STATE ESTIMATION FOR UNCERTAIN SYSTEMS WITH ARBITRARY QUADRATIC NONLINEARITY

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Abstract
This paper deals with the state estimation problem for uncertain dynamical control systems with a special structure, in which the nonlinear terms in the right-hand sides of related differential equations are quadratic in state coordinates. We do not impose the condition of positive definiteness of the quadratic nonlinearity in the system. The external ellipsoidal estimates of reachable sets of the system are constructed assuming that initial system states are unknown but bounded.

Key words
Control systems, nonlinearity, uncertainty, ellipsoidal estimates.

1 Introduction

In this paper we study control systems with unknown but bounded uncertainties related to the case of a set-membership description of uncertainty [Schweppe, 1973; Krasovskii and Subbotin, 1974; Bertsekas, 1995; Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2014; Chernousko, 1994; Walter and Pronzato, 1997; Milanese and Vicino, 1991; Milanese, Norton, Piet-Lahanier and Walter, 1996; Witsenhausen, 1968; Gusev, 2012]. The motivation to consider the set-membership approach is that in traditional formulations the characterization of parameter uncertainties requires assumptions on mean, variances or probability density function of errors. However in many applied areas ranged from engineering problems in physics to economics as well as to biological and ecological modeling it occurs that a stochastic nature of the error sequence is questionable [Apreutesei, 2009; August and Koeppl, 2012; August, Lu and Koeppl, 2012; Ceccarelli, Di Marco, Garulli, and Giannitrapani, 2004; Kuntsevich and Volosov, 2015]. For instance, in case of limited data or after some non-linear transformation of the data, the presumed stochastic characterization is not always valid. Hence, as an alternative to a stochastic characterization a so-called bounded-error characterization, also called set-membership approach, has been proposed and intensively developed in the last decades.

The solution of many control and estimation problems under uncertainty involves constructing reachable sets and their analogs. For models with linear dynamics under such set-membership uncertainty there are several constructive approaches which allow finding effective estimates of reachable sets. We note here two of the most developed approaches to research in this area. The first one is based on ellipsoidal calculus [Chernousko, 1994; Kurzhanski and Valyi, 1997; Polyak, Nazin, Durieu and Walter, 2004; Ovseevich and Taraban'ko, 2007] and the second one uses the interval analysis [Walter and Pronzato, 1997].

Many applied problems are mostly nonlinear in their parameters and the set of feasible system states is usually non-convex or even non-connected. The key issue in nonlinear set-membership estimation is to find suitable techniques, which produce related bounds for the set of unknown system states without being too computationally demanding. Some approaches to the nonlinear set-membership estimation problems and discrete approximation techniques for differential inclusions through a set-valued analogy of well-known Euler's method were developed in [Chahma, 2003; Dontchev and Lempio, 1992; Baier, Gerdts and Xausa, 2013; Häckl, 1996; Mazurenko, 2012].

In this paper the modified state estimation approaches which use the special quadratic structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus [Kurzhanski and Valyi, 1997; Chernousko, 1994] are presented. The special case when the quadratic form in the equations of dynamics of the controlled system may be not positive definite is studied. The studies in this direction are motivated also by applications [Kuntsevich and Volosov, 2015; Volosov and Kuntsevich, 2012].

Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented. The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory.

2 Problem Formulation

In this section we introduce the following basic notations. Let R^n be the n–dimensional Euclidean space, $R^{n\times n}$ stands for the set of all $n \times n$–matrices and $x'y$ be the usual inner product of $x, y \in R^n$ with prime as a transpose, $\|x\| = (x'x)^{1/2}$. We denote as $B(a, r)$ the ball in R^n, $B(a, r) = \{x \in R^n : \|x-a\| \le r\}$, I is the identity $n \times n$-matrix. Denote by $E(a, Q)$ the ellipsoid in R^n, $E(a, Q) = \{x \in R^n : (Q^{-1}(x-a), (x-a)) \le 1\}$ with center $a \in R^n$ and symmetric positive definite $n \times n$–matrix Q, for any $n \times n$–matrix $M = \{m_{ij}\}$ denote

