_{20}g_{11}+816f_{30}g_{20}g_{11}- 2260f_{11}g_{20}f_{20}f_{02} - 82f_{11}g_{20}g_{11}^2 + 2792g_{20}^2f_{20}f_{02} + 96f_{13}f_{11} + 9f_{21}^2 - 1536f_{02}f_{12}g_{02} + 928f_{20}g_{20}g_{21} + 620f_{02}g_{20}g_{11}g_{02} - 924g_{12}g_{20}^2 + 1424g_{20}g_{02}^3 - 396f_{02}^2g_{12} - 546g_{30}g_{11}^2 + 804g_{20}^2g_{11}^2 - 180f_{02}^2g_{30} - 2f_{02}g_{11}^3 - 528f_{13}g_{02} - 18g_{30}f_{11}^2 + 6f_{21}f_{11}^2 + 864f_{30}f_{12} - 198g_{12}g_{11}^2 + 4040f_{02}^3f_{20} + 369f_{11}^2g_{02}^2 - 330f_{11}g_{11}^2g_{02} - 1728g_{20}f_{30}f_{02} + 90f_{21}f_{02}g_{11} + 216f_{02}g_{11}g_{02}^2 + 126f_{11}^2g_{20}g_{02} - 90f_{02}g_{11}g_{12} + 1392f_{20}f_{30}f_{11} + 1614f_{03}f_{02}g_{11} + 5488f_{20}^2g_{20}g_{02} + 32f_{11}g_{21}g_{11} - 294g_{30}f_{20}g_{11} + 1332f_{11}^2f_{20}f_{02} - 1876f_{11}f_{02}^2g_{02} + 1308f_{02}f_{20}f_{21} - 3840f_{20}f_{30}g_{02} + 800f_{20}g_{21}g_{02}+692f_{11}^2f_{02}^2-300f_{03}g_{20}^2-288f_{30}g_{21}+ 2768f_{20}^3f_{02} - 58f_{11}^3g_{02} - 144g_{20}f_{31} - 528g_{31}f_{20} - 616f_{20}^3g_{11} + 984f_{20}^2f_{21} + 384g_{20}g_{22} - 144f_{02}g_{13} + 90f_{21}g_{30}+2016f_{02}f_{04}-336f_{11}g_{04}-366f_{02}g_{30}g_{11}+ 96f_{02}f_{11}g_{21} - 18f_{02}f_{11}^2g_{11} + 468f_{03}g_{20}g_{02} - 96f_{02}g_{20}g_{21} + 144f_{41} - 720g_{50} + 720f_{05} + 144f_{23} - 144g_{14} - 144g_{32} + 816f_{30}g_{11}g_{02} + 510f_{11}g_{12}g_{02} + 444f_{21}g_{20}g_{02} - 500g_{20}f_{11}f_{02}^2 - 222f_{11}f_{21}g_{02}).$$

To compute L_3 in the general case, applying the above-mentioned algorithms, it is necessary to treat the symbolic expressions, involving more than two millions of symbols. Therefore, to overcome the restrictions, arising while using a main memory in the packets of symbolic computations, we consider the following case

$$f_{20} = f_{30} = f_{40} = f_{50} = f_{60} = f_{70} = 0.$$

General system (22) is reduced to this form by the change

$$y_{old} = y_{new}+f_2x^2+f_3x^3+f_4x^4+f_5x^5+f_6x^6+f_7x^7,$$

where

$f_2 = f_{20}$,

$f_3 = f_{30} + f_{20} f_{11}$,

$f_4 = f_{02} f_{20}^2 + f_{40} + f_{11} f_{30} + f_{20} f_{11}^2 + f_{20} f_{21}$,

$f_5 = f_{20}f_{31} + f_{50} + 3f_{11}f_{02}f_{20}^2 + f_{11}f_{40} + f_{11}^2 f_{30} + f_{20}f_{11}^3 + 2f_{11}f_{20}f_{21} + f_{21}f_{30} + 2f_{20}f_{02}f_{30} + f_{12}f_{20}^2$,

$f_6 = f_{22}f_{20}^2 + f_{20}f_{41} + f_{03}f_{20}^3 + f_{60} + 3f_{21}f_{02}f_{20}^2 + 2f_{21}f_{11}f_{30} + 3f_{21}f_{20}f_{11}^2 + 2f_{31}f_{20}f_{11} + 6f_{11}^2 f_{02}f_{20}^2 + 3f_{11}f_{12}f_{20}^2 + 2f_{20}f_{02}f_{40} + 6f_{11}f_{20}f_{02}f_{30} + 2f_{20}^3 f_{02}^2 + 2f_{20}f_{12}f_{30} + f_{21}f_{40} + f_{20}f_{21}^2 + f_{31}f_{30} + f_{11}f_{50} + f_{11}^2 f_{40} + f_{11}^3 f_{30} + f_{20}f_{11}^4 + f_{02}f_{30}^2$,

$f_7 = 3f_{21}f_{11}^2 f_{30} + 2f_{20}f_{02}f_{50} + 2f_{41}f_{20}f_{11} + 4f_{21}f_{20}f_{11}^3 + 6f_{20}^2 f_{12} f_{11}^2 + 3f_{11}f_{20}f_{21}^2 + 2f_{31}f_{20}f_{21} + 3f_{02}f_{11}f_{30}^2 + 3f_{31}f_{20}f_{11}^2 + 6f_{20}^2 f_{02}^2 f_{30} + 3f_{03}f_{20}^2 f_{30} + 2f_{02}f_{30}f_{40} + 2f_{20}f_{22}f_{30} + 2f_{11}f_{21}f_{40} + 3f_{20}^2 f_{02} f_{31} + 3f_{20}^2 f_{12}f_{21} + 4f_{03}f_{20}^3 f_{11} + 2f_{31}f_{11}f_{30} + 4f_{20}^3 f_{12}f_{02} + 3f_{20}^2 f_{22}f_{11} + 10f_{20}^3 f_{02}^2 f_{11} + 2f_{20}f_{12}f_{40} + 10f_{20}^2 f_{02}f_{11}^3 + f_{13}f_{20}^3 + f_{21}^2 f_{30} + f_{32}f_{20}^2 + f_{20}f_{51} + f_{41}f_{30} + f_{12}f_{30}^2 + f_{31}f_{40} + f_{11}f_{60} + f_{11}^2 f_{50} + f_{11}^3 f_{40} + f_{11}^4 f_{30} + f_{20}f_{11}^5 + f_{21}f_{50} + f_{70} + 6f_{20}f_{12}f_{11}f_{30} + 6f_{20}f_{02}f_{11}f_{40} + 12f_{20}f_{02}f_{11}^2 f_{30} + 12f_{20}^2 f_{02}f_{11}f_{21} + 6f_{20}f_{02}f_{21}f_{30}$.

Note that this change is nonsingular and does not change the Lyapunov quantities of system since

