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STOPPING TIMES AND PROBLEMS OF MOTION CORRECTION

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Abstract

This paper deals with using of optimal stopping times in problems of optimal correction of the motion. The theory of Markov optimal stopping times has been considered, for example, in [Shiryayev, 1978; Chow at al., 1971]. On the other hand, a problem of motion correction for systems with incomplete information consists in the accumulation of measured data and the subsequent choice of a new control for remaining time interval. A determinate version of the problem of motion correction can be found in [Kurzhanski, 1977]. Here we consider multistage linear control systems with Gaussian noises and additive uncertainties. Using the results of convex analysis and the theory of Kalman filtering, we obtain the optimal minimax stopping times for the completion of observation and for the transition to a new control action. A simple one-dimensional example is examined for the purpose of an illustration. An application to the alignment problem in the theory of inertial navigation is also considered. In addition, we show some simulation results.

Key words

Stopping time, correction of motion, control problem.

1 Introduction

Stopping times are widely used in applications of the theory of random processes, in financial mathematics, and in the control theory. Let us give a mathematical preliminaries. Given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with increased family of σ-algebras $\mathcal{F}_t$, $t \in 0 : N$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F}$, consider the sequence of random $\mathcal{F}_t$-measurable values f_t,, $\mathrm{E}|f_t| < \infty$, where E is the expectation. The integer random value $\tau \in \{0, 1, \dots\}$ is called *the stopping time* if $\{\tau = t\} \in \mathcal{F}_t$. The set of all stopping times with the property $t \le \tau \le N$ (P-a.s.) is denoted by $\mathcal{M}_t^N$. If $\tau \in \mathcal{M}_0^N$ we set $f_\tau = \sum_{i=0}^N f_i I_{\{\tau=i\}}$, where I_A is the indicator function. Let $x \wedge y = \min\{x, y\}$ and $\bar{f} = \operatorname{ess\,sup}_\alpha f^\alpha$, where $\{f^\alpha, \alpha \in \mathcal{A}\}$ is any family of $\mathcal{F}$-measurable functions. By definition, put $\bar{f}(\omega) = \operatorname{ess\,sup}_\alpha f^\alpha(\omega)$, if $\bar{f}(\omega) \ge f^\alpha(\omega)$ (P-a.s.), $\forall \alpha$, $\bar{f}(\omega)$ is $\mathcal{F}$-measurable, and if $h(\omega)$ is another function, satisfying the inequality, then $\bar{f}(\omega) \le h(\omega)$ (P-a.s.). Such a function $\bar{f}(\omega)$ there exists, [Shiryayev, 1978]. A notion of $\operatorname{ess\,inf}_\alpha f^\alpha(\omega)$ is defined similarly. Let us define recursively the values $b_N^N = f_N$, $b_t^N = f_t \wedge \mathrm{E}(b_{t+1}^N|\mathcal{F}_t)$ for $t \in N-1 : 0$, and the stopping time $\tau_t^N = \min\{t \le i \le N : f_i = b_i^N\}$. In this paper, the following result is used.

Theorem ([Chow at al., 1971; Shiryayev, 1978]). *Let* $V_t^N = \inf\{\mathrm{E}f_\tau : \tau \in \mathcal{M}_t^N\}$ *and* $\mathrm{E}|f_t| < \infty$. *Then the following properties are hold:*

(a) $\tau_t^N \in \mathcal{M}_t^N$;
(b) $\mathrm{E}(f_{\tau_t^N}|\mathcal{F}_t) = b_t^N$;
(c) $\mathrm{E}(f_\tau|\mathcal{F}_t) \ge \mathrm{E}(f_{\tau_t^N}|\mathcal{F}_t) = b_t^N$, $\forall \tau \in \mathcal{M}_t^N$;
(d) $b_t^N = \operatorname{ess\,inf}\{\mathrm{E}(f_\tau|\mathcal{F}_t) : \tau \in \mathcal{M}_t^N\}$, *in particular*, $b_0^N = \inf\{\mathrm{E}f_\tau : \tau \in \mathcal{M}_0^N\} = \mathrm{E}f_{\tau_0^N} = V_0^N$;
(e) $V_t^N = \mathrm{E}b_t^N$, $V_N^N = \mathrm{E}f_N$.

