ROBUST LFR-BASED TECHNIQUE FOR STABILITY ANALYSIS OF PERIODIC SOLUTIONS

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Abstract: Linear matrix inequality based techniques, most often used for robust analysis of linear systems, are applied to the stability analysis of periodic equilibrium trajectories of nonlinear systems. Results are derived by linear-fractional representation of the nonlinearities and taking into account parametric uncertainties in the same time. Numerically exploitable formulas are obtained by discretization. An academic example illustrates the entire methodology.

Keywords: Robustness, Periodic systems, LMI, LFR.

1. INTRODUCTION

Stability analysis of nonlinear systems is a wide and important field of research. Among many, the fundamental results by Lyapunov indicate that a first analysis of the linearized model gives an information on local stability which needs in a second step to be characterized in terms of stable initial conditions. Following this close-to-equilibrium strategy many results have been derived inspired from robustness conditions for linear systems. The key idea is then to include the nonlinear system in some linear representation with uncertainties that include both nonlinear terms and other modeling uncertainties. The 'size' of the uncertainties is then a representative of the stable state domain. Among such modeling techniques one can cite the polytopic differential inclusions of (Boyd *et al.*, 1994), linear-fractional representations (LFR) as in (El Ghaoui and Scorletti, 1996) or descriptor type representations with auxiliary state in (Coutinho *et al.*, 2002). In the paper, LFR modeling is adopted.

Most results inspired from robust analysis of linear systems consider the case of equilibrium points. The aim of the paper is to illustrate how this methodology extends as well for the case of periodic stable trajectories. As said in (Bittanti and Colaneri, 2007), many processes are required to follow some periodic trajectory and this induces periodic models for linear or quasi-linear approaches. In such a case, at the difference with equilibrium point analysis, the LFR techniques and the related Linear Matrix Inequality (LMI) formulas cannot be applied directly to the obtained periodic continuous-time models. Indeed, (Farges, 2006), the time-varying characteristics need to be at some point discretized to formulate finite-dimentional problems. Similarly to (Kim *et al.*, 2006), the adopted discretization of the periodic continuous-time LFR is performed with an artificial sampling of the exogenous LFR signals. But at the difference of this last results, the discretization is done in a first-order hold manner following the example of (Imbert, 2001) which makes the results less conservative.

The paper is organized as follows. First the discrete-time periodic LFR modeling is defined. In section 3 the stability analysis results are then formulated. They are all LMI-based and inspired from quadratic separation type of results (Iwasaki and Hara, 1998; Peaucelle *et al.*, 2007; Scherer, 2005). Finally, a section is devoted to an illustrative academic example and some conclusions are given.

Notations: A^T is the transpose of the matrix A. $\mathtt{1}$ and $\mathtt{0}$ are respectively the identity and the zero matrices of appropriate dimensions. For Hermitian matrices, $A > (\geq) B$ if and only if $A-B$ is positive (semi) definite. Subscripts ^{c} are used in some cases to differentiate continuous-time signals $x^c(t)$ from their discrete-time sampled value $x(k)$.

2. PERIODIC LFR FOR PERIODIC TRAJECTORIES

2.1 LFT-modeling of linearized systems around a periodic trajectory

Let a continuous-time non-linear system described by the differential equation $\dot\eta(t) = f(\eta(t), \nu(t))$ and assume a T-periodic solution $\eta_s(t)$ of the system driven by a given T-periodic control law $\nu_s(t+T) = \nu_s(t)$

$$\dot\eta_s(t) = f(\eta_s(t), \nu_s(t)) \ , \ \eta_s(t+T) = \eta_s(t) \ \forall t \geq 0 \ .$$

