FREE BEAM VIBRATION ANALYSIS BASED ON THE METHOD OF INTEGRODIFFERENTIAL RELATIONS

Saurin V.V., Kostin G.V.

Institute for Problems in Mechanics of the Russian Academy of Sciences

Abstract: Equations describing small free oscillations of a rectilinear elastic beam with a rectangular cross section have been obtained within the framework of the linear theory of elasticity and solved using the method of integrodifferential relations (IDR). The influence of geometry and elastic characteristics on the frequencies and shapes of free beam oscillations is studied. It is shown that the longitudinal motions admit two types of displacement and internal stress fields. The lateral oscillations obtained are specified by two frequency bands, which correspond to different types of the characteristic equation roots. Numerical examples of free beam oscillations are presented and discussed.

Keywords: Distributed-parameter systems, Eigenvalue problems, Frequency characteristics, Verification.

1. INTRODUCTION

The beam theory based on the intuitive hypotheses which have been proposed by J. Bernoulli in the end of the 17th century (Donnell, 1976) occupies an important place among the simplified theories in the solid mechanics. In spite of the fact that the beam theory is applicable for a wide class of engineering problems, it does not take into account the important mechanical characteristics of the elastic structures like shear and anisotropic properties of the material. The qualifying formulae taken into account the influence of Poisson's ratio have been proposed for static (Timoshenko beam, see Timoshenko (1956)) and dynamic problems (Rayleigh correction, see Rayleigh (1926). Below the approach based on the expansions of unknown stress and displacements functions with respect to small parameter (the ratio of the beam height to its length) is used in order to deduce the new equations described the free beam vibrations (Kostin and Saurin (2005), Kostin and Saurin, (2006 a,b,c)).

2. STATEMENT OF THE PROBLEM

Let us consider a plate occupying a rectangular region Ω with a boundary γ. The plate has the height h, length l, and a constant thickness (for certainty, equal to unity). The boundary γ is assumed to be free of external loads. The stress-strain state of an isotropic body is described by a two-dimensional system of linear elasticity equations (Timoshenko and Goodier (1956)):

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0, \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0; \quad (1)$$

$$\varepsilon_x^0 = \frac{\partial u}{\partial x}, \varepsilon_y^0 = \frac{\partial v}{\partial y}, \varepsilon_{xy}^0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \quad (2)$$

$$\begin{gathered} \varepsilon_x^0 = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right), \varepsilon_y^0 = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right), \\ \varepsilon_{xy}^0 = \frac{\tau_{xy}}{2G}, \quad G = \frac{E}{2(1+\mu)}. \end{gathered} \quad (3)$$

The boundary conditions can be written as

$$\begin{gathered} \sigma_x n_x + \tau_{xy} n_y = 0, \\ \tau_{xy} n_x + \sigma_y n_y = 0, \quad (x, y) \in \gamma. \end{gathered} \quad (4)$$

where σ_x, σ_y, and τ_{xy} are components of the stress tensor $\boldsymbol{\sigma}$; ε_x^0, ε_y^0, and ε_{xy}^0 are components of the strain tensor $\boldsymbol{\varepsilon}^0$; u and v are components of the displacement vector **u**; f_x and f_y are components of the volume force vector **f**; n_x and n_y are the components of the unit vector **n**, which is normal to the boundary γ; and E, G, and μ are the Young's modulus, shear modulus, and Poisson's ratio, respectively.

We assume that (i) the body can perform small elastic oscillations relative to the state of equilibrium and (ii) the force vector **f** is determined by inertial forces caused by the motions of internal points of the plate:

$$f_x = -\rho \frac{\partial^2 u}{\partial t^2}, \quad f_y = -\rho \frac{\partial^2 v}{\partial t^2}, \tag{5}$$

where ρ is the material density.