Thus, the value τ_0^N is the optimal stopping time on the interval $0 : N$. But, in many problems of stochastic optimization, the probability measure P is not known. Suppose that we have a family $\{\mathrm{P}_\alpha, \alpha \in \mathcal{A}\}$ of probability measures. In this case, define the recursive values

$$b_N^N = f_N, \; b_t^N = f_t \wedge \operatorname*{ess\,sup}_\alpha \mathrm{E}_\alpha(b_{t+1}^N|\mathcal{F}_t), \quad \text{for } t \in N-1 : 0. \tag{1}$$

When $\operatorname{ess\,sup}_\alpha \mathrm{E}_\alpha|f_t| < \infty$, we introduce the value

$$\tau_t^N = \min\{t \le i \le N : f_i = b_i^N\} \tag{2}$$

that called the *optimal minimax stopping time* on the interval $t : N$ under the uncertain family of measures $\{\mathrm{P}_\alpha, \alpha \in \mathcal{A}\}$.

The Theorem and formulas (1), (2) are used in searching of the instant for the transition from observation to control in the problem of motion correction for multistage system

$$\begin{aligned} x_i &= A_i x_{i-1} + B_i u_i + C_i v_i + \xi_i, \\ y_i &= G_i x_{i-1} + w_i + \eta_i, \end{aligned} \tag{3}$$

where $x_i \in R^n$ is the unknown state vector; $y_i \in R^m$ is the observable vector; A_i, B_i C_i, G_i are matrices of appropriate dimensions. The initial Gaussian vector $x_0 \sim \mathcal{N}(x_0^1, \gamma_0)$ has uncertain mean value x_0^1 and does not depend on sequences $\xi_i \sim \mathcal{N}(0, \Xi_i)$, $\eta_i \sim \mathcal{N}(0, \Gamma_i)$. Suppose that $\mathrm{cov}(\xi_i, \eta_i) = Q_i$, and $\mathrm{cov}(\xi_i, \eta_j) = 0$, $\mathrm{cov}(\xi_i, \xi_j) = 0$, $\mathrm{cov}(\eta_i, \eta_j) = 0$ under $i \neq j$. The uncertain parameters x_0^1, v_i, w_i, which can be considered as the parameter α in (1), (2), are limited by either the geometric constraints

$$x_0^1 \in X_0, \quad v_i \in V_i, \quad w_i \in W_i, \tag{4}$$

where X_0, V_i, W_i are convex and compact sets, or, the joint quadratic constraints

$$\|x_0^1\|_{P_0}^2 + \sum_{i=1}^{N} \left(\|v_i\|_{F_i}^2 + \|w_i\|_{R_i}^2 \right) \leq \mu^2. \tag{5}$$

We suppose $\|x\|_P^2 = x'Px$. The symbol $'$ means the transposition. Matrices P_0, F_i, R_i are symmetric and positive. The controls u_i will be formed depending on the set of vectors $y^i = \{y_1, \dots, y_i\}$ under the rule given below. The estimation problems for the systems like (3) with mixed (determinate and random) disturbances was considered by the author in [Ananiev, 2007; Ananiev, 2010].

2 Problem of Motion Correction

Let the control u_i belong to $U_i \subset R^p$, where U_i is a convex compact set. The set of admissible controls $\{u_t, \dots, u_N\}$ is denoted by $u(t : N)$. If $t = 1$, then instead of $u(1 : N)$ we write the symbol u^N. The analogous designations are used for v_i, w_i. The whole set of uncertain parameters in system (3) will be denoted by $z^N = \{x_0^1, v^N, w^N\}$.

2.1 Correction without parametric uncertainty

Suppose at first that constraints (4), (5) give a single valued set z^N (constraints (4) are one-valued, and in (5) we have $\mu = 0$). We also suppose that the set of open-loop controls u^N is given. Let $t \in 1 : N$ be the defined instant and $\mathcal{F}_t = \sigma(y_1, \dots, y_t)$ be the σ-algebra generated by measurements. We can set a problem on motion correction of system (3), which consists in of the replacement of old controls with new $\mathcal{F}_t$-measurable controls $u(t+1 : N) \in U_{t+1} \times \cdots \times U_N$ for the purpose of minimization of terminal functional. Namely, we solve the following problem

$$\mathrm{E}\|Dx_N\|^2 \to \min_{u(t+1:N)}, \tag{6}$$

where $\|\cdot\|$ is the Euclidean norm, D is a matrix. Consider the solution of problem (6) in detail. We have $\mathrm{E}\|Dx_N\|^2 = \mathrm{EE}(\|Dx_N\|^2 | \mathcal{F}_t) = \mathrm{tr} DP_{N,t}D' + \mathrm{E}\|D\hat{x}_{N,t}\|^2$, where tr is the trace of matrix, $P_{N,t}$ is a solution of the matrix system

$$\begin{gathered} P_{i,t} = A_i P_{i-1,t} A_i' + \Xi_i, \quad P_{t,t} = \gamma_t, \\ i \in t+1 : N, \end{gathered} \tag{7}$$