Choosing $x^c = \eta_s - \eta$ and $u^c = \nu_s - \nu$, a linearized model of the system around the periodic trajectory is defined as $\dot x^c(t) \simeq A^c(t)x^c(t) + B^c(t)u^c(t)$ where $[A^c(t) \ B^c(t)] = J_f(\eta_s(t), \nu_s(t))$ is the Jacobian of f along the periodic trajectory. This linearized model is continuous-time T-periodic. Not to neglect totally the non-linear terms of the differential equation, assume there exists a near linear formulation of the form

$$\dot x^c(t) = A^c(t)x^c(t) + B^c(t)u^c(t) + B^c_\Omega(t)w^c_\Omega(t)$$

where the additional term

$$w^c_\Omega(t) = \Omega^c_{rat}(x(t))(C^c_\Omega(t)x^c(t) + D^c_{\Omega u}(t)u^c(t))$$

is a rational function of state dependent coefficients gathered in a matrix $\Omega^c(x(t))$:

$$\Omega^c_{rat}(x(t)) = \Omega^c(x(t))(\mathtt{1} - D^c_{\Omega\Omega}(t)\Omega^c(x(t)))^{-1} \ .$$

This rational expression of the model is assumed be such that Ω^c is bounded if x^c is bounded and the assumptions concerning this property are specified precisely in the following. Note that such rational modeling may be exact, especially in case f is a rational function of the state. The case of polynomial non-linear functions is illustrated on the numerical example in the last section of the paper.

At this stage the non-linear model is described at the vicinity of the periodic trajectory η_s as a Linear-Frational Representation (LFR)

$$\begin{array}{l} \dot x^c(t) = A^c(t)x^c(t) + B^c_\Omega(t)w^c_\Omega(t) + B^c(t)u^c(t) \\ z^c_\Omega(t) = C^c_\Omega(t)x^c(t) + D^c_{\Omega\Omega}(t)w^c_\Omega(t) + D^c_{\Omega u}u^c(t) \\ w^c_\Omega(t) = \Omega^c(x(t))z^c_\Omega(t) \ . \end{array}$$

The system parameters which are gathered in the matrices A^c, B^c_Ω, C^c_Ω and $D^c_{\Omega\Omega}$ may not be known exactly or may vary slowly (aging). To take these uncertainties (assumed constant) into account the model is enriched in the following way

$$\begin{array}{c} \begin{pmatrix} \dot x^c(t) \\ z^c_\Delta(t) \\ z^c_\Omega(t) \end{pmatrix} = M^c(t) \begin{pmatrix} x^c(t) \\ w^c_\Delta(t) \\ w^c_\Omega(t) \\ u^c(t) \end{pmatrix} \\ w^c_\Delta(t) = \Delta z^c_\Delta(t) \quad , \quad w^c_\Omega(t) = \Omega^c(x(t))z^c_\Omega(t) \end{array} \tag{1}$$

where Δ is a matrix that gathers all uncertain parameters and where the matrix defining the system parameters is T-periodic $M^c(t+T) = M^c(t)$ and partitioned as

$$M^c(t) = \begin{bmatrix} A^c(t) & B^c_\Delta(t) & B^c_\Omega(t) & B^c(t) \\ C^c_\Delta(t) & D^c_{\Delta\Delta}(t) & D^c_{\Delta\Omega}(t) & D^c_{\Delta u}(t) \\ C^c_\Omega(t) & D^c_{\Omega\Delta}(t) & D^c_{\Omega\Omega}(t) & D^c_{\Omega u}(t) \end{bmatrix} \ .$$

This modeling is always possible if the uncertain parameters enter the model as rational functions. For the sake of clarity of presentation a simple case with one scalar uncertainty is assumed:

$$\Delta = \delta \mathtt{1}_r \quad , \quad \underline{\delta} \leq \delta \leq \overline{\delta} \ . \tag{2}$$

More elaborated uncertainty matrices Δ can be considered following for example the methodologies given in (Iwasaki and Hara, 1998; Scherer, 2005).

2.2 Sampled discrete-time periodic model

We are interested at this stage to give a discrete-time version of this model. More precisely, the system is assumed to be controlled via a discrete-time calculator with a T-periodic sampling strategy defined by the N-periodic sequence $\{T_s(k)\}_{k\geq 0}$ such that

$$T_s(N) = T \quad , \quad T_s(k+N) = T_s(k) \ .$$

One such sampling strategy can be uniformly spaced over the period T of the system ($T_s(k) = kT/N$). An other strategy may be to sample thinner at the time intervals when the model parameters vary faster. Such situation is for example appropriate for satellite orbit control where, in case of elliptic orbits, the model parameters evolve faster when the satellite is closer to the earth, (Farges, 2006; Farges *et al.*, 2007; Theron *et al.*, 2007).