3. THE METHOD OF INTEGRODIFFERENCIAL RELATIONS

The problem is solved using the method of IDR, which basic ideas are described by Kostin and Saurin, 2005, Kostin and Saurin, 2006a,b,c. According to this approach Hooke's relations (3) are replaced by an integral equation

$$\Phi = \int_{\Omega} \xi : \xi d\Omega = 0, \xi = \mathbf{C}^{-1} : \sigma - \boldsymbol{\varepsilon}^0, \tag{6}$$

where $\mathbf{C}$ is the tensor of elastic moduli ($C_{ijkl} = C_{ijlk} = C_{klij}$). The components of the stress tensor $\boldsymbol{\sigma}$ and displacement vector $\mathbf{u}$ are considered as unknown functions. The linear elasticity problems are solved using the equivalent variation formulation,

$$\Phi[\boldsymbol{\sigma}, \mathbf{u}] \to \min_{\boldsymbol{\sigma}, \mathbf{u}} \tag{7}$$

under differential constraints (1), (2) and boundary conditions (4). In order to describe free oscillations of the elastic plate we represent the unknown components of the stress tensor $\boldsymbol{\sigma}$ and displacement vector $\mathbf{u}$ as expansions in powers of the ratio y/l:

$$\begin{aligned} \sigma_x &= e^{i\omega t} \sum_{j=0}^{1} \sigma_x^{(j)}(x) \left(\frac{y}{l} \right)^j, \\ \tau_{xy} &= e^{i\omega t} \sum_{j=0}^{2} \tau_{xy}^{(j)}(x) \left(\frac{y}{l} \right)^j, \\ \sigma_y &= e^{i\omega t} \sum_{j=0}^{3} \sigma_y^{(j)}(x) \left(\frac{y}{l} \right)^j; \end{aligned} \tag{8}$$

$$\begin{aligned} u &= e^{i\omega t} \left[u_0(x) + u_1(x) \frac{y}{l} \right], \\ v &= e^{i\omega t} \left[v_0(x) + v_1(x) \frac{y}{l} \right]; \end{aligned} \tag{9}$$

where ω is the unknown frequency. In expansion (8), the stress component σ_x is a linear function of y. The choice of the power of approximations for the functions τ_{xy} and σ_y is determined by the condition of nontrivial solutions for equilibrium equations (1). Substituting expansions (8) and (9) into Eqs. (1) and equating the coefficients at the corresponding powers of y/l to zero, we obtain the following system of five differential equations:

$$\begin{aligned} &\frac{d\sigma_x^{(0)}}{dx} + \tau_{xy}^{(1)} + \rho\omega^2 u_0 = 0, \\ &\frac{d\sigma_x^{(1)}}{dx} + 2\tau_{xy}^{(2)} + \rho\omega^2 u_1 = 0, \\ &\frac{d\tau_{xy}^{(0)}}{dx} + \sigma_y^{(1)} + \rho\omega^2 v_0 = 0, \\ &\frac{d\tau_{xy}^{(1)}}{dx} + 2\sigma_y^{(2)} + \rho\omega^2 v_1 = 0, \\ &\frac{d\tau_{xy}^{(2)}}{dx} + 3\sigma_y^{(3)} = 0. \end{aligned} \tag{10}$$

For the boundary conditions $\sigma_y(x, \pm h/2) = \tau_{xy}(x, \pm h/2) = 0$ equilibrium equations (10) can be solved with respect to the unknown functions u_0, u_1, v_0, v_1 and $\sigma_y^{(1)}$:

$$\begin{aligned} u_0 &= -\frac{1}{\rho\omega^2} \frac{d\sigma_x^{(0)}}{dx}, \\ u_1 &= -\frac{1}{\rho\omega^2} \left(\frac{d\sigma_x^{(1)}}{dx} - \frac{8l\tau_{xy}^{(0)}}{h^2} \right), \\ v_0 &= -\frac{2}{3\rho\omega^2} \frac{d\tau_{xy}^{(0)}}{dx}, \\ v_1 &= \frac{8l\sigma_y^{(0)}}{\rho\omega^2 h^2}, \quad \sigma_y^{(1)} = -\frac{l}{3} \frac{d\tau_{xy}^{(0)}}{dx}. \end{aligned} \tag{11}$$

Then, variational problem (7) is reduced to minimization of the functional Φ in (6), which depends only on the four unknown stress functions $\sigma_x^{(0)}$, $\sigma_x^{(1)}$, $\tau_{xy}^{(0)}$, $\sigma_y^{(0)}$ and obeys the following boundary conditions:

$$\begin{aligned} &\sigma_x^{(0)}(0) = \sigma_x^{(0)}(l) = \sigma_x^{(1)}(0) = \sigma_x^{(1)}(l) \\ &= \tau_{xy}^{(0)}(0) = \tau_{xy}^{(0)}(l) = 0. \end{aligned} \tag{12}$$