$$Tr(M) = \sum_{i=1}^{i=n} m_{ii}.$$

Consider the following system

$$\begin{gathered}\dot{x} = A(t)x + f(x)d + u(t),\\ x_0 \in X_0,\ t_0 \le t \le T,\end{gathered} \tag{1}$$

where $x, d \in R^n$, $\|x\| \le K$ $(K > 0)$, the $n \times n$–matrix $A(t)$ is assumed to be continuous on $t \in [t_0, T]$ and $f(x) = x'Bx$ is scalar function, with a symmetric $n \times n$–matrix B,

$$u(t) \in U, \quad U \subset R^m \quad \text{for a.e. } t \in [t_0, T]. \tag{2}$$

We will assume that X_0 in (1) is an ellipsoid, $X_0 = E(a, Q)$, with a symmetric and positive definite matrix Q and with a center a. Let the absolutely continuous function $x(t) = x(t, u(\cdot), t_0, x_0)$ be a solution to (1).

We will study the solutions of the system (1)–(2) in the framework of the theory of uncertain dynamical systems (differential inclusions [Aubin and Frankowska, 1990; Filippov, 1988; Panasyuk, 1990]) through the techniques of trajectory tubes [Kurzhanski and Filippova, 1993; Kurzhanski and Varaiya, 2014]:

$$\begin{gathered}X(\cdot) = X(\cdot;\ t_0, X_0) = \bigcup \{\ x(\cdot) =\\ x(\cdot, u(\cdot), t_0, x_0) \mid\ x_0 \in X_0,\ u(\cdot) \in U\ \}.\end{gathered} \tag{3}$$

The reachable set $X(t)$ of the system (1) at time t $(t_0 < t \le T)$ is defined as the cross-section of the trajectory tube (3)

$$\begin{gathered}X(t) = X(t;\ t_0, X_0) =\\ \{\ x \in R^n :\ \exists\, x_0 \in X_0,\ \exists\, u(\cdot) \in U\ \text{ such that}\\ x = x(t) = x(t;\ u(\cdot),\ x_0)\ \},\ \ t_0 < t \le T.\end{gathered}$$

The main problem of the paper is to find the external ellipsoidal estimate $E(a^+(t), Q^+(t))$ (with respect to the inclusion of sets) of the reachable set $X(t)$ $(t_0 < t \le T)$ by using the analysis of a special type of nonlinear control systems with uncertain initial data.

3 Preliminaries

We will need some auxiliary constructions and results which will be used in the following.

3.1 Nonlinearity defined by a positive definite quadratic form

Consider the nonlinear control system

$$\begin{gathered}\dot{x} = A(t)x + f(x)d + u(t),\\ x_0 \in X_0 = E(a_0, Q_0), \quad t_0 \le t \le T,\end{gathered} \tag{4}$$

where $x \in R^n$, $\|x\| \le K$ $(K > 0)$, $A(t) \in R^{n\times n}$ is a given continuous matrix, $u(t) \in U = E(\hat{a}, \hat{Q})$; d, a_0, $\hat{a}$ are given n-vectors, a scalar function $f(x)$ has a form $f(x) = x'Bx$, matrices B, Q_0, $\hat{Q}$ are symmetric and positive definite.

Denote the maximal eigenvalue of the matrix $B^{1/2}Q_0B^{1/2}$ by k^2, it is easy to see this k^2 is the smallest number for which the inclusion $X_0 \subseteq E(a_0, k^2B^{-1})$ is true. The following result describes the external ellipsoidal estimate of the reachable set $X(t) = X(t; t_0, X_0)$ $(t_0 \le t \le T)$.

