# SWING UP A DOUBLE PENDULUM BY SIMPLE FEEDBACK CONTROL

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**Abstract**

A real double pendulum is steered to a neighborhood of the upper equilibrium position by a simple swing-up feedback control. The moment of force generated at the suspension point is used as a control. Good controllability of the pendulum is demonstrated. The theoretical conclusions and the efficiency of the control algorithms are confirmed by the experiment.

**Key words**

Double pendulum, swing-up feedback control, experiment.

## 1 Introduction

The problem of controlling an inverted double pendulum is considered. A feedback control (Reshmin, 2005) is applied for steering this system to a neighborhood of the upper unstable equilibrium position. Proposed control law is an extension of the approaches (Reshmin, 1997; Reshmin and Chernousko, 1998) to the underactuated systems. In (Reshmin and Chernousko, 1998), the problem of controlling a nonlinear system of order $2n$ is reduced to a problem of controlling a system of $n$ simple independent second-order equations. After that all nonlinearities are considered as independent perturbations, and the control for each subsystem is constructed based on the game-theoretic approach (Krasovskii, 1970).

The main problems that arise in the solution of the control problem for the investigated system are connected with the fact that it is an essentially nonlinear dynamic system. Dynamic interaction between different degrees of freedom is typical of this system. Another complicating factor is that the number of controls in the system is half as many as the number of degrees of freedom.

Many publications have been devoted to the control of pendulum systems. Frequently, using such well-known systems, new control methods are approved, and their operation and efficiency is demonstrated. In (Formal'skii, 2006), the problem of stabilizing the vertical unstable equilibrium position of a double pendulum with a fixed suspension point was solved. In (Schaefer and Cannon, 1996), the problem of stabilizing an inverted multi-link pendulum was solved using horizontal movements of the suspension point. Some other control methods applicable to double pendulum were developed in (Absil and Sepulchre, 2001; Fantoni *et al.*, 2000; Rubi *et al.*, 2002; Sanfelice and Teel, 2007). Note that the distinctive feature of our swing-up control strategy is use of the game-theoretic approach.

## 2 Double pendulum

We consider a double pendulum controlled by a torque applied to the suspension axis, see Fig. 1. The pendulum consists of two rigid links $B_1$ and $B_2$. The revolute joint $O_1$ with a horizontal axis attaches the link $B_1$ to a fixed base $B_0$. The links $B_1$ and $B_2$ are connected by the revolute joint $O_2$ the axis of which is parallel to that of $O_1$. The motion of such a system occurs in a vertical plane. The center of mass $C_1$ of the link $B_1$

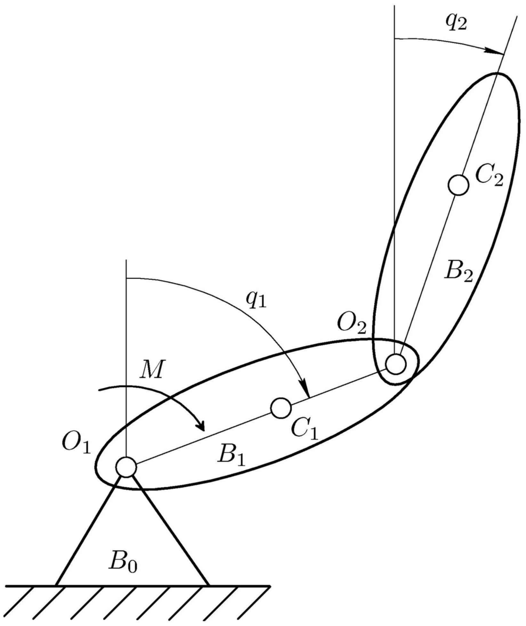


Figure 1. Double pendulum.

lies on the ray $O_1O_2$. The center of mass $C_2$ of the link $B_2$ does not lie on the axis of the joint $O_2$. The system is controlled by the torque $M$ applied to the joint $O_1$.