Let us analyze in more detail the structure of functional Φ. Using expansions of the stress (8) and displacement (9) functions, this functional can be written in the following form:

$$\Phi = \int_{\Omega} \left[H_1^2 + H_2^2 + 2H_3^2 \right] d\Omega, \tag{13}$$

where

$$
\begin{aligned}
H_1 &= \frac{du_0}{dx} - \frac{\sigma_x^{(0)} - \mu\sigma_y^{(0)}}{E} \\
&+ \left(\frac{du_1}{dx} - \frac{\sigma_x^{(1)} - \mu\sigma_y^{(1)}}{E}\right)\frac{y}{l} \\
&- \frac{4\mu\sigma_y^{(0)}}{E}\frac{y^2}{h^2} - \frac{4\mu\sigma_y^{(1)}}{E}\frac{y^3}{lh^2}, \\
H_2 &= \frac{v_1}{l} - \frac{\sigma_y^{(0)} - \mu\sigma_x^{(0)}}{E} - \frac{\sigma_y^{(1)} - \mu\sigma_x^{(1)}}{E}\frac{y}{l} \\
&+ \frac{4\sigma_y^{(0)}}{E}\frac{y^2}{h^2} + \frac{4\sigma_y^{(1)}}{E}\frac{y^3}{lh^2}, \\
H_3 &= \frac{u_1}{2l} + \frac{1}{2}\frac{dv_0}{dx} - \frac{\tau_{xy}^{(0)}}{2G} \\
&+ \frac{1}{2}\frac{dv_1}{dx}\frac{y}{l} + \frac{2\tau_{xy}^{(0)}}{G}\frac{y^2}{h^2}.
\end{aligned} \tag{14}
$$

The condition that the integral in Eq. (13) is zero implies that functions H_i $(i = 1,2,3)$ vanish in the region Ω. Therefore, the coefficients of the corresponding powers of y/l in expressions (14) also vanish everywhere in Ω. Using the finite-dimensional representation of stresses (8) and displacements (9) in terms of unknown functions $\sigma_x^{(0)}$, $\sigma_x^{(1)}$, $\tau_{xy}^{(0)}$, and $\sigma_y^{(0)}$ given by (11), we can satisfy only linear coefficients with respect to y in Eqs. (14)

$$
\begin{aligned}
&\frac{E}{\rho\omega^2}\frac{d^2\sigma_x^{(0)}}{dx^2} + \sigma_x^{(0)} - \mu\sigma_y^{(0)} = 0, \\
&\frac{8\sigma_y^{(0)}}{\rho\omega^2 h^2} - \frac{\sigma_y^{(0)} - \mu\sigma_x^{(0)}}{E} = 0;
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&12h^2\omega_0^2\frac{d^2\sigma_x^{(1)}}{dx^2} + l\left(24\omega_0^2 - \mu\omega^2\right)\frac{d\tau_{xy}^{(0)}}{dx} \\
&+12\omega^2\sigma_x^{(1)} = 0, \\
&lh^2\omega_0^2\frac{d^2\tau_{xy}^{(0)}}{dx^2} + 6h^2\omega_0^2\frac{d\sigma_x^{(1)}}{dx} \\
&+3l\left((1+\mu)\omega^2 - 4\omega_0^2\right)\tau_{xy}^{(0)} = 0;
\end{aligned} \tag{16}
$$

where ω_0 is a characteristic frequency defined by the relation $\omega_0^2 = E/\rho h^2$. Equations (15) and (16) form a system of ordinary differential equations with respect to $\sigma_x^{(0)}$, $\sigma_x^{(1)}$, $\tau_{xy}^{(0)}$, and $\sigma_y^{(0)}$, which have to be solved taking into account boundary conditions (12). It should be noted that Eqs. (15) approximately describe the tension and compression of the plate (longitudinal oscillations), while Eqs. (16) describe its bending (lateral oscillations).