$\hat{x}_{N,t}$ is a solution of the forecast system

$$\begin{gathered} \hat{x}_{i,t} = A_i \hat{x}_{i-1,t} + B_i u_i + C_i v_i, \quad \hat{x}_{t,t} = \hat{x}_t, \\ i \in t+1 : N. \end{gathered} \tag{8}$$

The initial states γ_t, $\hat{x}_t$ of systems (7), (8) are defined by Kalman filter equations, [Liptser and Shiryayev, 2000]:

$$\begin{gathered} \hat{x}_i = A_i \hat{x}_{i-1} + B_i u_i + C_i v_i + K_i (y_i - G_i \hat{x}_{i-1} \\ -w_i), \ \hat{x}_0 = x_0^1, \ i \in 1 : t, \quad K_i = (A_i \gamma_{i-1} G_i' \\ +Q_i)\Delta_i^-, \quad \Delta_i = \Gamma_i + G_i \gamma_{i-1} G_i', \\ \gamma_i = A_i \gamma_{i-1} A_i' + \Xi_i - K_i \Delta_i K_i'. \end{gathered} \tag{9}$$

Here Δ^- is the pseudoinverse matrix for Δ. For the solution of problem (6) one have to find $\min \|D\hat{x}_{N,t}\|^2$ over the all controls $u(t+1 : N)$ in system (8) with given initial condition $\hat{x}_t$. We have

$$\begin{gathered} \min_{u(t+1:N)} \|D\hat{x}_{N,t}\|^2 = \Big(\max_{\|l\| \leq 1} \Big\{ l' D A_t^N \hat{x}_t + \\ \sum_{i=t+1}^{N} \left(l' D A_i^N C_i v_i - \rho(-l | A_i^N B_i U_i) \right) \Big\} \Big)^2, \end{gathered} \tag{10}$$

where $A_t^N = A_N \cdots A_{t+1}$, $A_t^t = id$, $\rho(l|U) = \max_{u \in U} l'u$ is the support function of the set U, [Rockafellar, 1970]. If l_0 is a maximizer in problem (10), then the optimal controls $u^0(t+1 : N)$ satisfy the condition of minimum $\min_{u \in U_i} l_0' u = l_0' u_i^0$.

For the definition of Markov stopping time of transition to a new control, we introduce the designation

$$g_t(\hat{x}_t) = \min_{u(t+1:N)} \|D\hat{x}_{N,t}\|^2 + \mathrm{tr} DP_{N,t}D'. \tag{11}$$

Using linearity of the equations, we can write the equalities $y_i = y_i^0 + y_i^u + y_i^1$, $\hat{x}_i = \hat{x}_i^0 + x_i^u + x_i^1$, where vectors are formed by the systems

$$
\begin{gathered}
\hat{x}_i^0 = A_i\hat{x}_{i-1}^0 + K_i\zeta_i,\ y_i^0 = G_i\hat{x}_{i-1}^0 + \zeta_i,\\
\hat{x}_0^0 = 0;\ x_i^u = A_i x_{i-1}^u + B_i u_i,\ y_i^u = G_i x_{i-1}^u,\\
x_0^u = 0;\ x_i^1 = A_i x_{i-1}^1 + C_i v_i,\\
y_i^1 = G_i x_{i-1}^1 + w_i.
\end{gathered} \tag{12}
$$

Here $\zeta_i \sim \mathcal{N}(0,\Delta_i)$ is the innovation sequence of independent Gaussian values. Using Markov property of systems (9), (12), we recursively form the functions

$$
\begin{gathered}
s_N(x) = g_N(x),\ s_{t-1}(x) = g_{t-1}(x) \wedge \mathrm{E}s_t(A_t x\\
+K_t\zeta_t + B_t u_t + C_t v_t),\ t \in N:1,
\end{gathered} \tag{13}
$$

where the expectation is applied to the value ζ_t.

By means of the Theorem from Introduction we come to the conclusion.