$$y_{new}(0) = y_{old}(0) = h, y_{new}(T) = y_{old}(T).$$

For $\widetilde{T}_5$ we have the following expression

$$\begin{aligned}
\widetilde{T}_5 = &-\frac{\pi}{4320}(23130 f_{03} f_{02}^2 g_{11} + 2340 g_{20}^2 f_{02}^2 g_{11} +\\
&438 f_{11}^2 g_{11} g_{20} g_{02} - 1110 f_{11} g_{11} f_{21} g_{02} - 720 g_{32} g_{11} +\\
&2304 f_{04} g_{11}^2 - 144 f_{22} g_{11}^2 + 1200 g_{11} f_{12}^2 -\\
&1020 g_{12} f_{02}^3 + 1296 f_{04} f_{03} + 576 g_{02}^2 g_{13} + 576 f_{04} g_{02}^2 -\\
&288 f_{22} g_{12} - 960 f_{02}^2 g_{21} g_{02} - 1648 g_{21} g_{11}^2 g_{02} -\\
&810 f_{03} g_{30} f_{02} - 1856 g_{21} g_{20} g_{11}^2 + 2880 f_{22} f_{02}^2 +\\
&384 g_{31} g_{02}^2 + 1296 f_{03} g_{13} + 960 f_{22} g_{20}^2 - 1296 g_{30} f_{04} -\\
&432 g_{12} g_{13} + 720 f_{41} g_{11} + 4725 f_{02} g_{30}^2 + 1000 f_{02}^3 g_{20}^2 +\\
&960 f_{12} g_{20}^2 g_{02} + 8640 g_{40} g_{02} f_{02} + 16840 g_{20}^3 g_{11} g_{02} -\\
&8420 f_{02}^3 f_{11} g_{02} - 1530 g_{30} f_{02}^2 g_{11} - 20724 g_{30} g_{20}^2 g_{11} +\\
&144 g_{12} f_{11} g_{21} + 2160 g_{40} g_{11} f_{11} - 2286 g_{12} f_{02}^2 g_{11} -\\
&2160 g_{02} f_{31} g_{11} + 149 g_{11}^5 + 1002 f_{03} f_{02} f_{11}^2 +\\
&144 f_{12} f_{21} f_{11} - 4056 f_{02} g_{02}^2 g_{12} - 128 f_{02} g_{11}^2 f_{11}^2 +\\
&12096 f_{04} f_{02} g_{11} - 318 f_{02} f_{11}^2 g_{12} + 4580 g_{02} f_{02}^2 g_{20} g_{11} +\\
&1014 f_{11}^2 g_{20} f_{02} g_{02} - 9026 f_{11} f_{02}^2 g_{11} g_{02} -\\
&2032 f_{02} g_{21} g_{11} g_{02} - 864 f_{02} g_{13} g_{11} - 954 g_{12} f_{21} g_{11} +\\
&1856 f_{02} g_{02}^2 g_{11}^2 - 432 f_{11} g_{30} f_{12} + 2790 f_{11} g_{11} g_{12} g_{02} +\\
&114 f_{11} f_{21} g_{20} g_{11} + 7120 g_{02}^3 f_{02} g_{20} + 3360 g_{04} f_{02} g_{20} -\\
&132 g_{20}^2 f_{21} g_{11} + 7200 f_{02} f_{03} g_{11}^2 + 3988 g_{20} g_{11}^3 g_{02} -\\
&378 f_{03} g_{30} g_{11} + 144 f_{11}^2 g_{13} + 2000 g_{02}^4 g_{11} +\\
&960 g_{31} g_{20}^2 + 1440 f_{04} g_{20}^2 + 864 g_{30} g_{21} g_{20} -\\
&1440 f_{02}^2 g_{20} g_{21} - 2400 f_{02}^2 f_{12} g_{20} - 240 f_{02} f_{13} g_{20} +\\
&1920 g_{20} g_{11} g_{22} + 3400 f_{02}^3 g_{20} g_{02} - 720 f_{04} g_{02} f_{11} -\\
&3600 g_{50} f_{02} + 792 f_{11} g_{20}^2 g_{11} g_{02} + 8640 g_{40} g_{02} g_{11} +\\
&288 g_{21} g_{20} g_{12} - 864 g_{21} g_{02}^2 g_{20} - 864 f_{03} g_{21} g_{20} +\\
&5040 f_{02}^2 f_{11} f_{12} + 432 f_{11} g_{30} g_{21} - 4188 f_{11} g_{02}^2 g_{11} g_{20} +\\
&4740 f_{03} f_{02} g_{20} g_{02} - 4290 f_{11} f_{03} g_{11} g_{02} -\\
&6540 g_{12} f_{02} g_{20} g_{02} - 810 g_{12} f_{21} f_{02} - 846 f_{02} f_{03} g_{12} -\\
&6720 f_{02}^2 f_{12} g_{02} - 2640 f_{13} g_{02} g_{11} + 960 g_{31} g_{02} g_{20} -\\
&144 g_{13} f_{11} g_{20} - 240 f_{11} g_{22} g_{11} - 290 f_{11}^3 g_{11} g_{02} +\\
&2364 f_{21} g_{11} g_{20} g_{02} + 144 g_{12} g_{02} g_{21} + 900 f_{02} f_{03} g_{20}^2 +\\
&3500 f_{11} f_{02} g_{20}^3 - 2000 g_{20} f_{12} f_{02} g_{11} + 48 g_{21} g_{02}^2 f_{11} +\\
&3344 g_{20}^2 f_{02} g_{11}^2 - 2500 f_{02}^3 f_{11} g_{20} - 58 f_{11}^3 g_{20} f_{02} +\\
&480 f_{02} f_{13} f_{11} + 288 f_{21} g_{02} f_{12} - 1440 g_{02} g_{20}^2 g_{21} -\\
&2640 f_{13} g_{02} f_{02} + 13860 g_{02}^2 f_{02} g_{20}^2 + 426 f_{03} f_{11}^2 g_{11} -\\
&186 f_{11}^2 g_{30} g_{11} + 112 f_{11} g_{21} g_{11}^2 + 400 g_{20} f_{12} g_{11}^2 +\\
&10080 g_{20} g_{40} g_{11} - 1302 f_{21} f_{02} f_{11} g_{02} - 432 f_{11} f_{03} g_{21} +\\
&1440 g_{20}^2 g_{13} - 1962 g_{30} g_{11}^3 + 432 f_{21} g_{13} -\\
&750 g_{11}^3 g_{12} + 480 f_{13} f_{11} g_{11} + 3884 f_{11} g_{20}^3 g_{11} -\\
&240 f_{02} g_{22} f_{11} - 1680 g_{04} g_{11} f_{11} - 1854 g_{30} f_{02} f_{11} g_{02} +\\
&48 g_{02} g_{20} f_{11} g_{21} - 6006 f_{11} g_{30} g_{20} g_{11} -\\
&2160 g_{02} f_{31} f_{02} - 1440 g_{30} g_{11}^2 f_{02} - 2090 f_{02}^2 g_{20} f_{11} g_{11} -\\
&3560 f_{11} g_{02}^3 g_{11} - 18900 g_{30} f_{02} g_{20}^2 - 288 f_{11} f_{12} g_{02}^2 -\\
&3660 g_{12} f_{02} g_{20}^2 + 6600 f_{03} f_{02} g_{02}^2 - 9720 f_{02} g_{02}^2 g_{30} -\\
&318 g_{12} f_{11}^2 g_{11} - 3176 g_{02}^3 f_{02} f_{11} - 48 f_{11}^2 f_{12} g_{20} +\\
&2286 f_{02} g_{12} g_{30} + 3732 f_{03} g_{20} g_{11} g_{02} + 2400 g_{22} g_{02} f_{02} -\\
&720 f_{02} f_{31} g_{20} - 4632 g_{02}^2 g_{11} g_{12} + 1557 f_{11}^2 f_{02} g_{02}^2 -\\
&720 f_{31} g_{11} g_{20} + 1440 g_{02} g_{20} g_{13} + 432 f_{12} f_{03} f_{11} +\\
&144 g_{11} g_{31} f_{02} + 7312 g_{20} g_{02}^3 g_{11} + 688 f_{02} f_{11} g_{21} g_{11} +\\
&11520 f_{04} f_{02}^2 + 432 f_{21} f_{04} + 3429 f_{03}^2 g_{11} +\\
&720 f_{41} f_{02} + 3840 g_{04} g_{02} f_{02} - 23940 g_{20} f_{02} g_{30} g_{02} +\\
&1440 f_{04} g_{02} g_{20} - 5388 g_{12} g_{20}^2 g_{11} - 144 g_{21} g_{02} f_{21} +\\
&192 g_{21} f_{11}^2 g_{02} - 144 f_{11} f_{04} g_{20} + 3360 g_{20} g_{04} g_{11} +\\
&15400 g_{20}^3 f_{02} g_{02} - 144 f_{12} f_{11}^2 g_{02} - 288 g_{12} g_{02} f_{12} -\\
&4146 f_{03} f_{02} f_{11} g_{02} - 8124 g_{20} g_{11} g_{12} g_{02} +\\
&414 f_{11} g_{20} g_{12} g_{11} - 96 g_{31} f_{11} g_{20} + 1584 f_{02} f_{22} g_{11} -\\
&522 g_{30} f_{02} f_{11}^2 + 5320 f_{02}^2 g_{02}^2 g_{11} + 656 f_{12} g_{11}^2 g_{02} -\\
&240 g_{20} f_{13} g_{11} - 1710 f_{03} g_{12} g_{11} - 434 f_{11}^3 f_{02} g_{02} -\\
&480 g_{02} f_{22} f_{11} + 270 f_{21} f_{02} f_{11}^2 + 14724 g_{02}^2 g_{20}^2 g_{11} +\\
&7700 f_{02}^5 - 720 g_{14} f_{02} + 2000 g_{02}^4 f_{02} - 960 g_{21} g_{20}^3 +\\
&53 f_{11}^4 f_{02} + 7700 g_{20}^4 f_{02} + 384 g_{02}^2 f_{22} + 720 f_{23} f_{02} +\\
&240 g_{21}^2 g_{11} + 18900 f_{02}^3 f_{03} + 3940 f_{02}^3 f_{11}^2 +\\
&384 f_{12} g_{02}^3 - 864 g_{30} g_{31} - 864 g_{30} f_{22} + 333 f_{02} f_{21}^2 +\\
&1200 f_{02} f_{12}^2 + 477 f_{02} g_{12}^2 + 1736 g_{02}^2 g_{11}^3 + 720 f_{23} g_{11} +\\
&3156 g_{20}^2 g_{11}^3 - 432 f_{04} g_{12} + 864 f_{03} f_{22} - 720 g_{14} g_{11} +\\
&5200 f_{02}^3 g_{02}^2 + 3600 f_{05} g_{11} + 288 g_{31} f_{21} + 96 g_{31} f_{11}^2 +\\
&909 g_{12}^2 g_{11} + 462 f_{21} g_{11}^3 - 1296 g_{30} g_{13} + 4725 f_{03}^2 f_{02} -\\
&432 g_{21} f_{03} g_{02} + 288 f_{21} f_{02} g_{11}^2 + 1854 f_{02}^2 f_{21} g_{11} +\\
&3870 g_{12} g_{30} g_{11} - 864 g_{02} f_{12} g_{30} + 3624 f_{21} f_{02} g_{02}^2 -\\
&3180 f_{11} g_{02}^2 f_{02} g_{20} - 48 f_{11}^2 g_{20} g_{21} + 585 f_{02} g_{20}^2 f_{11}^2 +\\
&324 f_{03} g_{20}^2 g_{11} - 144 f_{11} f_{21} g_{21} + 246 g_{20} f_{03} f_{11} g_{11} -\\
&96 f_{11} f_{22} g_{20} + 864 g_{02} f_{03} f_{12} - 7216 f_{12} f_{02} g_{11} g_{02} -\\
&624 f_{02} f_{11} g_{11}^2 g_{02} + 384 g_{02} g_{20} f_{12} f_{11} +\\
&4592 f_{02} g_{20} g_{11}^2 g_{02} - 126 f_{21} f_{02} g_{20} f_{11} +\\
&3180 f_{21} f_{02} g_{20} g_{02} + 144 f_{04} f_{11}^2 + 4192 f_{02} f_{12} f_{11} g_{11} -\\
&2144 g_{21} f_{02} g_{20} g_{11} - 1680 g_{04} f_{02} f_{11} -\\
&330 f_{02} f_{03} g_{20} f_{11} + 1800 f_{11} g_{20}^2 f_{02} g_{02} -\\
&432 f_{02} g_{20} f_{11} g_{11}^2 - 25332 g_{30} g_{20} g_{11} g_{02} -\\
&510 f_{11} g_{11} g_{30} g_{02} + 960 g_{02} g_{20} f_{22} - 720 g_{13} g_{02} f_{11} +\\
&432 g_{02} g_{30} g_{21} + 240 f_{02} g_{21}^2 + 3180 f_{21} f_{02}^3 + 576 g_{31} g_{11}^2 +\\
&5 f_{11}^4 g_{11} + 8660 g_{20}^4 g_{11} + 288 f_{22} f_{21} - 48 f_{11}^3 g_{21} -\\
&5 f_{02} g_{11}^4 + 1242 f_{03} g_{11}^3 + 154 f_{11}^2 g_{11}^3 + 144 g_{13} g_{11}^2 +\\
&45 f_{21}^2 g_{11} - 3600 g_{11} g_{50} + 3600 f_{05} f_{02} + 48 f_{11}^3 f_{12} -\\
&720 g_{32} f_{02} - 900 g_{30} f_{02}^3 + 864 f_{03} g_{31} + 11200 f_{02}^4 g_{11} -\\
&288 g_{31} g_{12} + 720 f_{02}^2 g_{13} + 96 f_{22} f_{11}^2 - 192 g_{02}^3 g_{21} +\\
&5045 f_{02}^3 g_{11}^2 + 1391 f_{02}^2 g_{11}^3 + 5589 g_{30}^2 g_{11} + 720 g_{31} f_{02}^2 +\\
&1854 f_{03} f_{21} f_{02} + 540 g_{20}^2 f_{02} f_{21} + 2160 f_{02} g_{40} f_{11} +\\
&1845 g_{02}^2 f_{11}^2 g_{11} - 480 g_{31} g_{02} f_{11} + 6024 g_{02}^2 f_{03} g_{11} +\\
&30 f_{11}^2 f_{21} g_{11} - 288 f_{02} g_{12} g_{11}^2 - 1170 f_{11} g_{11}^3 g_{02} +
\end{aligned}$$