4. LONGITUDINAL VIBRATIONS

Equations (15) can be reduced to an ordinary differential equation with respect to the unknown stress function $\sigma_x^{(0)}$:

$$
\frac{d^2\sigma_x^{(0)}}{dx^2} + \lambda^2\sigma_x^{(0)} = 0, \tag{17}
$$

$$
\lambda^2 = \frac{\omega^2}{\omega_0^2 h^2}\left(1 - \frac{\omega^2\mu^2}{\omega^2 - 8\omega_0^2}\right). \tag{18}
$$

The other unknown stress function $\sigma_y^{(0)}$ is found from the second equation of system (15). In contrast to the classical equation describing longitudinal oscillations of a beam in terms of the displacement function u, expression (17) contains a parameter λ, which nonlinearly depends on the frequency ω and the system parameters μ, h, and ω_0^2. The values of $\lambda^2(\omega)$ are positive for $\omega \in (0,\omega_1) \cup (\omega_2,\infty)$ and negative for $\omega \in (\omega_1,\omega_2)$, where

$$
\begin{aligned}
&\omega_1^2 = 8\omega_0^2, \quad \lambda(\omega_1) = \infty, \\
&\omega_2^2 = \frac{8\omega_0^2}{1-\mu^2}, \quad \lambda(\omega_2) = 0.
\end{aligned} \tag{19}
$$

It can be shown that boundary value problem (12), (17) has no nontrivial solutions for $\lambda^2(\omega) \le 0$, while for $\lambda^2(\omega) > 0$ the solution is $\sigma_x^{(0)} = c\sin(\lambda x)$.

The characteristic equation for eigenfrequencies ω can be written as

$$
\frac{\omega}{\omega_0}\sqrt{1 - \frac{\omega^2\mu^2}{\omega^2 - 8\omega_0^2}} = \varepsilon n, \varepsilon = \frac{\pi h}{l}, n \in \mathbb{N}. \tag{20}
$$

This equation has two positive roots ω_+ and ω_-, which can be expressed analytically as functions of integer $n \ge 0$:

$$
\begin{aligned}
&\omega_\pm(n) = \sqrt{\frac{8 + \varepsilon^2 n^2 \pm \sqrt{R}}{2(1-\mu^2)}}\omega_0, \\
&R = 64 + \varepsilon^2 n^2\left[\varepsilon^2 n^2 - 16(1 - 2\mu^2)\right].
\end{aligned} \tag{21}
$$

Figure 1 (solid curves) shows the plots of $\omega_\pm(n)$ calculated for the following parameters: $\omega_0 = 1$, $\varepsilon = \pi/10$, and $\mu = 0.3$. The horizontal dashed lines indicate the values $\omega = \omega_1 = 2\sqrt{2} \approx 2.828$ and $\omega = \omega_2 \approx 2.965$. The sloped dashed line represents the classical solution for a beam with the same parameters. The eigenvalues are determined from these plots for the corresponding integers n. It should be noted that the behavior of $\omega_-(n)$ for the first several n was established and explained by Rayleigh.