Theorem 1. *Let the parameters z^N in* (3)–(5) *be fixed and the stopping time be of the form $\tau_t^N = \min\{t \le i \le N : g_i(\hat{x}_i) = s_i(\hat{x}_i)\}$. Then under $f_t = g_t(\hat{x}_t)$, $b_t^N = s_t(\hat{x}_t)$ the properties* (a) – (e) *of the theorem from Introduction hold.*

Remark 1. Suppose that the open-loop control u^N is a minimizer of problem (6) (when $t = 0$) and the control is not recalculated. Then the value $f_t = \mathrm{E}(\|Dx_N\|^2|\mathcal{F}_t)$ is a martingale and, therefore, $\mathrm{E}f_\tau \equiv \mathrm{E}f_1$ for any stopping time $\tau \in \mathcal{M}_0^N$, [Liptser and Shiryayev, 2000]. The value $g_t(\hat{x}_t)$ is not a martingale. Consequently the problem about a finding of the optimal stopping time for system (3) makes sense.

2.2 Correction under parametric uncertainty

Suppose that the uncertain parameters z^N in (3) are restricted by constraints (4) or (5). Now under given set u^N of open-loop controls, the value $\mathrm{E}\|Dx_N\|^2 = \mathrm{tr}DP_{N,t}D' + \mathrm{E}\|D\hat{x}_{N,t}\|^2$ in problem (6) is uncertain. There are several approaches to the possible solution. The most simple approach consists in the decision of a minimax problem

$$
\max_{z^N} \|D\hat{x}_{N,t}\|^2 \to \min_{u(t+1:N)}. \tag{14}
$$

The minimum in (14) will majorize expression (10) for any parameters z^N and it can be used as approximation from above. Let us calculate the minimum under constraints (4). We use the methods of the convex analysis [Rockafellar, 1970]. Denote the matrix $A_i - K_iG_i$ by $\mathbf{A}_i$ and the product $\mathbf{A}_N\cdots\mathbf{A}_{i+1}$ by $\mathbf{A}_i^N$; $\mathbf{A}_i^i = \mathbf{id}$. Using equations (8), (9), we can solve problem (14) and obtain the minimum in the form

$$
\begin{gathered}
r_t(\hat{x}_t^*) = \Big(\max_{\|l\|\le 1}\Big\{l'DA_t^N\hat{x}_t^* + d_t(l)\\
-\sum_{i=t+1}^{N}\rho(-l|DA_i^N B_iU_i)\Big\}\Big)^2,
\end{gathered} \tag{15}
$$

where $\hat{x}_t^*$ is a solution of the system

$$
\hat{x}_i^* = \mathbf{A}_i\hat{x}_{i-1}^* + B_iu_i + K_iy_i,\ \hat{x}_0^* = 0, \tag{16}
$$

and the value $d_t(l)$ is defined by the formula

$$
\begin{gathered}
d_t(l) = \mathrm{conc}\Big(\rho(l|DA_t^N\mathbf{A}_0^tX_0)\\
+\sum_{i=1}^{t}\big(\rho(l|DA_t^N\mathbf{A}_i^tC_iV_i) + \rho(-l|DA_t^N\mathbf{A}_i^tK_iW_i)\big)\\
+\sum_{i=t+1}^{N}\rho(l|DA_i^NC_iV_i)\Big).
\end{gathered} \tag{17}
$$

In formula (17) the symbol $\mathrm{conc}f(l)$ means the least concave function majorizing the $f(l)$ over the unite ball. For constraints (5) instead of formula (17) we get

$$
\begin{gathered}
d_t(l) = \mu\cdot\mathrm{conc}\Big(l'DA_t^N\big(\mathbf{A}_0^tP_0^{-1}\mathbf{A}_0^{t'}\\
+\sum_{i=1}^{t}\big(\mathbf{A}_i^tC_iF_i^{-1}C_i'\mathbf{A}_i^{t'}\\
+\mathbf{A}_i^tK_iR_i^{-1}K_i'\mathbf{A}_i^{t'}\big)\big)A_t^{N'}D'l\\
+\sum_{i=t+1}^{N}l'DA_i^NC_iF_i^{-1}C_i'A_i^{N'}D'l\Big)^{1/2}.
\end{gathered} \tag{18}
$$

Note that $\mathrm{conc}(l'Al)^{1/2} = (m(A)(1-l'l) + l'A'l)^{1/2}$, where $m(A) = \max_{\|l\|\le 1} l'A'l$ for matrices $A' = A \ge 0$. In this connection the concave hull in the formula (18) can be calculated explicitly. The same as in subsection 2.1, if l_0 is a maximizer in problem (15) with function (17) or (18), the optimal controls $u^0(t+1:N)$ satisfy the condition of minimum $\min_{u\in U_i} l_0'u = l_0'u_i^0$.