$2400g_{02}g_{11}g_{22} + 3082f_{02}^2f_{11}^2g_{11} + 162g_{11}f_{21}g_{30} +$
$153f_{11}^2g_{20}^2g_{11} - 414f_{02}f_{21}g_{30} - 10104g_{30}g_{02}^2g_{11} -$
$384f_{11}g_{20}^2g_{21} - 288g_{21}f_{21}g_{20} + 3840g_{02}g_{04}g_{11} +$
$1782g_{12}f_{02}f_{11}g_{02} - 5670g_{20}f_{02}g_{30}f_{11} +$
$414f_{02}g_{12}g_{20}f_{11} + 38f_{11}^3g_{20}g_{11} + 1920f_{02}g_{22}g_{20} +$
$480f_{11}f_{12}g_{20}^2 + 960f_{12}g_{02}^2g_{20} - 144g_{12}f_{12}f_{11} +$
$3240g_{02}^2g_{11}f_{21} + 10080g_{20}g_{40}f_{02} - 266g_{20}f_{11}g_{11}^3 +$
$558f_{03}f_{21}g_{11} - 272f_{11}f_{12}g_{11}^2)$.

We obtain the coefficients g_{03} and g_{05} from the conditions $L_1 = L_2 = 0$

$g_{03} = \frac{1}{3}(g_{11}g_{20} - f_{11}f_{02} - 2g_{02}f_{02} + g_{11}g_{02} - f_{12} - g_{21})$,

$g_{05} = \frac{1}{45}(6g_{20}f_{02}f_{21} + 2g_{11}^3g_{20} - 9g_{02}^2g_{21} +$
$9g_{21}g_{30} + 9g_{21}f_{03} - 6f_{13}g_{11} + 15g_{20}g_{31} + 27g_{02}g_{13} +$
$9g_{13}g_{20} - 6g_{22}f_{02} + 18g_{02}^2f_{12} - 6f_{21}f_{12} + 12f_{22}g_{02} +$
$9g_{11}g_{40} + 21g_{02}g_{31} + 15g_{11}g_{20}^3 - 6g_{12}f_{12} +$
$3g_{11}g_{22} - 7g_{11}^2g_{20}f_{02} - 15f_{13}f_{02} - 9g_{21}g_{20}f_{11} +$
$6g_{20}f_{02}g_{02}f_{11} + 9g_{20}g_{11}g_{02}f_{11} + 3g_{02}f_{11}f_{12} -$
$15g_{20}^2g_{21} + 9g_{11}g_{20}^2f_{11} - 6g_{12}g_{11}g_{20} + 7g_{11}g_{21}f_{02} -$
$5g_{11}f_{02}f_{12} - 21g_{11}g_{02}g_{30} - 8g_{11}^2g_{02}f_{02} - 3g_{20}g_{11}f_{21} +$
$24g_{11}g_{20}^2g_{02} - 30g_{02}f_{02}f_{03} - 15g_{11}g_{20}f_{02}^2 -$
$24g_{11}g_{20}g_{30} + g_{11}^2f_{11}f_{02} + 18f_{11}f_{02}g_{02}^2 +$
$6g_{20}f_{02}g_{12} - 5g_{11}f_{02}^2f_{11} + 18g_{12}g_{02}f_{02} - 9f_{11}f_{02}f_{21} -$
$15f_{11}f_{02}f_{03} + 3f_{02}g_{20}f_{11}^2 + 3g_{02}f_{02}f_{11}^2 - 9g_{11}g_{20}f_{03} +$
$9g_{11}g_{02}^2g_{20} - 12g_{11}f_{03}g_{02} - 10g_{11}g_{02}f_{02}^2 +$
$6g_{02}g_{20}f_{12} + 12g_{02}f_{02}f_{21} - 9g_{21}g_{02}f_{11} - 6f_{11}f_{02}g_{12} -$
$9g_{12}g_{11}g_{02} - 6g_{11}f_{03}f_{11} - 6g_{30}g_{11}f_{11} + 3g_{20}f_{12}f_{11} -$
$24g_{21}g_{20}g_{02} - 9f_{04}f_{11} + 3g_{12}g_{21} + 6g_{20}f_{22} + 6f_{11}g_{31} -$
$9f_{32} - 9g_{41} - 9f_{14} - 9g_{23} - 2g_{11}^2g_{21} + 2g_{11}^3g_{02} -$
$9f_{31}f_{02} - 18g_{02}f_{04} - 15g_{11}g_{04} - 3f_{02}f_{11}^3 + g_{11}^2f_{12} -$
$3f_{22}f_{11} + 15g_{21}f_{02}^2 + 3g_{21}f_{21} - 3f_{12}f_{11}^2 - 60f_{02}g_{04})$.