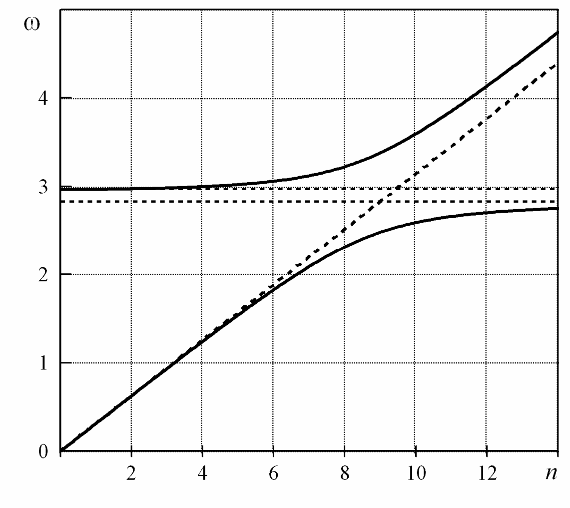

Fig. 1. Longitudinal eigenfrequencies ω versus n

Figure 2 presents the shapes of stress eigenfunctions $\sigma_x^{(0)}$ and $\sigma_y^{(0)}$ for the values of parameters indicated above. Here, solid curves show the distributions of $\sigma_x^{(0)}$ for $n=1$ and $n=9$, the dashed curve shows $\sigma_y^{(0)}$ that corresponds to ω_+ for $n=1$ (the values of $\sigma_y^{(0)}$ for $\omega=\omega_-(1)$ are not shown because $\left|\sigma_y^{(0)}/\sigma_x^{(0)}\right| \ll 1$), and the dash–dot curve shows $\sigma_y^{(0)}$ for $\omega=\omega_-(9)$. An important feature of these longitudinal oscillations is that $\sigma_y^{(0)}/\sigma_x^{(0)}<0$ for the lower branch $\omega_-(n)$ and $\sigma_y^{(0)}/\sigma_x^{(0)}>0$ for the upper branch $\omega_+(n)$. It is worth also noting that, in the system with the indicated parameters, the maximum values of $\sigma_x^{(0)}$ and $\sigma_y^{(0)}$ for the two branches of solution (21) at $n=9$ are almost equal.

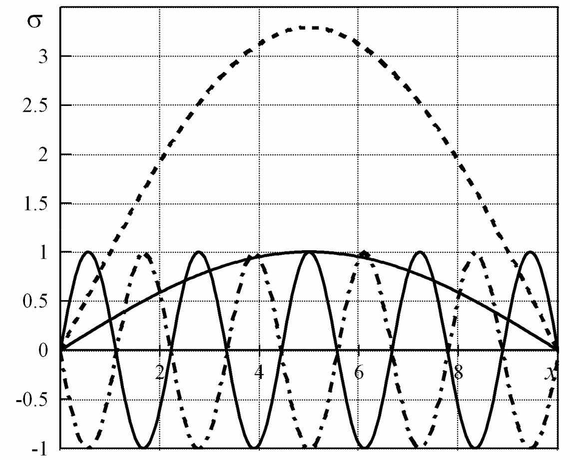

Fig. 2. Longitudinal eigenstresses σ_x and σ_y

5. LATERAL VIBRATIONS

Explicitly expressing a stress component $\tau_{xy}^{(0)}$ from system (16) as

$$\begin{aligned} &\tau_{xy}^{(0)} = A\frac{d^3\sigma_x^{(1)}}{dx^3} + B\frac{d\sigma_x^{(1)}}{dx}, \\ &A = \frac{h^4\omega_0^4}{l\left((1+\mu)\omega^2 - 4\omega_0^2\right)\left(\mu\omega^2 - 24\omega_0^2\right)}, \\ &B = \frac{h^2\omega_0^2\left[24\omega_0^2 + (2-\mu)\omega^2\right]}{2l\left((1+\mu)\omega^2 - 4\omega_0^2\right)\left(\mu\omega^2 - 24\omega_0^2\right)}, \end{aligned} \tag{22}$$

we obtain a differential equation of the fourth order for $\sigma_x^{(1)}$:

$$\begin{aligned} &2h^4\omega_0^4\frac{d^4\sigma_x^{(1)}}{dx^4} + (8+5\mu)h^2\omega_0^2\omega^2\frac{d^2\sigma_x^{(1)}}{dx^2} \\ &+6\omega^2\left((1+\mu)\omega^2 - 4\omega_0^2\right)\sigma_x^{(1)} = 0. \end{aligned} \tag{23}$$

Note that the denominator in expression (22) vanished at

$$\omega = \omega_{3,4},\ \omega_3^2 = \frac{4\omega_0^2}{1+\mu},\ \omega_4^2 = \frac{24\omega_0^2}{\mu}. \tag{24}$$