Input digital-to-analog conversion is modeled as a zero-order hold operator ($u^c(t) = u^c(T_s(k))$ on each interval $[T_s(k),\ T_s(k+1)]$) and output analog-to-digital conversion is a sampler. To define the discretized model define the following vectors

$$\begin{array}{ll} x(k) = x^c(T_s(k)) & , \tilde{w}_\star(k) = w^c_\star(T_s(k)) \\ \tilde{z}_\star(k) = z^c_\star(T_s(k)) & , u(k) = u^c(T_s(k)) \end{array}$$

where $\star$ is among $\{\Delta, \Omega\}$. Moreover, let the following assumptions, all related to the hypothesis of a sufficiently fast sampling with respect to the system dynamics.

Assumptions 1. The model parameters are approximated as constant and equal to their median value on the intervals $t \in [T_s(k),\ T_s(k+1)]$:

$$M(t) \simeq \tilde{M}(k) = M^c(0.5(T_s(k) + T_s(k+1)))$$

with the appropriate partitioning

$$\tilde{M}(k) = \begin{bmatrix} \tilde{A}(k) & \tilde{B}_\Delta(k) & \tilde{B}_\Omega(k) & \tilde{B}(k) \\ \tilde{C}_\Delta(k) & \tilde{D}_{\Delta\Delta}(k) & \tilde{D}_{\Delta\Omega}(k) & \tilde{D}_{\Delta u}(k) \\ \tilde{C}_\Omega(k) & \tilde{D}_{\Omega\Delta}(k) & \tilde{D}_{\Omega\Omega}(k) & \tilde{D}_{\Omega u}(k) \end{bmatrix} .$$

Assumptions 2. The exogenous signals are approximated on each sampling interval in a first-order hold manner:

$$\begin{array}{l} w^c_\star(t) \\ \simeq \tilde{w}_\star(k) + \dfrac{t - T_s(k)}{T_s(k+1) - T_s(k)} (\tilde{w}_\star(k+1) - \tilde{w}_\star(k)) \end{array}$$

for $\star$ among $\{\Delta, \Omega\}$, $t \in [T_s(k),\ T_s(k+1)]$.

This last assumption follows the methodology proposed in (Imbert, 2001). It happens to be less conservative than zero-order hold discretization adopted in (Kim *et al.*, 2006; Ma and Iglesias, 2002). The disadvantage is the necessity to have some knowledge on the future sampled signals at time $T_s(k+1)$. The sampling method therefore conduces to the next assumption.

Assumptions 3. There exist two scalars $q > 0$, $\beta > 1$, an N-periodic sequence of matrices $Q(k)$ and an N-periodic sequence of sets $\mathbf{\Xi}_\gamma(k)$ such that if the initial conditions at the sample of time $T_s(k)$ satisfy the quadratic constraint

$$x^T(k) Q(k) x(k) \leq q \tag{3}$$

then for any elements $\Xi(k) \in \mathbf{\Xi}_q(k)$ and $\hat{\Xi}(k) \in \mathbf{\Xi}_{\beta q}(k)$, the matrices $\Omega(x(k))$ and $\Omega(x(k+1))$ satisfy the quadratic constraints:

$$\begin{array}{c} \begin{bmatrix} 1 \\ \Omega(x(k)) \end{bmatrix}^T \Xi(k) \begin{bmatrix} 1 \\ \Omega(x(k)) \end{bmatrix} \leq 0 \\ \begin{bmatrix} 1 \\ \Omega(x(k+1)) \end{bmatrix}^T \hat{\Xi}(k) \begin{bmatrix} 1 \\ \Omega(x(k+1)) \end{bmatrix} \leq 0 \end{array} \tag{4}$$

The sets $\mathbf{\Xi}_\gamma(k)$ are supposed to be described by LMIs.