Then for the third Lyapunov quantity we obtain

$\mathrm{L}_3 = \frac{\pi}{1728}(30g_{02}g_{11}^3g_{30} + 6g_{21}^3 + 36f_{02}f_{11}g_{12}g_{20}^2 -$
$1080g_{02}^2g_{11}g_{20}^3 - 585g_{11}f_{11}g_{20}^4 - 140g_{02}g_{11}f_{02}^4 +$
$99f_{12}g_{02}f_{11}g_{30} + 1278g_{20}^2g_{11}f_{11}g_{30} +$
$1575g_{02}^2g_{20}g_{11}g_{30} - 99f_{02}g_{20}f_{11}f_{31} -$
$198f_{21}g_{20}^2g_{02}f_{02} - 90f_{12}g_{02}g_{20}^3 + 144g_{30}f_{21}g_{20}f_{02} +$
$540g_{21}g_{20}^4 - 54f_{12}g_{12}^2 + 216g_{21}g_{30}^2 + 630f_{02}g_{11}^2g_{20}^3 +$
$189g_{20}f_{11}g_{02}g_{11}f_{02}^2 - 108f_{12}g_{02}f_{11}g_{20}^2 +$
$135g_{02}^3g_{11}g_{30} - 261f_{02}g_{02}f_{11}f_{31} + 36g_{20}g_{04}g_{21} +$
$72g_{32}g_{20}g_{11} + 126g_{21}g_{20}g_{22} + 306g_{12}f_{11}g_{20}g_{02}f_{02} +$
$18g_{02}g_{20}f_{11}f_{22} + 9g_{21}f_{21}^2 + 4g_{02}g_{11}^5 + 4g_{20}g_{11}^5 -$
$60g_{20}^3g_{11}^3 - 4g_{21}g_{11}^4 + 210g_{21}f_{02}^4 - 576f_{03}g_{30}g_{20}g_{11} +$
$27g_{21}g_{12}^2 - 378f_{12}g_{02}^2f_{03} - 6g_{11}^3g_{20}f_{21} -$
$639g_{02}^2g_{21}g_{30} - 9f_{11}^2f_{12}g_{30} + 384f_{02}f_{12}g_{02}g_{21} +$
$6g_{21}g_{11}^2f_{21} + 9f_{12}f_{11}^4 + 24g_{11}^3g_{20}g_{30} - 540g_{11}g_{20}^5 +$
$405g_{30}g_{40}g_{11} - 144f_{11}f_{03}g_{11}g_{30} + 9f_{02}f_{11}^5 +$
$2f_{12}g_{11}^4 + 63f_{02}g_{02}^2g_{20}f_{11}^2 - 63f_{02}g_{02}g_{12}f_{11}^2 -$
$90f_{11}g_{30}g_{11}g_{12} - 54g_{12}^2f_{11}f_{02} - 27g_{12}f_{32} -$
$105f_{11}f_{02}^3f_{21} - 420f_{02}^2f_{03}g_{11}g_{02} + 234g_{30}f_{02}g_{02}g_{12} +$
$261g_{20}^3g_{11}f_{21} + 270g_{02}^3g_{20}g_{21} - 54g_{40}f_{11}g_{02}f_{02} +$
$42g_{20}g_{11}^2f_{11}g_{21} + 234f_{12}g_{02}^2g_{30} - 504f_{03}g_{21}g_{20}g_{02} +$
$54f_{02}f_{11}^3f_{21} + 84f_{03}g_{20}g_{11}^2f_{02} - 180g_{51}f_{11} +$
$210g_{02}f_{11}g_{20}f_{02}^3 - 45f_{23}g_{20}g_{11} - 180g_{04}f_{11}g_{20}f_{02} -$
$81g_{13}g_{22} + 81f_{22}f_{31} - 360g_{02}^2g_{21}f_{02}^2 - 81g_{22}g_{31} -$
$84f_{02}g_{02}^2f_{11}g_{11}^2 - 204f_{02}f_{11}g_{30}g_{11}^2 - 18g_{20}f_{12}f_{11}^3 -$
$30f_{12}f_{21}g_{11}^2 + 240g_{02}f_{02}^2g_{11}f_{21} + 315g_{13}f_{12}f_{02} -$
$216f_{03}g_{41} + 63f_{02}g_{02}g_{20}f_{11}^3 + 180g_{30}g_{20}g_{11}f_{11}^2 +$
$420f_{11}g_{20}^2g_{11}f_{02}^2 + 216f_{02}g_{11}g_{20}g_{13} - 81g_{12}g_{23} -$
$237f_{02}g_{21}g_{11}f_{11}g_{20} - 105f_{02}^2f_{12}g_{11}^2 -$
$72g_{20}f_{12}f_{11}f_{21} + 18f_{02}g_{02}^2g_{20}f_{21} - 135g_{33}g_{20} +$
$495g_{11}g_{50}g_{02} + 135g_{30}g_{21}g_{12} + 168g_{12}g_{02}g_{11}f_{02}^2 +$
$36f_{13}f_{21}g_{11} + 342f_{02}g_{02}g_{20}f_{31} + 54g_{31}f_{21}f_{11} +$
$99f_{11}^2g_{30}g_{02}f_{02} + 102f_{02}g_{21}^2g_{02} - 54f_{11}^2g_{40}g_{11} -$
$90g_{20}g_{21}f_{11}f_{21} - 18f_{02}g_{02}g_{12}f_{21} + 90f_{23}f_{12} +$
$210f_{02}f_{12}^2g_{02} + 210f_{22}f_{02}^2g_{20} - 180g_{14}g_{20}f_{02} -$
$18g_{11}^2g_{20}g_{13} - 639g_{21}g_{02}f_{11}g_{30} + 369f_{02}g_{21}g_{11}g_{30} -$
$210f_{02}^3f_{13} + 45g_{14}g_{21} - 840g_{04}f_{02}^3 + 72f_{02}f_{11}f_{21}^2 +$
$315f_{03}g_{20}g_{11}f_{02}^2 - 225g_{14}g_{11}g_{02} - 234f_{13}g_{02}f_{11}f_{02} -$
$99g_{11}g_{20}^2f_{31} - 135g_{50}g_{21} + 135g_{20}^2f_{32} - 306g_{21}g_{12}g_{20}^2 -$
$9g_{20}g_{11}f_{21}^2 + 144f_{21}g_{21}f_{02}^2 - 12g_{20}f_{11}g_{02}g_{11}^3 +$
$9g_{20}f_{12}f_{11}g_{11}^2 - 81f_{32}g_{30} + 189f_{04}f_{13} -$
$84f_{12}g_{02}^2g_{11}^2 + 105f_{02}f_{03}f_{12}g_{11} + 315g_{11}g_{02}^2f_{11}g_{30} +$
$228g_{20}g_{11}^2g_{02}g_{21} - 27f_{41}g_{21} + 72g_{02}^2g_{21}g_{11}^2 -$
$207g_{20}f_{03}f_{11}g_{21} - 51f_{02}^2g_{02}f_{11}g_{21} - 70g_{11}f_{11}f_{02}^4 -$
$455g_{20}g_{11}^2f_{02}^3 + 105f_{11}^2g_{20}f_{02}^3 + 455g_{21}g_{11}f_{02}^3 +$
$288g_{22}g_{02}g_{20}f_{02} + 36f_{12}f_{21}^2 - 135g_{60}g_{11} +$
$27f_{22}g_{20}g_{11}^2 + 9f_{32}f_{11}^2 - 54g_{30}g_{21}g_{11}^2 - 15f_{11}^2f_{12}g_{11}^2 +$
$162g_{11}g_{20}f_{11}^2f_{02}^2 + 360g_{02}^2g_{20}g_{11}f_{02}^2 - 180f_{13}g_{02}^2g_{11} +$
$423g_{02}f_{03}g_{31} - 423g_{02}f_{03}g_{11}g_{30} - 345f_{02}^2g_{11}f_{11}g_{30} -$
$270g_{02}^3g_{11}g_{20}^2 - 90f_{11}^2f_{21}g_{20}f_{02} + 21f_{12}f_{02}g_{11}^3 -$
$9f_{22}g_{12}f_{11} - 132g_{02}g_{11}^3g_{20}^2 + 18f_{02}f_{11}^3g_{20}^2 -$
$420g_{20}g_{21}f_{11}f_{02}^2 - 540g_{31}g_{20}^3 + 54g_{41}f_{11}^2 + 180g_{24}f_{02} -$
$885g_{30}g_{11}^2g_{02}f_{02} + 105f_{22}f_{02}^2f_{11} + 63g_{21}g_{02}f_{13} +$
$1890f_{02}g_{06} + 105g_{20}f_{12}f_{11}f_{02}^2 + 585g_{21}f_{11}g_{20}^3 -$
$135f_{31}g_{20}f_{12} + 72g_{12}f_{11}g_{31} + 108f_{21}g_{40}g_{11} -$
$414f_{41}f_{02}g_{02} + 18f_{11}f_{03}g_{11}g_{12} - 54g_{30}f_{02}g_{22} -$
$54f_{21}g_{40}f_{02} - 48f_{03}g_{21}g_{11}^2 - 156f_{02}g_{21}g_{11}f_{11}g_{02} +$
$18g_{02}g_{11}f_{11}^2f_{02}^2 - 288g_{20}^2g_{02}^2f_{11}f_{02} + 9f_{23}g_{21} +$
$135f_{06}f_{11} - 270g_{02}^3g_{20}f_{12} - 1005g_{02}g_{30}g_{11}f_{02}^2 +$
$108g_{30}g_{11}g_{22} - 81f_{14}f_{11}g_{02} - 90f_{23}f_{02}g_{20} +$
$81f_{11}g_{20}g_{23} + 144f_{02}g_{02}f_{11}g_{20}g_{30} +$
$1080g_{21}g_{02}f_{11}g_{20}^2 + 18g_{12}f_{13}g_{11} - 504g_{02}^2g_{21}f_{02}g_{11} +$
$405g_{23}g_{02}^2 - 315f_{02}^3f_{31} - 45f_{21}g_{02}f_{11}g_{21} +$
$54g_{11}g_{12}g_{22} - 90g_{12}g_{20}^3f_{02} - 9f_{11}^2g_{22}f_{02} +$
$81f_{22}f_{13} - 6f_{02}f_{22}g_{21} + 210g_{12}f_{02}^3g_{20} +$
$522g_{41}g_{02}^2 + 315f_{03}^2f_{11}f_{02} + 450g_{50}g_{20}g_{11} -$
$288g_{12}g_{20}^2g_{02}f_{02} - 315g_{20}g_{51} - 54f_{03}g_{02}g_{11}f_{11}^2 +$
$396f_{11}g_{20}g_{41} + 972g_{02}g_{20}g_{41} + 63f_{12}g_{02}g_{20}f_{11}^2 +$
$48f_{03}g_{20}g_{11}^3 + 12g_{21}g_{02}f_{11}g_{11}^2 + 54g_{11}f_{03}f_{31} -$
$198g_{02}f_{21}^2f_{02} + 6f_{02}g_{21}g_{11}f_{11}^2 + 81f_{04}f_{31} + 9f_{14}g_{21}^2 -$
$30f_{02}g_{21}^2f_{11} + 450g_{41}g_{20}^2 + 3f_{22}g_{11}g_{21} - 210f_{02}^3g_{22} +$
$345g_{31}f_{02}^2f_{11} - 108g_{12}f_{02}g_{22} + 90g_{20}^2f_{21}g_{11}f_{11} +$
$99g_{20}g_{21}f_{31} - 9f_{12}g_{02}^2f_{11}^2 + 108g_{12}^2f_{02}g_{20} -$
$36f_{11}g_{22}g_{20}f_{02} - 969g_{30}g_{20}g_{11}^2f_{02} - 135g_{02}^2g_{11}g_{22} -$
$45f_{32}g_{11}^2 - 288g_{22}g_{02}g_{20}g_{11} - 54f_{13}g_{02}f_{11}g_{11} -$
$495g_{51}g_{02} - 210f_{02}^2g_{11}f_{11}f_{03} + 54g_{30}g_{13}f_{11} -$
$315f_{02}^2f_{32} - 27f_{31}g_{12}f_{02} + 216g_{30}g_{20}g_{13} +$
$36f_{12}g_{30}g_{11}^2 - 63f_{12}g_{02}f_{11}g_{12} - 135g_{20}^3g_{13} -$
$315f_{02}^2g_{23} + 54f_{33}g_{11} + 3105g_{02}g_{20}^2g_{30}g_{11} +$
$756f_{04}g_{04} + 177f_{12}f_{02}g_{11}f_{11}g_{20} - 243g_{02}g_{30}g_{11}f_{21} -$
$45f_{11}f_{02}g_{11}^2f_{21} + 171f_{13}f_{21}f_{02} - 90f_{13}f_{11}g_{20}f_{02} -$
$54f_{11}g_{30}f_{21}g_{11} - 54g_{33}f_{11} - 315g_{31}g_{02}^2f_{11} +$
$243g_{31}f_{21}g_{02} + 30f_{02}g_{21}^2g_{20} + 153f_{13}f_{11}^2f_{02} +$
$18g_{11}^3g_{20}g_{12} - 135f_{05}g_{21} + 66f_{21}f_{02}g_{02}g_{11}^2 +$
$234f_{02}g_{02}f_{11}g_{20}f_{21} + 351g_{30}g_{02}g_{13} + 117g_{04}f_{11}^2f_{02} -$