## 3 Equations of motion

The motion of this system is governed by Lagrange's equations

$$(m_2 l_1^2 + I_1)\,\ddot q_1 + m_2 l_1 l_{g2}\cos(q_2 - q_1)\,\ddot q_2$$
$$-m_2 l_1 l_{g2}\sin(q_2 - q_1)\,\dot q_2^2 = M + G_1^0 \sin q_1 + v_1$$

$$m_2 l_1 l_{g2}\cos(q_2 - q_1)\,\ddot q_1 + I_2\,\ddot q_2$$
$$+m_2 l_1 l_{g2}\sin(q_2 - q_1)\,\dot q_1^2 = G_2^0 \sin q_2 + v_2$$

$$G_1^0 = g(m_1 l_{g1} + m_2 l_1), \qquad G_2^0 = g m_2 l_{g2} \tag{1}$$

where $q_i$ is the angle of deflection of the link $B_i$ from the vertical; $l_{gi}$ is the length of the segment $O_iC_i$; $l_1$ is the length of the segment $O_1O_2$; $m_i$ is the mass of the link $B_i$; $I_i$ is the moment of inertia of the link $B_i$ about the axis of the joint $O_i$; $G_i^0 \sin q_i$ is the torque created by the gravity force at the joint $O_i$; $v_i$ is the torque created by the disturbances at the joint $O_i$; and $g$ is the acceleration due to gravity.

The control torque $M$ is subjected to the constraint

$$|M| \le M^0 \tag{2}$$

where $M^0$ is a positive constant. Constraints are also imposed on the disturbances

$$|v_1| \le v_1^0, \qquad |v_2| \le v_2^0 \tag{3}$$

where $v_1^0 \ge 0$ and $v_2^0 \ge 0$ are given constants.

Introduce the dimensionless variables

$$t' = \left(\frac{M^0}{m_2 l_1 l_{g2}}\right)^{1/2} t, \qquad M' = \frac{M}{M^0}$$
$$G_i^{0\prime} = \frac{G_i^0}{M^0}, \qquad v_i' = \frac{v_i}{M^0}, \qquad v_i^{0\prime} = \frac{v_i^0}{M^0} \tag{4}$$
$$\alpha = \frac{I_1 + m_2 l_1^2}{m_2 l_1 l_{g2}}, \qquad \beta = \frac{I_2}{m_2 l_1 l_{g2}}$$

If we omit the superscript $'$ in the notation of the new variables $t'$, $M'$, $G_i^{0\prime}$, $v_i'$, and $v_i^{0\prime}$ then equations (1) and constraint (2) take the form

$$\alpha\,\ddot q_1 + \cos(q_2 - q_1)\,\ddot q_2 = M + w_1$$
$$\cos(q_2 - q_1)\,\ddot q_1 + \beta\,\ddot q_2 = w_2$$
$$w_1 = G_1^0 \sin q_1 + \sin(q_2 - q_1)\,\dot q_2^2 + v_1 \tag{5}$$
$$w_2 = G_2^0 \sin q_2 - \sin(q_2 - q_1)\,\dot q_1^2 + v_2$$

$$|M| \le 1 \tag{6}$$

but at the same time the form of constraints (3) does not change.

## 4 First integrals

Consider now situation (coordinated mode) when control $M$ is chosen so that

$$x = q_2(t) - q_1(t) \equiv \text{const} \tag{7}$$

Substituting (7) into the second equation of (5), we obtain

$$(\beta + \cos x)\ddot q_1 = G_2^0 \sin(q_1 + x) - \sin x\,\dot q_1^2 + v_2 \tag{8}$$

It is worth mentioning that equation (8) looks like the equation of motion of a common physical pendulum to which the term -$\sin x\,\dot q_1^2$ has been added. This term is quadratic in the velocity $\dot q_1$.

If additionally to (7), we assume that

$$v_2(t) \equiv \text{const} \tag{9}$$

then equation (8) can be integrated

$$\exp\left(q_1 \tan\varphi\right) \times \left[\dot q_1^2 + \frac{G_2^0 \cos(q_1 + x + \varphi)\sin\varphi - v_2}{\sin x}\right] = \text{const}$$

$$\varphi = \arctan\left(\frac{2\sin x}{\beta + \cos x}\right) \tag{10}$$

In the simplest case, when

$$x = 0 \qquad \text{and} \qquad v_2 = 0$$

equation (8) completely coincides with the equation of motion of the equivalent single pendulum

$$(\beta + 1)\ddot q_1 = G_2^0 \sin q_1 \tag{11}$$

that has the first integral

$$E_0 = \frac{\beta+1}{2}\dot q_1^2 + G_2^0(\cos q_1 - 1) \equiv \text{const} \tag{12}$$

We consider $E_0(q_1, \dot q_1)$ as a virtual energy of the double pendulum when $q_1(t) \approx q_2(t)$ and $v_2(t) \approx 0$.