The same as in subsection 2.1, for the definition of the Markov stopping time of the transition to a new control, we introduce the designation

$$
g_t(\hat{x}_t^*) = r_t(\hat{x}_t^*) + \mathrm{tr}DP_{N,t}D'. \tag{19}
$$

The main difficulty of using of function (19) consists in the fact that the forecast of the signal y_t under the observable data y^{t-1} contains uncertain parameters. In this connection we will consider the random attainability domain of the system

$$
\begin{gathered}
\hat{x}_i^* = A_i\hat{x}_{i-1}^* + B_iu_i + K_i(\zeta_i + w_i + G_i\tilde{x}_{i-1}),\\
\tilde{x}_i = \mathbf{A}_i\tilde{x}_{i-1} + C_iv_i - K_iw_i,\ \tilde{x}_0 \in X_0,
\end{gathered} \tag{20}
$$

which is equivalent to (16).

We define the functions

$$
\begin{gathered}
s_N(x) = g_N(x),\ s_{t-1}(x) = g_{t-1}(x)\\
\wedge\max_{z^t}\mathrm{E}s_t(A_tx + B_tu_t + K_t(\zeta_t + w_t\\
+G_t\tilde{x}_{t-1})),\quad t \in N:1,
\end{gathered} \tag{21}
$$

where the E is applied to ζ_t, in order to receive the following result.

Theorem 2. *In the parametric uncertain case, let the stopping time be of the form* $\tau_t^N = \min\{t \le i \le N : g_i(\hat{x}_i^*) = s_i(\hat{x}_i^*)\}$*. Then without uncertainties this stopping time coincides with one in the theorem 1. For* $f_t = g_t(\hat{x}_t^*)$*,* $b_t^N = s_t(\hat{x}_t^*)$*, the value* τ_t^N *is the optimal minimax stopping time in the sense of formulas* (1), (2)*. For any set* z^N *of parameters, the estimation* $\mathrm{E}\min_{u(\tau_0^N+1:N)} \mathrm{E}(\|Dx_N\|^2|y^{\tau_0^N}) \le \mathrm{E}g(\hat{x}^*_{\tau_0^N})$ *from above is valid.*

Remark 2. After the reaching of the instant τ_0^N when the control is changed, it is possible to continue observation and again to trace the stoping time. In that case we receive the process of *multiple correction*.

2.3 Example

Let us consider a simple one-dimensional example for the purpose of an illustration. Let the equations look like

$$x_i = x_{i-1} + u_i + v_i,\ y_i = x_{i-1} + w_i + \eta_i,$$
$$|u_i| \le \alpha,\ |v_i| \le \beta,\ |w_i| \le \delta,$$

where the random values $\eta_i \sim \mathcal{N}(0,\Gamma)$ and $x_0 \sim \mathcal{N}(0,\gamma_0)$ are present. For given example we have

$$\gamma_i = \gamma_0\Gamma/(\Gamma + i\gamma_0),\ K_i = \gamma_{i-1}/(\Gamma + \gamma_{i-1})$$
$$= \gamma_i/\Gamma,\ P_{N,t} = \gamma_t,\ \hat{x}_t = \gamma_t \sum_{i=1}^{t}(y_i - w_i)/\Gamma.$$

If $\beta = \delta = 0$, the function (11) is of the form

$$g_t(x) = \gamma_t + \begin{cases} 0,\ |x| \le \alpha(N-t); \\ \big(|x| - \alpha(N-t)\big)^2,\ |x| > \alpha(N-t). \end{cases}$$

After the calculation of functions (13) and the reaching of the instant τ_0^N, the control is found by the formula

$$u_i^0 = \begin{cases} -\alpha \cdot \mathrm{sign}(\hat{x}_{\tau_0^N}),\ |\hat{x}_{\tau_0^N}| > \alpha(N-t), \\ -\hat{x}_{\tau_0^N}/(N-t),\ |\hat{x}_{\tau_0^N}| \le \alpha(N-t). \end{cases}$$

Let $N = 50$, $\alpha = \gamma_0 = 1$, $\Gamma = 0.5$. Here the functions $g_{N-1}(x)$ and $s_{N-1}(x)$ coincide. Therefore the instant $i = N-1$ is always the stopping time, but in some random realizations the stopping time can happen earlier. The structure of functions $s_{N-4}(x)$ and $g_{N-4}(x)$ are shown on Fig.1. The solution in the parametric case is provided analogously according to formulas (15) – (21).