$504g_{04}f_{02}g_{11}^2 - 45f_{42}f_{11} + 504f_{04}f_{02}g_{20}g_{11} -$
$1140g_{21}g_{02}g_{20}f_{02}^2 + 63f_{02}f_{14}g_{11} + 18g_{13}g_{11}f_{12} -$
$54f_{11}^4g_{02}f_{02} + 144g_{02}g_{20}g_{30}f_{12} - 81g_{04}g_{31} -$
$27f_{02}f_{22}g_{11}f_{11} + 63f_{04}g_{11}f_{12} - 63f_{02}^2g_{11}g_{22} -$
$270f_{11}g_{11}g_{30}^2 + 270f_{11}g_{30}g_{31} - 246g_{20}f_{02}g_{02}f_{12}g_{11} +$
$108f_{41}f_{12} - 45g_{12}g_{20}g_{11}f_{21} + 171f_{03}f_{11}f_{02}f_{21} +$
$81f_{03}g_{21}f_{21} + 180g_{20}^2g_{11}f_{11}g_{12} + 405g_{02}^2g_{20}g_{11}g_{12} +$
$180g_{11}g_{50}f_{11} + 54f_{11}f_{03}g_{13} - 252f_{02}g_{11}f_{32} -$
$108f_{11}^2g_{20}g_{31} - 90g_{20}^3f_{21}f_{02} + 225g_{20}f_{11}g_{02}g_{11}g_{12} -$
$12f_{12}f_{02}g_{11}f_{11}^2 + 630f_{03}^2g_{02}f_{02} - 9f_{02}f_{11}^3g_{12} +$
$18f_{02}f_{11}^2g_{20}g_{12} - 30f_{12}^2g_{21} - 30f_{13}g_{11}^3 +$
$54g_{04}f_{22} - 531g_{02}g_{30}g_{11}g_{12} - 216f_{03}g_{02}g_{11}g_{12} +$
$45g_{24}g_{11} - 660f_{02}g_{21}g_{11}g_{20}^2 + 18g_{20}f_{12}f_{11}g_{12} -$
$126g_{12}g_{20}g_{11}f_{02}^2 - 189g_{40}g_{31} - 90g_{20}^3f_{22} +$
$81f_{21}f_{04}f_{11} - 27g_{32}g_{21} + 351g_{20}^3g_{11}g_{12} +$
$420f_{02}^2f_{12}f_{11}g_{02} - 48f_{02}g_{21}g_{11}^3 + 135f_{02}f_{33} +$
$54g_{13}f_{13} - 18g_{20}g_{11}f_{11}f_{31} + 180f_{02}g_{02}g_{30}f_{21} +$
$252g_{02}g_{11}f_{03}^2 - 15f_{02}f_{11}^3g_{11}^2 - 105f_{11}f_{02}^3g_{11}^2 -$
$198f_{02}g_{02}f_{03}f_{11}^2 - 396g_{21}g_{02}g_{20}f_{21} +$
$27f_{02}g_{02}^2f_{11}f_{21} + 72f_{03}g_{20}g_{11}g_{02}f_{11} + 6g_{22}g_{11}^3 +$
$315f_{03}f_{13}f_{02} + 27f_{21}f_{32} - 54g_{31}f_{31} - 351g_{33}g_{02} -$
$351g_{30}g_{20}g_{11}f_{21} + 270g_{02}^3f_{22} + 135g_{21}g_{02}g_{20}f_{11}^2 +$
$72g_{02}f_{03}f_{22} + 36f_{11}f_{03}f_{22} - 54f_{24}g_{20} - 414f_{42}g_{02} -$
$45f_{02}f_{11}g_{30}f_{21} - 180g_{02}^2f_{03}g_{11}f_{11} - 90f_{13}g_{02}g_{20}f_{02} +$
$27f_{24}f_{11} - 675g_{15}g_{02} + 81g_{30}g_{04}g_{11} - 18f_{22}g_{12}g_{02} +$
$48g_{20}f_{02}g_{11}^4 + 54g_{32}f_{12} + 135g_{11}g_{12}g_{40} -$
$54g_{02}f_{11}^3f_{12} - 108g_{12}g_{41} + 540f_{02}g_{02}^2f_{11}g_{12} -$
$18f_{02}f_{11}g_{30}g_{12} + 42f_{03}f_{12}g_{11}^2 - 180g_{21}g_{02}f_{11}g_{12} -$
$135g_{31}g_{02}^3 - 45f_{11}^2g_{20}f_{03}f_{02} - 144g_{02}f_{03}f_{21}g_{11} +$
$27f_{04}f_{11}^3 - 270g_{20}f_{11}g_{02}^3f_{02} - 54f_{12}g_{30}g_{12} +$
$174f_{02}f_{22}g_{20}g_{11} + 228g_{11}g_{02}^2f_{11}f_{02}^2 - 72f_{22}f_{21}g_{20} -$
$24g_{21}g_{12}g_{11}^2 - 81g_{30}g_{23} + 216f_{03}g_{21}g_{30} - 270f_{14}g_{02}^2 -$
$207f_{12}g_{02}f_{11}f_{21} + 21f_{02}^2f_{11}g_{11}^3 - 48f_{02}^2g_{11}f_{11}^3 -$
$162g_{04}g_{02}g_{21} + 360f_{02}g_{40}g_{11}^2 - 70f_{12}g_{11}f_{02}^3 +$
$135f_{51}f_{02} - 81g_{40}g_{13} - 135g_{20}g_{15} + 54g_{12}f_{14} +$
$126f_{03}f_{11}f_{02}g_{12} - 6f_{02}f_{11}g_{12}g_{11}^2 - 216g_{30}g_{41} +$
$750g_{20}^3g_{11}f_{02}^2 + 792f_{02}g_{11}g_{02}g_{13} - 540g_{20}^2f_{11}g_{31} +$
$270f_{12}f_{11}g_{02}^3 + 405g_{12}g_{02}g_{13} + 117g_{32}g_{11}g_{02} -$
$195f_{02}^2g_{11}f_{11}f_{21} + 216g_{31}f_{21}g_{20} - 54g_{40}f_{12}g_{02} +$
$36f_{12}g_{02}^2f_{21} + 63f_{12}g_{02}^2f_{11}g_{20} - 270f_{02}g_{02}^2g_{20}g_{12} -$
$9f_{11}g_{22}f_{12} - 135g_{02}^2g_{21}f_{21} + 504f_{03}g_{04}g_{11} +$
$45g_{31}f_{12}f_{02} - 84f_{03}g_{21}f_{02}g_{11} - 108f_{03}g_{21}f_{11}g_{02} -$
$54g_{20}f_{04}f_{11}g_{02} - 135g_{13}f_{11}g_{20}g_{02} - 33g_{11}f_{12}^2g_{02} +$
$270f_{06}g_{02} - 102g_{20}g_{21}^2g_{11} - 216f_{21}f_{04}g_{02} +$
$54g_{42}f_{02} + 27f_{14}f_{11}^2 - 63g_{04}f_{11}g_{20}g_{11} -$
$420g_{02}f_{03}f_{02}^3 - 210f_{03}f_{11}f_{02}^3 - 900g_{14}f_{02}g_{02} +$
$54f_{21}f_{14} - 210g_{20}g_{11}f_{02}^4 + 216g_{02}^2g_{20}f_{03}g_{11} -$
$891g_{20}g_{11}g_{30}^2 + 945g_{06}g_{11} - 189g_{04}g_{13} + 54f_{04}g_{22} -$
$216f_{24}g_{02} - 135f_{03}f_{32} + 33f_{12}g_{02}f_{11}g_{11}^2 -$
$231f_{12}f_{02}g_{11}g_{12} + 315f_{02}f_{15} - 450g_{41}f_{02}^2 +$
$180f_{15}g_{11} - 84g_{04}g_{11}^3 - 27f_{21}g_{23} - 108f_{21}g_{41} +$
$54f_{11}f_{03}f_{21}g_{11} + 45f_{11}^2f_{12}f_{21} + 171f_{32}g_{02}^2 +$
$396f_{12}g_{02}g_{20}g_{12} + 24g_{20}f_{02}^2g_{11}f_{21} + 18f_{23}g_{11}f_{11} -$
$54g_{40}f_{22} - 27g_{42}g_{11} + 630f_{05}f_{02}g_{02} - 198g_{02}g_{20}^2f_{22} -$
$504g_{40}f_{11}g_{02}g_{11} - 81g_{20}g_{11}g_{12}^2 + 504f_{02}g_{02}^2g_{20}g_{11}^2 -$
$18f_{22}g_{20}f_{11}^2 - 42g_{31}g_{20}g_{11}^2 + 210f_{12}f_{22}f_{02} +$
$315f_{02}^2g_{20}g_{13} - 90g_{02}f_{03}g_{20}f_{11}f_{02} - 36g_{11}^2g_{23} -$
$135f_{03}g_{23} + 81f_{34} + 135f_{16} + 135g_{61} + 135f_{52} +$
$135g_{25} + 81g_{43} + 945g_{07} + 27g_{13}f_{21}g_{20} -$