The matrices $\Xi(k)$ are basically defined to describe in a quadratic fashion the sets where lie the matrices $\Omega(x(k))$ if $x(k)$ lies in an ellipsoid (3). $\hat{\Xi}(k)$ describe identically the sets where are expected to lie the matrices $\Omega(x(k+1))$, that is at the following sample of time. As there is no possibility at this stage to guarantee that the states did not diverge between the two samples of time, the assumption is that the states do not escape from the origin by more than a factor β.

Based on the three assumptions one can integrate quite easily the differential equations. To do so define the notations $A(k) = e^{\tilde{A}(k)(T_s(k+1) - T_s(k))}$,

$$A_1(k) = \int_{T_s(k)}^{T_s(k+1)} e^{\tilde{A}(k)\tau} d\tau \ ,$$

$$A_2(k) = \int_{T_s(k)}^{T_s(k+1)} \frac{t - T_s(k+1)}{T_s(k+1) - T_s(k)} e^{\tilde{A}(k)\tau} d\tau \ ,$$

$A_3(k) = A_2(k) - A_1(k)$, $B(k) = A_1(k)\tilde{B}(k)$ and for $\star$ and $\diamond$ chosen among $\{\Delta, \Omega\}$ define

$$\begin{array}{l} B_\star(k) = \begin{bmatrix} A_2(k) & A_3(k) \end{bmatrix} \tilde{B}_\star(k) \ , \\ C_\diamond(k) = \begin{bmatrix} \tilde{C}_\diamond(k) \\ \tilde{C}_\diamond(k+1) A(k) \end{bmatrix} \ , \\ D_{\diamond\star}(k) \\ = \begin{bmatrix} \tilde{D}_{\diamond\star}(k) & 0 \\ 0 & \tilde{D}_{\diamond\star}(k+1) \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{C}_\diamond(k+1) \end{bmatrix} B_\star(k) \ , \\ w_\star(k) = \begin{pmatrix} \tilde{w}_\star(k) \\ \tilde{w}_\star(k+1) \end{pmatrix} \ , \ z_\star(k) = \begin{pmatrix} \tilde{z}_\star(k) \\ \tilde{z}_\star(k+1) \end{pmatrix} \ . \end{array}$$

With these notations and assumptions, the sampled system is such that

$$\begin{array}{c} \begin{pmatrix} x(k+1) \\ z_\Delta(k) \\ z_\Omega(k) \end{pmatrix} = M(k) \begin{pmatrix} x(k) \\ w_\Delta(k) \\ w_\Omega(k) \\ u(k) \end{pmatrix} \\ w_\Delta(k) = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} z_\Delta(k) = \delta 1_{2r} z_\Delta(k) \\ w_\Omega(k) = \begin{bmatrix} \Omega(x(k)) & 0 \\ 0 & \Omega(x(k+1)) \end{bmatrix} z_\Omega(k) \end{array} \tag{5}$$

where $M(k+N) = M(k)$ is the N-periodic sequence of the model parameters partitioned as

$$M(k) = \begin{bmatrix} A(k) & B_\Delta(k) & B_\Omega(k) & B(k) \\ C_\Delta(k) & D_{\Delta\Delta}(k) & D_{\Delta\Omega}(k) & D_{\Delta u}(k) \\ C_\Omega(k) & D_{\Omega\Delta}(k) & D_{\Omega\Omega}(k) & D_{\Omega u}(k) \end{bmatrix} .$$

3. ROBUST STABILITY ANALYSIS

The N-periodic discrete-time system (5) is uncertain in LFT form and subject to two types of uncertainties. The first one is a constant parametric uncertainty Δ. The second is a time-varying uncertainty described by a quadratic constraint (4). Due to the fact that the quadratic constraint is valid only for bounded values of the state, stability of the system need to be proved along with a guarantee that the state remains bounded in the ellipsoids defined in (3).