## 5 Problem statement

The following control problem can be formulated.

**Problem 1.** *Find a feedback control $M(q_1, \dot q_1, q_2, \dot q_2)$ that satisfies (6) and steers system (5) from the given initial state*

$$q_1(0) = \pi, \qquad q_2(0) = \pi$$
$$\dot q_1(0) = 0, \qquad \dot q_2(0) = 0 \tag{13}$$

*to the prescribed neighborhood of the upper equilibrium position*

$$|q_1 - 2\pi k| < \varepsilon, \quad |q_2 - 2\pi m| < \varepsilon$$
$$k, m = 0, \pm 1, \pm 2, \ldots \tag{14}$$
$$|\dot q_1| < \varepsilon', \quad |\dot q_2| < \varepsilon'$$

*where $\varepsilon$ and $\varepsilon'$ are given constants which can be arbitrarily small.*

**Remark 1.** *The goal is only to take the state of the system to a neighborhood of the upright equilibrium point. It is not necessary then to keep the pendulum for all time.*

We make certain simplifying assumptions concerning the possibilities of the control $M$ and the disturbances $v_1$ and $v_2$. It is assumed that on the one hand, constant $M^0$ in (2) is not too small and, on the other hand, constants $v_1^0$ and $v_2^0$ in (3) are not very large.

## 6 Swing-up feedback control

A bounded feedback control $M(q_1, \dot q_1, q_2, \dot q_2)$ which satisfies (2) and brings the system (5) from the initial state (13) to the terminal state (14) in a finite time for any admissible disturbances $v_1$ and $v_2$ satisfying (3) can be taken in the form

$$M = \text{sign}(\dot x - \tilde\psi), \qquad \dot x \neq \tilde\psi$$
$$M = \text{sign}(\dot x), \qquad \dot x = \tilde\psi \tag{15}$$

where $x$ is the angle between the links

$$x = q_2 - q_1, \qquad \dot x = \dot q_2 - \dot q_1 \tag{16}$$

and $\tilde\psi(x)$ is a switching function defined by the following relations:

$$\tilde\psi(x) = \psi(x - \tilde x)$$
$$\psi(\cdot) = -(2X|\cdot|)^{1/2}\text{sign}(\cdot) \tag{17}$$
$$\tilde x = -f \,\text{sign}\, \dot q_1, \qquad (\text{sign}\, 0 = -1)$$

Here, $X$ and $f$ are positive control parameters to be identified. This control is of the bang-bang type and switches between the two limiting vales, i.e., $M = \pm 1$. The switching curve $\dot x = \tilde\psi(x)$ consists of two parabolic arcs symmetric with respect to the point $(\tilde x, 0)$, see Fig. 2. Note that the variable $\tilde x$ and the velocity $\dot q_1$ change in sign simultaneously.

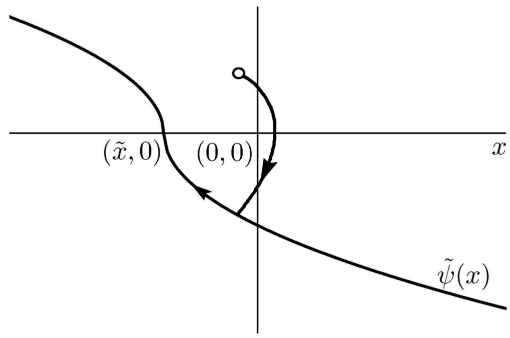


Figure 2. Switching curve in case $\dot q_1 > 0$.

In (Reshmin, 2005), we have proved that the control law of (15)–(17) can be used for the solution of Problem 1 and have obtained a system of inequalities (bounds) for the admissible values of the control parameters $X$ and $f$. It is necessary to find such control parameters that provide positive and at the same time not very high swing up intensity (increasing the oscillation amplitude). A specific procedure for choosing or calculating these parameters has been also proposed. Below, we only describe the main points of the proof.