Theorem 1 ([Filippova, 2009]). *Assume that $X_0 = E(a, k^2(B^{-1}))$ (with some $k > 0$), then for all $\sigma > 0$ and for $X(t_0 + \sigma) \subseteq X(t_0 + \sigma, t_0, X_0)$ we have the following upper estimate*

$$X(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0, 1), \tag{5}$$

where $\sigma^{-1}o(\sigma) \to 0$ when $\sigma \to +0$ and

$$\begin{gathered}a^+(\sigma) = a(\sigma) + \sigma\hat{a},\\ a(\sigma) = a + \sigma(A_0a + a'Bad + k^2d),\\ Q^+(\sigma) = (p^{-1} + 1)Q(\sigma) + (p + 1)\sigma^2\hat{Q},\\ Q(\sigma) = k_2^2(I + \sigma R)B^{-1}(I + \sigma R)',\\ R = A + 2da'B.\end{gathered} \tag{6}$$

and where p is the unique positive root of the equation

$$\sum_{i=1}^{n} \frac{1}{p + \alpha_i} = \frac{n}{p(p + 1)}$$

with $\alpha_i \ge 0$ $(i = 1, ..., n)$ being the roots of the following equations $|Q(\sigma) - \alpha\sigma^2\hat{Q}| = 0$.

The following result presents the continuous-type version of the Theorem 1.

Theorem 2 ([Filippova, 2010]). *The inclusion is true for any* $t \in [t_0, T]$

$$X(t; t_0, X_0) \subseteq E(a^+(t), r^+(t)B^{-1}), \tag{7}$$

where functions $a^+(t)$, $r^+(t)$ *are the solutions of the following system of ordinary differential equations*

$$\begin{gathered}\dot{a}^+(t) = A_0 a^+(t) + ((a^+(t))' B a^+(t) + \\ r^+(t))d + \hat{a}, \quad t_0 \le t \le T, \\ \dot{r}^+(t) = \max_{\|l\|=1} \Big\{ l' \big(2r^+(t) B^{1/2}(A_0 + \\ 2d(a^+(t))'B) B^{-1/2} + \\ q^{-1}(r^+(t)) B^{1/2} \hat{Q} B^{1/2}) \big) l \Big\} + q(r^+(t)) r^+(t), \\ q(r) = ((nr)^{-1} \mathrm{Tr}(B\hat{Q}))^{1/2},\end{gathered} \tag{8}$$

with initial state

$$a^+(t_0) = a_0, \quad r^+(t_0) = k^2.$$

Numerical algorithms basing on Theorem 1 and Theorem 2 and producing the external ellipsoidal tube $E^+(t) = E(a(t), Q(t))$ are given in [Filippova, 2012; Filippova and Matviychuk, 2014].

The following example shows that in nonlinear case the reachable sets may lose their convexity with increasing time $t > t_0$. Nevertheless the related external estimates given by Theorems 1–2 are ellipsoidal-valued (and therefore convex) and contain the reachable sets of the system (4).

Example 1. Consider the following control system

$$\begin{cases} \dot{x}_1 = 2x_1 + u_1, \\ \dot{x}_2 = 2x_2 + 4x_1^2 + x_2^2 + u_2, \end{cases} \qquad x_0 \in X_0, \quad t_0 \le t \le T. \tag{9}$$

Here $t_0 = 0$, $T = 0.25$, $X_0 = B(0,1)$ and $U = B(0,1)$. The trajectory tube $X(t)$ and its external ellipsoidal estimating tube $E(a(t), Q(t))$ are shown in Fig. 1.

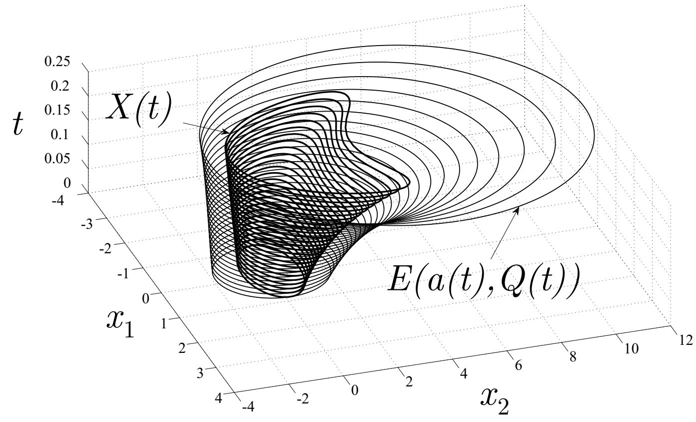

Figure 1. Trajectory tube X(t) and its estimating tube $E(a(t), Q(t))$.