According to the "quadratic separation" terminology of (Iwasaki and Hara, 1998; Peaucelle *et al.*, 2007), define for each sample k the constraints on the vertex separator $\Theta_\Delta(k)$ with respect to the uncertainty $1_2 \otimes \Delta = \delta 1_{2r}$

$$\begin{array}{c} \begin{bmatrix} 0 \\ 1_{2r} \end{bmatrix}^T \Theta_\Delta(k) \begin{bmatrix} 0 \\ 1_{2r} \end{bmatrix} \geq 0 \\ \begin{bmatrix} 1_{2r} \\ \underline{\delta} 1_{2r} \end{bmatrix}^T \Theta_\Delta(k) \begin{bmatrix} 1_{2r} \\ \underline{\delta} 1_{2r} \end{bmatrix} \leq 0 \\ \begin{bmatrix} 1_{2r} \\ \overline{\delta} 1_{2r} \end{bmatrix}^T \Theta_\Delta(k) \begin{bmatrix} 1_{2r} \\ \overline{\delta} 1_{2r} \end{bmatrix} \leq 0 \ . \end{array} \tag{6}$$

Define as well the constraints on the D-scaling type separator $\Theta_\Omega(k)$ with respect to the uncertainty $\mathrm{diag}(\Omega(x(k)), \Omega(x(k+1)))$

$$\begin{array}{l} \Xi(k) = \begin{bmatrix} \Xi_1(k) & \Xi_2(k) \\ \Xi_2^T(k) & \Xi_3(k) \end{bmatrix} \in \mathbf{\Xi}_q(k) \\ \hat{\Xi}(k) = \begin{bmatrix} \hat{\Xi}_1(k) & \hat{\Xi}_2(k) \\ \hat{\Xi}_2^T(k) & \hat{\Xi}_3(k) \end{bmatrix} \in \mathbf{\Xi}_{\beta q}(k) \\ \Theta_\Omega(k) = \begin{bmatrix} \Xi_1(k) & 0 & \Xi_2(k) & 0 \\ 0 & \hat{\Xi}_1(k) & 0 & \hat{\Xi}_2(k) \\ \Xi_2^T(k) & 0 & \Xi_3(k) & 0 \\ 0 & \hat{\Xi}_2^T(k) & 0 & \hat{\Xi}_3(k) \end{bmatrix} \ . \end{array} \tag{7}$$

Along with the notations

$$\begin{array}{l} N_x(k) = \begin{bmatrix} A(k) & B_\Delta(k) & B_\Omega(k) \\ 1 & 0 & 0 \end{bmatrix} \\ N_\Delta(k) = \begin{bmatrix} C_\Delta(k) & D_{\Delta\Delta}(k) & D_{\Delta\Omega}(k) \\ 0 & 1 & 0 \end{bmatrix} \\ N_\Omega(k) = \begin{bmatrix} C_\Omega(k) & D_{\Omega\Delta}(k) & D_{\Omega\Omega}(k) \\ 0 & 0 & 1 \end{bmatrix} \ . \end{array}$$

Invariant set and asymptotic stability results are now formulated.

Theorem 1. If there exists a solution $\Theta_\Delta(k)$, $\Theta_\Omega(k)$ such that for all $k = 1 \dots N$ the LMIs (6), (7) and

$$\begin{array}{l} N_x^T(k) \begin{bmatrix} Q(k+1) & 0 \\ 0 & -Q(k) \end{bmatrix} N_x(k) \\ < N_\Delta^T(k) \Theta_\Delta(k) N_\Delta(k) + N_\Omega^T(k) \Theta_\Omega(k) N_\Omega(k) \end{array} \tag{8}$$

hold, then for any initial conditions such that $x^T(0)Q(0)x(0) \leq q$ the system (5) with zero input $u_k = 0$ is such that $x^T(k)Q(k)x(k) \leq q$ holds for all $k \geq 0$.