The control (15)–(17) proposed above has a structure that tends to maintain a pendulum configuration conserving the angle between the links equal to $\tilde{x}$. However, by (17), the quantity $\tilde{x}$ and $\dot{q}_1$ reverse sign simultaneously. As a result, oscillating process can be divided into alternating stages corresponding to the coordinated ($x = \tilde{x}$) and the transient ($x \neq \tilde{x}$) control modes. Analysis of the first integral (10) shows that the amplitude of the oscillations increases during each stage with coordinated mode. The influence of the stages with the transient modes is inessential. For sufficiently small values $f$, the growth rate of the swing of the pendulum oscillations is low. As a result, there comes a time when the pendulum is in the vicinity of the upper equilibrium position with sufficiently small velocities of links. Thus, the source system (5) is in a neighborhood of the equilibrium position (14).

## 7 Modifications of the control law

The control of (15)–(17) can be modified in different ways.

Firstly, the control time in Problem 1 can be significantly shorten, if we replace constant positive parameter $f$ in (17) by a function

$$f(q_1, \dot{q}_1) = -\frac{f_0 E_0(q_1, \dot{q}_1)}{2G_2^0} \qquad (18)$$

where $f_0$ is a new constant parameter and $E_0$ is the virtual energy defined by (12). After such replacement, admissible control parameters $X$ and $f_0$ should be identified. Note that according to (18)

$$f = f_0 \qquad (19)$$

at the beginning and

$$f \approx 0 \qquad (20)$$

at the end of the control process. As a result, the swinging intensity (which can be sufficiently large at the beginning) gradually decreases during the control process.

Secondly, the control of (15)–(17) can be modified to perform both the swing up task and the local stabilization task. Numerical simulation for this more complex problem is quite satisfactory.

## 8 Experimental rig

The triple pendulum rig built on January 2005 in the Department of Automatics and Biomechanics, has been used for the experimental control of the double pendulum (see Fig. 3). For more details see works (Awrejcewicz *et al.*, 2008; Awrejcewicz *et al.*, 2007). The double pendulum rig has been obtained just by the elimination of the third link together with the elements of the third axis. The resulting double pendulum is used in swinging up to the upper vertical position and control experiments. The control has two two-stage regulation character, because the input signal (i.e. the external forcing), can have only two values: $M = \pm 1.718$Nm. The control system allows also switching off the external forcing (i.e. setting $M = 0$), but this possibility has not been used.

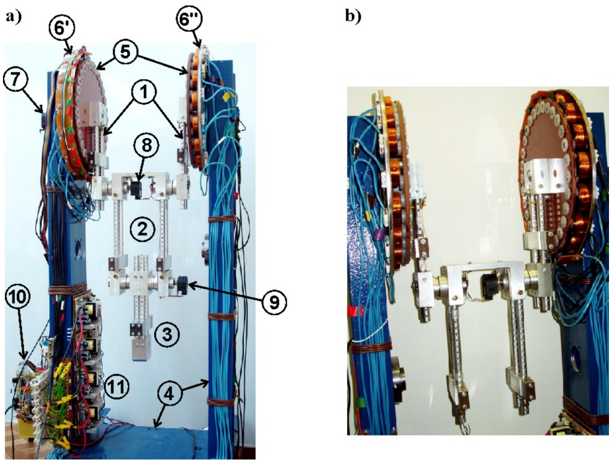


Figure 3. Experimental rig: a) triple pendulum; b) double pendulum. 1 - first link ($B_1$), 2 - second link ($B_2$), 3 - third link; 4 - stand; 5 - motor rotor; 6',6" - motor stators; 7, 8, 9 - rotational potentiomenters; 10 - electronic control system of the motor supplying; 11 - impulse feeders.

In order to obtain approximation of the pendulum parameters, two different cases of free motion of the double pendulum have been recorded. Then the parameter estimation has been performed by the use of special program developed by G. Kudra, giving the following results: $l_1 = 174$ mm, $m_1 = 3.68$ kg, $m_2 = 1.565$ kg, $l_{g1} = 60.2$ mm, $l_{g2} = 89.9$ mm, $I_1 = 0.0403$ kgm$^2$, $I_2 = 0.0140$ kgm$^2$. Since the double pendulum presented here is a special case of the triple pendulum, for more information about mathematical model, parameters and parameters' estimation procedure see works (Awrejcewicz *et al.*, 2008; Awrejcewicz *et al.*, 2007), where the triple pendulum was investigated.

## 9 The LabView control implementation

The special program has been developed by the use of the LabView environment, which controls the double pendulum due to the algorithm prepared by S. Reshmin. The voltage signals from the rotational potentiomenters are directly led to the National Instruments measurement card of type PXI-6052E. Figure 4 presents main block of the program.