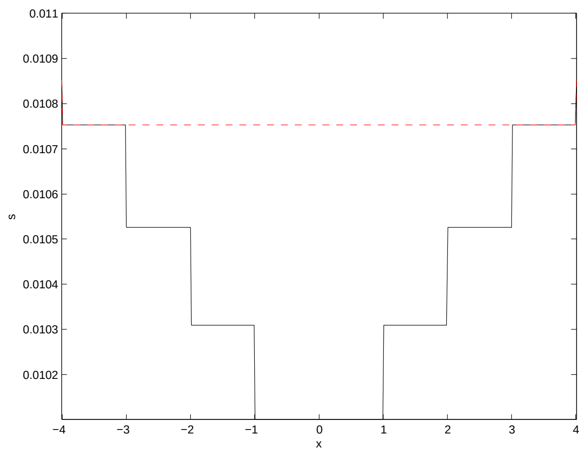

Figure 1. Structure of functions.

3 Application

We apply our consideration to the alignment problem from inertial navigation. Consider a transport ship-airplane system. The base coordinate system (b.c.s.) of the ship is correct. The axis 1 is directed along the parallel to the West. The axis 2 is the local vertical. The axis 3 is directed along the meridian to the North. The airplane dependent coordinate system (d.c.s.) with respect to b.c.s. is estimated by the Krylov (or Euler) angles. The sequence of clockwise rotations: θ_1, θ_3,

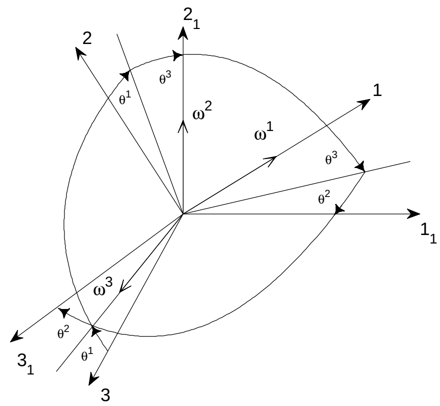

Figure 2. Alignment by rotation.

θ_2. Kinematic equations are of the form

$$\dot{\theta}_1 = \omega_1 - \dot{\theta}_2 \sin\theta_3, \quad \dot{\theta}_2 = (\omega_2 \cos\theta_1$$
$$-\omega_3 \sin\theta_1)/\cos\theta_3,\ \dot{\theta}_3 = \omega_2 \sin\theta_1 + \omega_3 \cos\theta_1,$$

where ω_i are projections of relative angular velocity. Under small angles (several degrees) these equations are well linearized. In the simplest case, the movement occurs on the Equator and $\theta_1 = \theta_2 \equiv 0$. Then only one angle $\theta \equiv \theta_3$ gives a deviation. The aim of the alignment is to match systems using the output of accelerometer's integrator along axis 2. When θ is small we obtain $\dot\theta = \Omega - \Omega_1 + \beta$, $\dot\beta = v$, $\dot y = g\theta + w + \dot\eta$, where Ω is absolute angular velocity of d.c.s., β is a slowly changed zero offset, w and η are determinate and random disturbances, g is the acceleration of gravity. The constraints are of the form

$$\int_0^T v(t)^2 dt/T \le \alpha^2, \quad \int_0^T w(t)^2 dt/T \le \delta^2.$$

Let $h = T/N$ be a time step. For discrete type of measurements we get the system

$$\theta_t = \theta_{t-1} + u_t + h\beta_{t-1} + v_{1t}, \quad \beta_t = \beta_{t-1} + v_{2t}, \quad y_t = hg\theta_{t-1} + w_t + \eta_t.$$