Proof: Let the notations

$$\begin{array}{l} \Upsilon_\Delta(\delta, k) = \begin{bmatrix} 1_{2r} & \delta 1_{2r} \end{bmatrix} \Theta_\Delta(k) \begin{bmatrix} 1_{2r} \\ \delta 1_{2r} \end{bmatrix} \\ \Upsilon_\Omega(\Omega, \Xi) = \begin{bmatrix} 1 & \Omega^T \end{bmatrix} \Xi \begin{bmatrix} 1 \\ \Omega \end{bmatrix} \end{array}$$

The first constraint in (6) implies that the inequality $\Upsilon_\Delta(\delta, k) \leq 0$ is convex with respect to δ. The two last constraints in (6) imply that the inequality holds for the extremal values of δ. Therefore due to convexity $\Upsilon_\Delta(\delta, k) \leq 0$ holds for all $\delta \in [\underline{\delta} \ \overline{\delta}]$.

According to Assumption 3 and due to the structure of $\Theta_\omega(k, q)$ given in (7), if $x^T(k)Q(k)x(k) \leq q$ then

$$\begin{array}{c} \Upsilon_\Omega(\Omega(x(k)), \Xi(k)) \leq 0 \\ \Upsilon_\Omega(\Omega(x(k+1)), \hat{\Xi}(k)) \leq 0 \ . \end{array}$$

Multiply (8) from the left-hand side by the vector $\left(x^T(k) \ w_\Delta^T(k) \ w_\Omega^T(k) \right)$ and from the right-hand side by its transpose to get, due to equations (5),

$$\begin{array}{l} x^T(k+1)Q(k+1)x(k+1) \\ \leq x^T(k)Q(k)x(k) \\ + z_\Delta^T(k) \Upsilon_\Delta(\delta, k) z_\Delta(k) \\ + \tilde{z}_\Omega^T(k) \Upsilon_\Omega(\Omega(x(k)), \Xi(k)) \tilde{z}_\Omega(k) \\ + \tilde{z}_\Omega^T(k+1) \Upsilon_\Omega(\Omega(x(k+1)), \hat{\Xi}(k)) \tilde{z}_\Omega(k+1) \ . \end{array}$$

According to the properties of the $\Upsilon_\star$ terms given above, if $x^T(k)Q(k)x(k) \leq q$ the last equation implies $x^T(k+1)Q(k+1)x(k+1) \leq q$ which proves the theorem by recurrence. ■

Theorem 2. If there exists a solution $P(k) \geq 0$, $\Theta_\Delta(k)$, $\Theta_\Omega(k)$ such that for all $k = 1 \dots N$ the LMIs (6), (7) and

$$\begin{array}{l} N_x^T(k) \begin{bmatrix} \hat{Q}(k+1) & 0 \\ 0 & -\hat{Q}(k) \end{bmatrix} N_x(k) \\ < N_\Delta^T(k) \Theta_\Delta(k) N_\Delta(k) + N_\Omega^T(k) \Theta_\Omega(k, q) N_\Omega(k) \end{array} \tag{9}$$

hold where $\hat{Q}(k) = Q(k) + P(k) > 0$ and $P(N+1) = P(1)$, then for any initial conditions such that $x^T(0)(Q(0) + P(N))x(0) \leq q$ the system (5) with zero input $u_k = 0$ is asymptotically stable.

Proof: First note that since $P(k) \geq 0$ the condition $x^T(k)\hat{Q}(k)x(k) \leq q$ implies the boundedness condition $x^T(k)Q(k)x(k) \leq q$. Based on the same methodology as in the proof of Theorem 1, assuming that $x^T(k)Q(k)x(k) \leq q$ implies that

$$x^T(k+1)\hat{Q}(k+1)x(k+1) \leq x^T(k)\hat{Q}(k)x(k) \ .$$

The Lyapunov function $V(x,k) = x^T(k)\hat{Q}(k)x(k)$ is positive definite and decreasing along the trajectories of the system thus proving asymptotic stability. ■