In order to swing up the pendulum from its downward stable equilibrium position to the upright unstable equi-

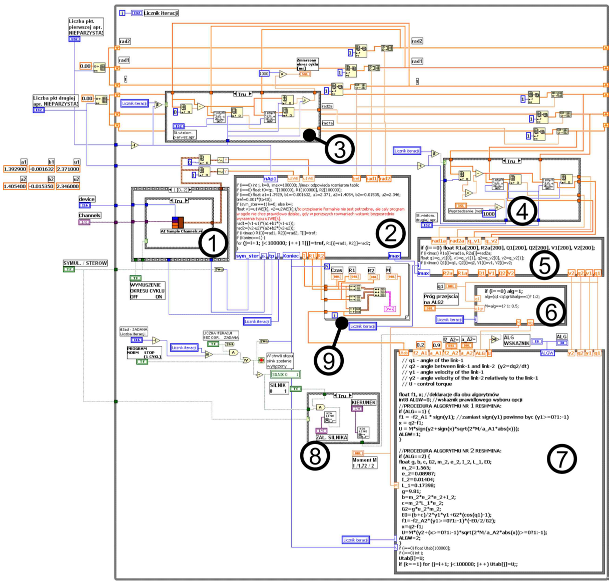

Figure 4. The main block diagram of the program for the pendulum control, developed by the use of the LabView system. Program components: 1 - the time and the rotational potentiometers voltage measurements, 2 - time and voltage processing, data tables creating, 3 - the first approximation of the angular positions data, 4 - the second approximation and the lead time choice, 5 - readout of the angular positions and velocities, 6 - the algorithm choice, 7 - the control algorithm, 8 - lead of the forcing turn signal, 9 - registration of angles and forcing in the control algorithm.

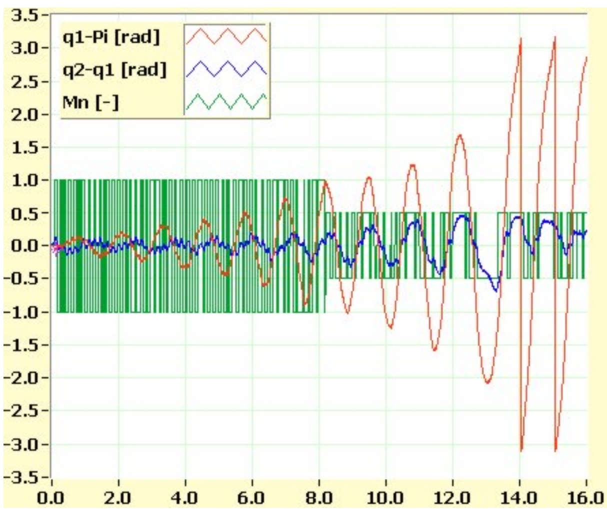


Figure 5. The experimental time history of the swinging up process.

librium, the experiment has been performed. In Fig. 5, the exemplary time history of the control process is presented. For easier observation of the angular position of the first link, the $q_1 - \pi$ (red line) is plotted instead of $q_1$ in Fig. 5. The gradual increase of the amplitude of the first link can be observed during the experiment till to the 14th second, when the link passes the upright vertical position slowing down a bit and then rotates again. For this moment, we cannot stabilize the upright vertical equilibrium position. Deviation of the angular position of each the link from the upright vertical equilibrium position during the motion do not exceed 0.5 rad (blue line in the diagram). The amplitude of the external forcing in each the experiment is the same and is equal to 1.718 Nm. The green line on the diagram presents the computational value $M_n$ used in the control algorithms.

## 10 Conclusions

The control presented above is applied to the real underactuated system with uncertainties. The control is a feedback control of the form $M(q_1, \dot{q}_1, q_2, \dot{q}_2)$ and is constructed for any parameters of the pendulum links. The geometric constraints imposed on the controlling torque produced at the suspension point are taken into account.

The approach is based on the decomposition of the system into simple subsystems. Methods of optimal control and differential games are used to obtain explicit formulas for feedback control. The control law proposed has a simple structure.

This approach does not assume the external forces to be known; they can be uncertain, and only bounds on them are essential. The obtained feedback control is robust, i.e., it can cope with additional small disturbances and parameter variations.

The control strategy presented in section 6 allows performing both the swing up and the local, for a short time instant, stabilization tasks (it will be shown on a video movie during presentation).


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