For vectors $v_t = [v_{1t}; v_{2t}]$ and numbers w_t we obtain the constraints

$$\sum_{i=1}^N \|v_i\|_F^2 \le \alpha^2 h^2 N, \quad \sum_{i=1}^N w_i^2 \le \delta^2 h^2 N, \qquad (22)$$

where $F^{-1} = [h^2/3, h/2; h/2, 1]$. Let the controls $u_i = \int_{t_{i-1}}^{t_i} (\Omega - \Omega_1)dt$ be also restricted by the constraint analogous to (22) for w with $\delta = \delta_u$. Here we have used standard Matlab designations for vectors and matrices. Let $(\mathrm{E}\theta_0^2)^{1/2} = 1\,\text{grad}$, $(\mathrm{E}\beta_0^2)^{1/2} = 12\,\text{grad/hour}$. Other numerical data: $T = 300\,\text{sec}$, $N = 300$, $\delta = 0.009\,\text{m/sec}^2$, $\alpha = 36\,\text{grad/hour}^2$, $(\mathrm{E}\eta_t^2)^{1/2} = 1/600\,\text{m/sec}$, $g = 9.81\,\text{m/sec}^2$, $\delta_u = 10^{-3}\,\text{rad/sec}$. The signal y_t is realized under $w_t \equiv -h\delta$, $v_t \equiv [h/2; 1]h\alpha$. The alteration of the cost $r = \sqrt{g_t(\hat x_t^*)}$ is shown on Fig. 3. The control is on Fig. 4. For given random event ω, the change of control happens at $t = 292$, $t = 299$.

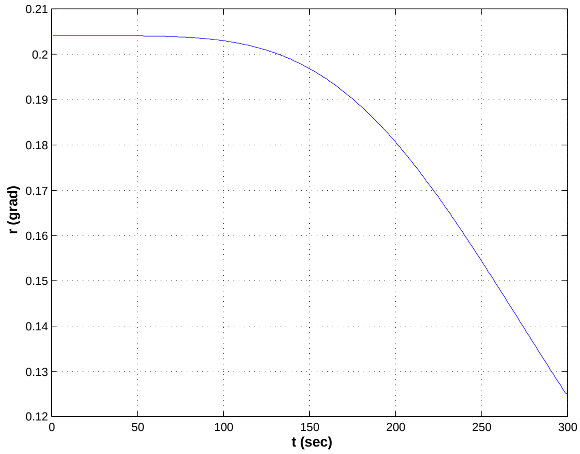

Figure 3. Alteration of the cost.

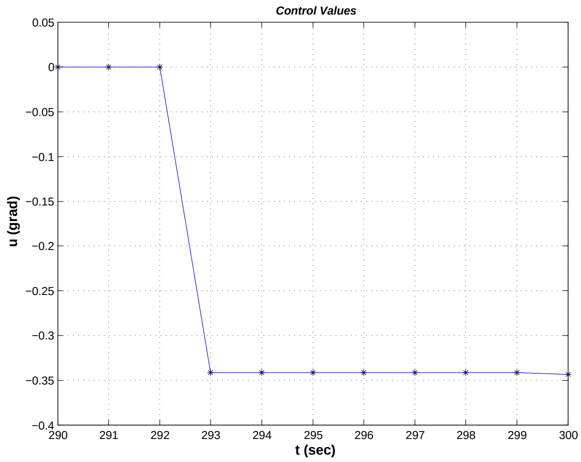

Figure 4. Control actions.

4 Continuous case

The same approach with the help of [Oksendal, 2000] can be applied to the system of the form

$$dx_t = (A(t)x_t + B(t)u + C(t)v)dt + \sigma_1(t)dB_t^1,$$
$$dy_t = (G(t)x_t + D(t)w)dt + \sigma_2(t)dB_t^2. \qquad (23)$$

For system (23) we write the Kalman-Busy filter

$$d\hat x_t = (A(t)\hat x_t + B(t)u + C(t)v)dt + P(t)G'(t)(\sigma_2\sigma_2')^{-1/2}d\hat y_t, \qquad (24)$$
$$dy_t = (G(t)\hat x_t + D(t)w)dt + (\sigma_2\sigma_2')^{1/2}d\hat y_t,$$

where $\hat y_t$ is the innovation Brownian process. To solve the problem like (6) introduce the generating operator

$$L = \sum_{i=1}^n (A(t)x + B(t)u + C(t)v)_i \partial/\partial x_i + \partial/\partial t + \sum_{i=1}^m (G(t)x + D(t)w)_i \partial/\partial y_i + 1/2 \sum_{i,j=1}^m (\sigma_2\sigma_2')_{i,j}\partial^2/\partial y_i\partial y_j + \frac{1}{2}\sum_{i,j=1}^n (PG'(\sigma_2\sigma_2')^{-1}GP)_{i,j}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{j=1}^m\sum_{i=1}^n (GP)_{j,i}\frac{\partial^2}{\partial x_i\partial y_j}, \qquad (25)$$

and the functional $J^\tau(s,x,y) = \mathrm{E}^{s,x,y}\|\hat x_\tau\|^2$, where τ is a continuous stopping time. Let $V = \{(s,x,y) : s > 0,\ x \in R^n,\ y \in R^m\}$ be a basic domain. The following conditions are sufficient for finding of the *optimal* *stopping time*. If there exists a function $\phi(s,x,y)$ such that:

(i) $\phi \in C^1(V) \cap C(\overline{V})$;
(ii) $\phi \le \|x\|^2$ and $\phi(0,x,y) = \|x\|^2$ if $(s,x,y) \in V$;
(iii) If $\mathcal{D} = \{(s,x,y) : \phi(s,x,y) < \|x\|^2\}$, then $\mathrm{E}^{s,x,y} \int_0^T \chi_{\partial\mathcal{D}}(t,\hat{x}_t,y_t)dt = 0, \forall(s,x,y) \in V$;
(iv) $\partial\mathcal{D}$ is a Lipshitz surface, and $\phi \in C^2(V \setminus \partial\mathcal{D})$;
(v) $L\phi \le 0$ if $(s,x,y) \in V \setminus \overline{\mathcal{D}}$, and $L\phi = 0$ if $(s,x,y) \in \mathcal{D}$;
(vi) $\tau_{\mathcal{D}} = \inf\{t > 0 : (t,\hat{x}_t,y_t) \notin \mathcal{D}\} < \infty$ a.s.,$\forall(s,x,y) \in V$,

then

$$\phi(s,x,y) = \inf_{\tau \le T} \mathrm{E}^{s,x,y}\|\hat{x}_\tau\|^2 = J^{\tau_{\mathcal{D}}}(s,x,y), \quad (26)$$

and $\tau^* = \tau_{\mathcal{D}}$ is an *optimal stopping time*. In (25) we substitute the optimal open-loop control $u(t,y(\cdot))$ that can be found as above. If the functions $v(\cdot), w(\cdot)$ in equations (23) are unknown, the above item (v) should be replaced by the following
(v') $\tilde{L}\phi \le 0$ if $(s,x,y) \in V \setminus \overline{\mathcal{D}}$, and $\tilde{L}\phi = 0$ if $(s,x,y) \in \mathcal{D}$; where

$$\begin{aligned}\tilde{L}\phi = \max_{v,w} \Big\{ \sum_{i=1}^{n} (A(s)x + B(s)u(s,y(\cdot)) \\ +C(s)v)_i \frac{\partial\phi}{\partial x_i} + \sum_{i=1}^{m} (G(s)x + D(s)w)_i \frac{\partial\phi}{\partial y_i} \Big\} \\ +\frac{\partial\phi}{\partial s} + \frac{1}{2} \sum_{i,j=1}^{m} (\sigma_2\sigma_2')_{i,j} \frac{\partial^2\phi}{\partial y_i \partial y_j} \quad (27) \\ +\frac{1}{2} \sum_{i,j=1}^{n} (PG'(\sigma_2\sigma_2')^{-1}GP)_{i,j} \frac{\partial^2\phi}{\partial x_i \partial x_j} \\ +\sum_{j=1}^{m} \sum_{i=1}^{n} (GP)_{j,i} \frac{\partial^2\phi}{\partial x_i \partial y_j}.\end{aligned}$$

5 Conclusion

We have considered the using of the Markov stopping times in problems of motion correction under mixed disturbances. It was supposed that the phase vector of the linear multistage system is non-observable, but we can observe a vector signal with noise at discrete instants. The expectation of the noise was uncertain and bounded by set-valued constraints. Here we have examined only multistage linear control systems with Gaussian noises and additive uncertainties. However, the same arguments can be applied to continuous control systems exited by Wiener processes. Using the results of convex analysis and the theory of Kalman filtering, we have obtained the optimal minimax stopping times for the completion of observation and for the transition to a new control action. The new control was obtained from minimax auxiliary open-loop control problem. The process of correction may be multiple. Thus, we have suggested the motion correction algorithm that consists in finding instants of correction when the old control is replaced with new one on the remaining time interval. A simple one-dimensional example was examined for the purpose of an illustration. An application to the alignment problem in the theory of inertial navigation was also considered. We also considered briefly the generalization of the theory to the continuous case.

Some problems need to be solved:

> *What is better*: to solve numerically the free boundary problem with partial derivatives or to use discretization?
> *Nonlinear cases*: needs one to use nonlinear filtering theory or something else? How to take into account uncertainties for the mixed disturbances case?

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