$45g_{30}f_{02}f_{12}g_{11} + 135g_{30}g_{21}f_{21} + 180f_{12}g_{14} -$
$180g_{20}f_{42} + 1350g_{21}g_{02}g_{20}^3 + 111g_{21}g_{20}g_{11}f_{12} +$
$495g_{02}^2g_{21}f_{11}g_{20} - 954g_{40}g_{20}^2g_{11} - 360f_{03}g_{20}^2g_{21} +$
$57f_{12}f_{22}g_{11} - 1260g_{04}f_{02}^2g_{11} - 9f_{13}g_{20}^2g_{11} +$
$216g_{11}f_{03}g_{40} - 36g_{04}g_{20}^2g_{11} + 420g_{02}^2f_{11}f_{02}^3 +$
$126f_{04}f_{02}g_{11}f_{11} + 1140g_{02}g_{20}^2g_{11}f_{02}^2 +$
$267g_{20}^2f_{11}g_{11}^2f_{02} + 210f_{12}g_{02}g_{20}f_{02}^2 - 96g_{21}^2g_{11}g_{02} +$
$15f_{12}^2g_{20}g_{11} - 90g_{32}f_{02}g_{20} + 63g_{04}f_{11}g_{21} - 18g_{11}^2g_{41} -$
$27g_{20}f_{14}f_{11} - 360f_{02}g_{41}g_{11} - 45f_{12}f_{11}g_{20}^3 -$
$72f_{21}^2g_{20}f_{02} - 90f_{02}g_{02}f_{11}g_{20}^3 - 126g_{04}g_{12}f_{02} -$
$171g_{22}g_{20}^2g_{11} - 63f_{03}f_{12}f_{11}g_{02} + 420g_{02}f_{02}^3g_{12} -$
$1800g_{21}g_{02}g_{20}g_{30} + 234f_{02}g_{02}^2f_{11}g_{30} +$
$1548g_{20}f_{11}g_{02}g_{11}g_{30} + 189g_{21}g_{12}f_{02}^2 -$
$1026g_{04}g_{02}f_{11}f_{02} - 117f_{03}g_{20}g_{11}f_{21} + 135g_{20}^2g_{23} +$
$18g_{40}g_{11}^3 - 93g_{21}f_{11}^2f_{02}^2 + 105f_{11}^2f_{12}f_{02}^2 +$
$450g_{30}g_{21}f_{02}^2 + 54f_{31}f_{12}f_{11} - 198g_{22}f_{11}g_{02}f_{02} -$
$168g_{02}g_{11}^3f_{02}^2 + 9f_{11}^3f_{22} + 81f_{03}g_{21}g_{12} +$
$90f_{12}g_{20}^2f_{21} + 9f_{11}g_{30}f_{22} - 1350g_{11}g_{02}g_{20}^4 -$
$1236g_{02}g_{20}f_{02}g_{11}g_{21} + 63f_{03}f_{12}f_{11}^2 - 750g_{20}^2g_{21}f_{02}^2 -$
$1494g_{20}g_{11}g_{02}g_{40} + 90f_{12}g_{12}g_{20}^2 - 12g_{20}^2g_{11}^3f_{11} -$
$135f_{11}^2g_{20}^2g_{11}g_{02} + 63f_{41}f_{02}f_{11} + 213f_{12}g_{02}g_{11}g_{21} -$
$90g_{14}g_{20}g_{11} + 126f_{03}^2g_{11}f_{11} - 315f_{03}g_{21}f_{02}^2 -$
$162f_{03}g_{20}g_{11}g_{12} - 234g_{32}f_{02}g_{02} + 540f_{12}g_{02}^2g_{12} +$
$210g_{02}f_{02}^3f_{11}^2 - 270f_{04}g_{12}g_{02} + 135g_{20}^2f_{02}f_{31} -$
$180g_{04}g_{20}f_{12} - 18f_{21}f_{02}g_{22} - 432g_{02}g_{20}^2g_{13} +$
$1260f_{03}g_{04}f_{02} - 180f_{41}f_{02}g_{20} + 144f_{04}g_{11}^2g_{02} -$
$54f_{21}f_{04}g_{20} - 123f_{02}f_{11}g_{21}f_{12} + 6f_{11}g_{21}g_{11}f_{12} -$
$27g_{20}f_{04}f_{11}^2 - 171f_{04}g_{21}g_{11} - 15f_{12}f_{02}g_{11}f_{11}g_{02} -$
$756g_{02}g_{11}g_{30}^2 + 147g_{21}g_{11}^2f_{02}^2 - 1080f_{11}g_{02}g_{11}g_{20}^3 -$
$468g_{30}g_{20}g_{11}g_{12} + 1005g_{31}f_{02}^2g_{02} - 522g_{40}g_{02}^2g_{11} +$
$378f_{04}f_{03}g_{02} - 54g_{20}f_{14}g_{02} + 360f_{03}g_{31}g_{20} -$
$54g_{12}f_{02}g_{40} + 117g_{04}f_{11}f_{12} - 315f_{02}^2f_{13}g_{11} +$
$81g_{21}g_{13}f_{02} - 30f_{12}f_{02}g_{11}g_{20}^2 - 147g_{11}^3g_{20}f_{02}^2 +$
$270f_{02}g_{02}^3f_{11}^2 - 54f_{12}g_{30}f_{21} - 495g_{11}g_{02}^2f_{11}g_{20}^2 +$
$108f_{03}f_{02}f_{11}^3 - 9f_{02}g_{02}^2f_{11}^3 + 156g_{20}^2g_{21}g_{11}^2 -$
$261f_{11}^2f_{21}g_{02}f_{02} - 30g_{21}^2g_{11}f_{11} + 72f_{02}f_{11}^2g_{20}g_{30} -$
$360f_{23}f_{02}g_{02} - 45g_{02}g_{11}f_{11}g_{22} - 350g_{02}f_{02}^3g_{11}^2 -$
$6f_{12}g_{12}g_{11}^2 + 261g_{12}g_{20}g_{31} - 99g_{02}f_{12}f_{13} -$
$84f_{02}g_{02}f_{03}g_{11}^2 + 450f_{02}^2g_{11}g_{40} - 288g_{22}g_{02}f_{12} -$
$54g_{02}f_{11}^2f_{22} - 153f_{11}g_{20}g_{12}g_{21} - 147f_{03}f_{11}f_{02}g_{11}^2 +$
$207f_{03}g_{20}^2g_{11}f_{11} + 45g_{22}g_{21}f_{11} + 675g_{30}g_{20}g_{31} +$
$162g_{04}g_{02}g_{20}g_{11} + 144g_{12}f_{02}g_{20}g_{30} +$
$204f_{02}g_{11}f_{11}g_{31} - 450f_{13}g_{02}^2f_{02} + 750f_{02}^2g_{20}g_{31} -$
$189f_{02}f_{13}g_{11}^2 - 90g_{20}g_{22}f_{12} + 756g_{30}g_{02}g_{31} +$
$117g_{21}g_{02}f_{31} - 45f_{02}f_{31}g_{11}^2 + 135f_{23}f_{02}f_{11} +$
$18g_{12}g_{20}f_{22} - 1026g_{04}g_{02}f_{12} + 12f_{11}g_{11}^2g_{31} -$
$36g_{31}g_{11}f_{12} - 27g_{40}f_{12}f_{11} + 63f_{11}^2f_{31}f_{02} +$
$210f_{02}^3g_{20}f_{21} + 414g_{12}g_{02}g_{31} + 18g_{20}g_{02}^2f_{22} -$
$1305g_{02}g_{20}^2g_{31} + 144g_{30}g_{20}f_{22} + 144g_{22}g_{02}g_{21} +$
$432g_{02}g_{20}g_{23} + 36f_{21}f_{02}g_{20}g_{12} + 609f_{02}g_{31}g_{20}g_{11} +$
$18g_{32}g_{11}f_{11} + 171g_{02}^2f_{31}f_{02} + 630f_{02}^2g_{02}g_{13} +$
$162g_{12}g_{13}g_{20} + 9g_{20}f_{13}g_{21} + 90g_{22}g_{20}^2f_{02} -$
$90g_{02}f_{11}^2g_{31} - 9f_{22}f_{11}g_{02}^2 + 315f_{02}^2g_{13}f_{11} +$
$216f_{12}g_{02}g_{20}f_{21} - 24g_{21}^2f_{12} + 180g_{30}g_{02}f_{22} +$
$171f_{04}g_{11}^2g_{20} + 270g_{22}g_{02}^2f_{02} - 30f_{03}g_{11}^3f_{11} +$
$144f_{11}f_{03}g_{31} + 315f_{05}f_{02}f_{11} - 405g_{20}g_{02}^2g_{13} +$
$36f_{22}f_{21}f_{11} + 126f_{03}f_{13}g_{11} + 360g_{02}g_{11}f_{05} -$
$612g_{02}g_{20}g_{12}g_{21} - 378f_{02}f_{03}g_{02}g_{12} +$
$33g_{02}f_{11}^2f_{02}g_{11}^2 - 111f_{12}f_{02}g_{11}f_{21} + 126g_{12}f_{13}f_{02} +$