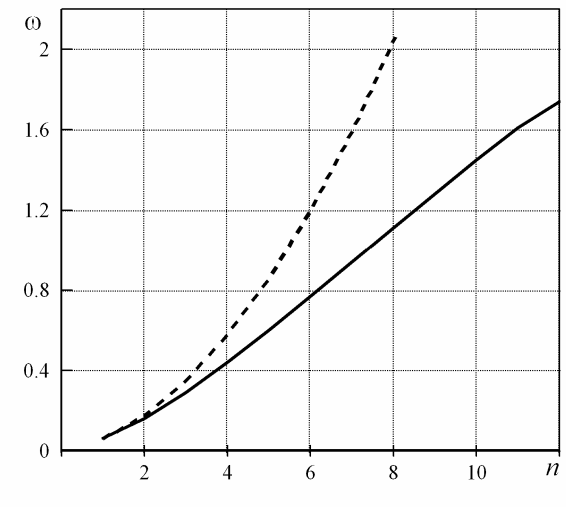

Fig. 3. Lateral eigenfrequencies ω and ω_c versus n

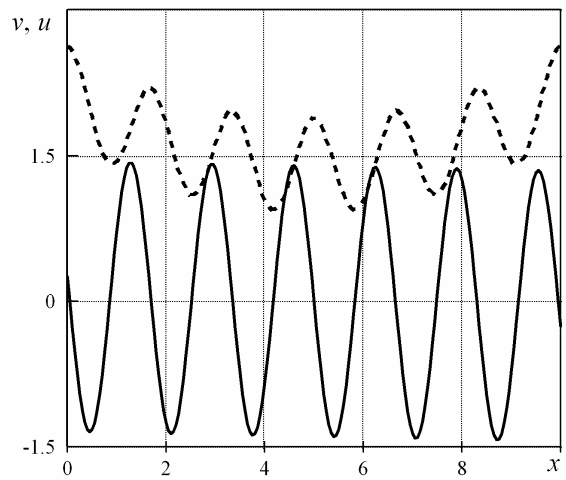

Fig. 4. Lateral eigenforms u and v

The roots $\kappa_i(\omega)$ ($i=1,2,3,4$) of the characteristic biquadratic equation can be analytically expressed:

$$\begin{aligned}&\kappa^2=-\frac{\omega}{4\omega_0 h^2}\Big[(8+5\mu)\omega\pm\sqrt{Q}\Big],\\&Q=(16+32\mu+25\mu^2)\omega^2+192\omega_0^2.\end{aligned}\qquad(25)$$

It is follows from this relation that there are two complex conjugate imaginary roots for any $\omega>0$. When $\omega<\omega_3$, the two remaining roots are real, while for $\omega>\omega_3$ they are imaginary. The eigenfrequencies ω are determined using the condition of nontrivial solutions for the boundary value problem (12), (23).

Figure 3 (solid curve) shows plots of the eigenfrequencies versus n of the first twelve eigenmodes for the plate parameters $h=1$, $l=10$, $\mu=0.3$, and $\omega_0=1$. For comparison, the dashed curve shows several lower eigenfrequencies of lateral oscillations for a classical beam with the same parameters. Figure 4 presents the shapes of the displacement eigenfunctions $u(x,h/2)$, (dashed curve) and $v(x,0)$ (solid curve) calculated using relations (9) and (11) with $\omega=\omega(12)$. It should be noted that the shape of oscillations at $n=1$ is characterized by bending displacements (shear deformations are virtually absent). As n increases, the shear component more significantly influences on the shape of free oscillations. In particular, the function $u(x,h/2)$ is positive at $n=12$.

5. CONCLUSIONS

Based on the method of integrodifferential relation it is shown that there are two different kinds of the eigendisplacements and internal eigenstresses for longitudinal vibrations of a rectilinear elastic beam. For lateral vibrations it is found two frequency zones corresponded to different solution types of the biquadrate characteristic equation.

ACKNOWLEDGMENTS

This research was supported by the Russian Foundation for Basic Research (grants Nos. 05-01-00563 and 05-08-18094) and the Program “State Support for Leading Scientific Schools” (NSh-1245.2006.1, NSh-9831.2006.1).

REFERENCES

Donnell, L.H. (1976). *Beams, plates and shells*. McGraw-Hill, New York.

Kostin, G.V. and V.V. Saurin (2005). Itegro-differential approach to solving problems of linear elasticity. *Doklady Physics*, **50**(10), pp. 535–538

Kostin, G.V. and V.V. Saurin (2006a) The method of integrodifferential relations for linear elasticity problems // Archive of Applied Mechanics. **76**(7–8), pp. 391–402.

Kostin, G.V. and V.V. Saurin (2006b) Variational approaches in the beam theory. *Mechanics of Solids.* **41**(1).

Kostin, G.V. and V.V. Saurin (2006c) Free Beam Oscillations. *Doklady Physics,* **51**(12), pp. *680–684.*

Strutt, J.W. (Baron Rayleigh) (1926) *Theory of sound.* V. 1. MacMillan, London.

Timoshenko, S. (1956) *Strength of materials. Pt 1. Elementary theory and problems.* D. Van Nostrand Reinhold, Princenton:

Timoshenko, S.P. and J.N. Goodier (1970) *Theory of elasticity*. McGraw, New York.