4. NUMERICAL EXAMPLE

Let the nonlinear system taken from (Jordan and Smith, 1987)

$$\dot{\eta}_1 = \eta_2 \ , \ \dot{\eta}_2 = -\eta_1 - (\eta_1^2 + \eta_2^2 - 1)\eta_2$$

which admits the following periodic solutions parameterized by the phase ϕ:

$$\eta_{s1}(t) = \cos(t+\phi) \quad , \quad \eta_{s2}(t) = -\sin(t+\phi) \ .$$

Assuming η_1 can be measured and the existence of a control input on the second differential equation, consider the modified (controlled) system

$$\begin{aligned}\dot{\eta}_1 = \eta_2 \ , \ \dot{\eta}_2 = -\eta_1 &- (\eta_1^2 + \eta_2^2 - 1)\eta_2 \\ &+(\kappa+\delta)(\cos(t) - \eta_1(t))\end{aligned}$$

affected by uncertainties δ and which admits a single periodic solution

$$\eta_{s1}(t) = \cos(t) \quad , \quad \eta_{s2}(t) = -\sin(t) \ .$$

Following the methodology exposed in the paper, this system around the periodic trajectory writes exactly as

$$\begin{aligned}\dot{x}^c(t) = &\begin{bmatrix} 0 & 1 \\ -1-\kappa+\sin 2t & \cos 2t - 1 \end{bmatrix} x^c(t) \\ &+ \begin{bmatrix} 0 \\ -1 \end{bmatrix} w^c_\Delta(t) \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\sin t & 2\cos t & -3\sin t & -1 \end{bmatrix} w^c_\Omega(t)\end{aligned}$$

with $z^c_\Delta = \begin{bmatrix} 1 & 0 \end{bmatrix} x^c$, $z^c_\Omega = x^c$, $\Delta = \delta$ and

$$\Omega^c(x^c) = \begin{bmatrix} x_1^c & 0 \\ x_2^c & 0 \\ 0 & x_2^c \\ 0 & x_1^{c2} + x_2^{c2} \end{bmatrix} .$$

Define the convex set $\mathbf{\Xi}_\gamma$ as composed of matrices with the following structure

$$\Xi_1 = -\gamma \begin{bmatrix} \alpha_1+\alpha_4 & \alpha_2+\alpha_5 \\ \alpha_2+\alpha_5 & 2\alpha_3+\alpha_6+\gamma\alpha_7 \end{bmatrix}$$

$$\Xi_2 = \begin{bmatrix} 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & 0 & \alpha_3 \end{bmatrix} \ , \ \Xi_3 = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1+\alpha_4 & \alpha_5 & 0 \\ 0 & \alpha_5 & \alpha_6 & 0 \\ 0 & 0 & 0 & \alpha_7 \end{bmatrix}$$

and where the scaling elements satisfy the LMI constraints:

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & 2\alpha_3 \end{bmatrix} \geq 0 \ , \ \begin{bmatrix} \alpha_4 & \alpha_5 \\ \alpha_5 & \alpha_6 \end{bmatrix} \geq 0 \ , \ \alpha_7 \geq 0 \ .$$

This set allows to define the properties expected in Assumption 3: if $x^{cT}x^c \leq \gamma$ (*i.e.* for $Q=1$) it is guaranteed that

$$\begin{aligned}\begin{bmatrix} 1 \\ \Omega^c(x^c) \end{bmatrix}^T \Xi \begin{bmatrix} 1 \\ \Omega^c(x^c) \end{bmatrix} = \\ (x_1^{c2}+x_2^{c2}-\gamma)\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & 2\alpha_3 \end{bmatrix} \\ +(x_2^{c2}-\gamma)\begin{bmatrix} \alpha_4 & \alpha_5 \\ \alpha_5 & \alpha_6 \end{bmatrix} \\ +((x_1^{c2}+x_2^{c2})^2-\gamma^2)\begin{bmatrix} 0 & 0 \\ 0 & \alpha_7 \end{bmatrix} \leq 0\end{aligned}$$

Discretization of assumptions 1 and 2 are applied to the system assuming an uniformly spaced sampling $T_s(k) = kT/N$, $\forall k \geq 0$ where $T = 2\pi$, and choosing $N = 20$ samples over the period. The 'control' gain is chosen at $\kappa = 4.5$ with $\overline{\delta} = -\underline{\delta} = 0.5$ uncertainty.