$$108f_{21}f_{02}f_{31} - 81f_{31}g_{30}f_{02} + 135f_{05}g_{20}g_{11} - 9g_{31}g_{21}f_{02} + 36g_{11}f_{12}^2f_{11} + 36f_{11}g_{20}f_{32} - 144f_{23}g_{11}g_{02} + 240f_{02}g_{11}g_{02}f_{22} - 288g_{02}f_{03}f_{21}f_{02} - 738f_{11}g_{20}g_{30}g_{21} - 135g_{02}g_{11}g_{12}^2 + 396g_{11}g_{02}g_{20}^2f_{21} - 270g_{02}^2g_{21}g_{12} - 9f_{11}^2f_{12}g_{12} - 828f_{03}f_{02}g_{02}^2f_{11} - 936g_{20}g_{02}^2g_{31} - 45f_{13}g_{20}f_{12} + 54g_{04}f_{21}f_{02} + 45g_{20}f_{11}g_{02}g_{11}f_{21} + 46g_{02}f_{02}g_{11}^4 - 348f_{02}g_{02}g_{12}g_{11}^2 + 156f_{02}g_{02}f_{11}g_{20}g_{11}^2 + 180f_{05}g_{11}f_{11} + 135f_{11}g_{02}g_{23} - 297f_{02}g_{11}g_{23} + 522g_{40}g_{21}g_{02} - 30g_{31}g_{11}^2g_{02} - 108g_{02}f_{02}g_{20}^2f_{11}^2 - 270f_{04}g_{02}^2f_{11} + 18g_{32}f_{02}f_{11} + 54f_{31}g_{30}g_{11} + 504g_{40}g_{20}g_{21} + 18f_{31}g_{21}f_{11} - 9f_{02}f_{11}^3g_{30} + 135g_{20}g_{02}^2g_{11}f_{21} - 213f_{12}g_{02}g_{11}^2g_{20} + 54g_{11}^2g_{02}g_{13} + 54g_{13}g_{11}g_{21} - 252f_{31}f_{02}^2g_{11} - 18f_{12}g_{12}f_{21} - 18f_{02}g_{20}f_{11}^4 + 30g_{20}f_{12}f_{02}g_{21} + 108f_{11}f_{12}f_{13} + 18f_{03}f_{11}^3g_{11} - 1200g_{20}f_{02}^2g_{30}g_{11} - 54f_{04}g_{12}g_{20} + 90g_{02}g_{11}f_{11}^2g_{30} + 72g_{20}f_{12}f_{11}g_{30} - 99f_{13}g_{02}g_{20}g_{11} - 909g_{31}f_{11}g_{20}g_{02} + 90f_{03}f_{22}g_{20} + 27f_{41}g_{20}g_{11} - 15f_{22}g_{11}^2f_{11} - 81g_{20}^2f_{11}g_{13} - 297f_{31}g_{02}f_{12} + 207f_{11}g_{40}g_{21} - 18f_{13}f_{11}g_{20}g_{11} + 63g_{04}g_{12}g_{11} + 12g_{02}g_{11}^3g_{12} - 288f_{12}g_{02}^2g_{20}^2 - 18g_{20}f_{03}f_{11}^2g_{11} - 45f_{02}f_{11}^2g_{20}^3 + 420f_{12}g_{02}^2f_{02}^2 + 18f_{13}f_{11}^2g_{11} + 99f_{02}g_{22}g_{11}^2 + 210g_{02}f_{02}^3f_{21} + 72g_{31}g_{11}g_{21} + 105f_{02}f_{12}^2f_{11} - 87f_{12}g_{20}^2g_{11}^2 - 261g_{20}^2g_{21}f_{21} + 270f_{02}g_{02}^3f_{21} + 756g_{12}g_{02}g_{11}g_{20}^2 + 180g_{14}f_{02}f_{11} - 12g_{30}g_{11}^3f_{11} + 189f_{04}f_{03}f_{11} + 123f_{02}g_{21}g_{11}f_{21} - 216g_{02}^2f_{03}g_{21} - 72f_{11}g_{22}g_{20}g_{11} + 9f_{02}g_{20}f_{11}^2g_{11}^2 + 156f_{02}g_{21}g_{11}g_{12} - 255g_{20}f_{02}g_{11}^2g_{12} - 231f_{02}^2g_{11}f_{11}g_{12} - 198f_{22}f_{21}g_{02} + 1080g_{02}^2g_{21}g_{20}^2 + 36g_{12}f_{21}g_{21} + 270g_{12}^2f_{02}g_{02} + 1665g_{20}^3g_{11}g_{30} - 93g_{20}f_{02}g_{11}^2f_{21} - 72g_{02}^2g_{20}g_{11}^3 + 24f_{03}g_{02}g_{11}^3 + 1134g_{20}^2g_{11}^2g_{02}f_{02} + 2f_{11}f_{02}g_{11}^4 + 885g_{11}g_{31}f_{02}g_{02} - 27f_{21}f_{02}f_{11}g_{12} - 27f_{11}^2g_{40}f_{02} + 135f_{03}g_{20}g_{13} + 360f_{03}g_{11}g_{20}^3 - 603g_{20}g_{11}f_{11}g_{40} + 504g_{02}g_{20}^2f_{03}g_{11} - 81f_{04}f_{11}^2g_{02} + 66g_{11}^2g_{02}f_{22} + 108g_{02}f_{03}g_{13} + 252f_{04}f_{02}g_{11}g_{02} - 135f_{11}^2g_{20}^3g_{11} + 210f_{22}f_{02}^2g_{02} + 228f_{12}g_{02}^2f_{02}g_{11} - 117g_{20}g_{11}g_{02}f_{31} + 342f_{32}g_{20}g_{02} - 9f_{22}g_{20}^2f_{11} + 504f_{11}g_{02}g_{41} + 36f_{11}g_{02}f_{32} + 81f_{21}g_{20}^2f_{11}f_{02} + 54f_{04}g_{12}f_{11} + 9f_{04}g_{11}^2f_{11} + 9f_{21}g_{11}g_{22} + 81f_{03}g_{22}g_{11} - 504f_{04}g_{21}f_{02} + 18f_{02}g_{11}g_{13}f_{11} - 72g_{30}g_{21}f_{11}^2 + 135g_{21}f_{11}^2g_{20}^2 + 18f_{11}^2f_{12}g_{20}^2 - 1125g_{30}g_{21}g_{20}^2).$$