Forgetting about the uncertainties Δ and the nonlinear terms Ω, the nominal linearized and sampled system $x(k+1) = A(k)x(k)$ happens to be stable. Indeed its monodromy matrix has all eigenvalues in the unit circle:

$$\lambda\left(A(N=20)\cdots A(2)A(1)\right) = \{ \ -0.1 \ , \ 0.002\} .$$

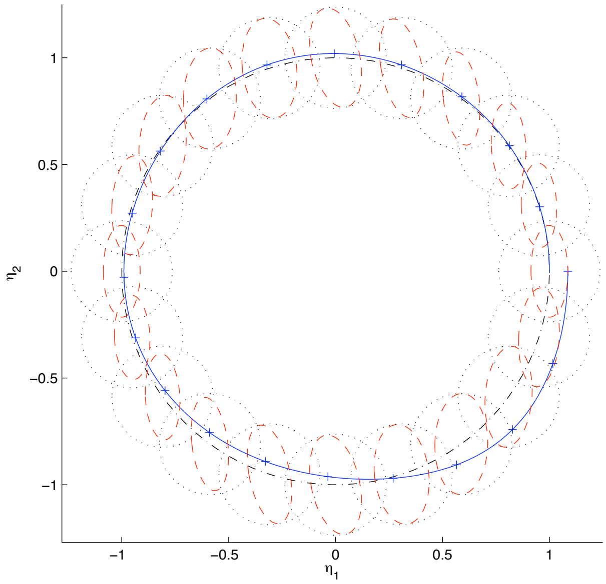
Assumption 3 is formulated with $Q(k) = 1$ for all $k \geq 0$ and a dilatation factor of $\beta = 1.5$. Theorem 2 is then applied for various values of q. A line-search procedure allows to find the maximal admissible bound on the initial condition $\overline{q} = 0.056$. For each tested value of q, the computation time is about 0.7 seconds. The computations were preformed on an PC with Intel Pentium D processor with 3.00GHz frequency and 1GB memory. The LMI constraints were declared using YALMIP (Löfberg, 2004) and solved using the latest version of SeDuMi (Sturm, 1999) (SeDuMi 1.1 available at `http://sedumi.mcmaster.ca/`).

For the maximal attained value $\overline{q} = 0.056$, the LMI problem is solved again while minimizing $\sum_{k=1}^N \mathrm{trace}P(k)$. This criterion is chosen as an heuristic for maximizing the stability domains $x^T(k)\hat{Q}(k)x(k) \leq q$. The computation time is then of 1.4 seconds and the obtained initial conditions for which asymptotic stability is guaranteed is the ellipse defined by

$$\begin{aligned} x^T(0)\hat{Q}(0)x(0) = \\ x^T(0)\begin{bmatrix} 7.5311 & 0.0545 \\ 0.0545 & 1.2153 \end{bmatrix} x(0) \leq 0.056 = \overline{q} \ .\end{aligned}$$

Figure 1 plots in state-space $(\eta_1, \ \eta_2)$ coordinates a simulation of the system over one period of time for $\delta = 0$. The equilibrium circle trajectory is plotted in dashed line. At each instant corresponding to the sampling $(t = kT/N)$ the position of the systems state is located using a '+' sign, the ellipse in which the state is guaranteed to lie is plotted in dashed lines and the circles defined by $x^T x \leq \overline{q}$ are plotted in dotted lines. Asymptotic oscillatory convergence is noticed with less than 0.85% error at time $t = 2\pi$. Other simulations show the same behavior for other admissible values of δ (not plotted to keep the figure readable).

Figure 1. Simulation of the non-linear system



To test the conservatism induced by robustness with respect to the uncertainty δ, the same tests are made with $\underline{\delta} = \overline{\delta} = 0$. In that case the maximal admissible bound is $\overline{q} = 0.083$.

5. CONCLUSIONS

Stability analysis of periodic equilibrium trajectories for nonlinear systems is solved using LMI-based robustness techniques. Although discretization assumptions are introduced and nevertheless the inherent conservatism of the LMI results, the numerical experiments show good promising results. Future contributions will be devoted to conservatism reduction.

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