3.2 The Lienard equation

Before the analysis of system and the computation of Lyapunov quantities a system is often reduced to more simple form. One of routine forms, to which the polynomial systems of different forms are reduced is the Lienard equation (see, for example. [Leonov, 1997; Leonov, 1998; Albarakati *et al.*, 2000; Leonov, 2006; Leonov, 2007; Leonov, 2008[1]]).

Assuming in (22)

$$f(x,y) \equiv 0,$$
$$\frac{dg(x,y)}{dy} = g_{x1}(x),\ g(x,0) = g_{x0}(x),\ \frac{dg_{x0}}{dx}(0) = 0,$$

we obtain the following system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + g_{x1}(x)y + g_{x0}(x), \end{aligned} \tag{23}$$

or the equivalent Lienard equation

$$\ddot{x} + x + \dot{x}g_{x1}(x) + g_{x0}(x) = 0.$$

Let be $g_{x1}(x) = g_{11}x + ...,\ g_{x0}(x) = g_{11}x^2 + ...$ Then

$$\mathrm{L}_1 = -\frac{\pi}{4}(g_{20}g_{11} - g_{21}).$$

If $g_{21} = g_{20}g_{11}$, then $\mathrm{L}_1 = 0$ and

$$\mathrm{L}_2 = \frac{\pi}{24}(3g_{41} - 5g_{20}g_{31} - 3g_{40}g_{11} + 5g_{20}g_{30}g_{11}).$$

If $g_{41} = \frac{5}{3}g_{20}g_{31} + g_{40}g_{11} - \frac{5}{3}g_{20}g_{30}g_{11}$, then $\mathrm{L}_2 = 0$ and
$\mathrm{L}_3 = -\frac{\pi}{576}(70g_{20}^3g_{30}g_{11} + 105g_{20}g_{51} + 105g_{30}^2g_{11}g_{20} + 63g_{40}g_{31} - 63g_{11}g_{40}g_{30} - 105g_{30}g_{31}g_{20} - 70g_{20}^3g_{31} - 45g_{61} - 105g_{50}g_{11}g_{20} + 45g_{60}g_{11})$.

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