3.2 Systems with two positive definite quadratic forms

Consider the following uncertain differential system with two positive definite quadratic forms in the dynamical equation

$$\begin{gathered}\dot{x} = Ax + f^{(1)}(x)d^{(1)} + f^{(2)}(x)d^{(2)}, \\ x_0 \in X_0, \ t_0 \le t \le T,\end{gathered} \tag{10}$$

where $d^{(1)}$ and $d^{(2)}$ are n-vectors and $f^{(1)}, f^{(2)}$ are scalar functions,

$$f^{(1)}(x) = x' B^{(1)} x, \ \ f^{(2)}(x) = x' B^{(2)} x,$$

with symmetric and positive definite matrices $B^{(1)}, B^{(2)}$. We assume also that $d_i^{(1)} = 0$ for $i = k+1, \ldots, n$ and $d_j^{(2)} = 0$ for $j = 1, \ldots, k$ where k $(1 \le k \le n)$ is fixed. This assumption means that the first k equations of the system (10) contain only the nonlinear function $f^{(1)}(x)$ (with some constant coefficients $d_i^{(1)}$) while $f^{(2)}(x)$ is included only in the equations with numbers $k+1, \ldots, n$.

We will assume as before that X_0 in (10) is an ellipsoid, $X_0 = E(a_0, Q_0)$. We need here some auxiliary results where necessary constructions and additional parameters will be defined.

Lemma 1 ([Filippova and Matviychuk, 2014]). *The following inclusion is true*

$$X_0 \subseteq E(a, k_1^2 (B^{(1)})^{-1}) \bigcap E(a, k_2^2 (B^{(2)})^{-1}) \tag{11}$$

where k_i^2 *is the maximal eigenvalue of the matrix* $(B^{(i)})^{1/2} Q (B^{(i)})^{1/2}$ $(i = 1, 2)$*. The following equalities are true*

$$\begin{gathered}\max_{z' B^{(1)} z \le k_1^2} z' B^{(2)} z = k_1^2 \lambda_{12}^2, \\ \max_{z' B^{(2)} z \le k_2^2} z' B^{(1)} z = k_2^2 \lambda_{21}^2,\end{gathered} \tag{12}$$

where λ_{12}^2 *and* λ_{21}^2 *are maximal eigenvalues of matrices* $(B^{(1)})^{-1/2} B^{(2)} (B^{(1)})^{-1/2}$ *and* $(B^{(2)})^{-1/2} B^{(1)} (B^{(2)})^{-1/2}$ *respectively.*

Theorem 3 ([Filippova and Matviychuk, 2014]). *For all* $\sigma > 0$ *and for* $X(t_0 + \sigma) = X(t_0 + \sigma, t_0, X_0)$

we have the following upper estimate

$$\begin{gathered} X(t_0+\sigma) \subseteq \\ E(a^{(1)}(\sigma), Q^{(1)}(\sigma)) \bigcap E(a^{(2)}(\sigma), Q^{(2)}(\sigma)) \\ + o(\sigma)B(0,1), \end{gathered} \quad (13)$$

where $\sigma^{-1}o(\sigma) \to 0$ *when* $\sigma \to +0$ *and*

$$\begin{gathered} a^{(1)}(\sigma) = a(\sigma) + \sigma k_1^2 \lambda_{12}^2 d^{(2)}, \\ a^{(2)}(\sigma) = a(\sigma) + \sigma k_2^2 \lambda_{21}^2 d^{(1)}, \\ a(\sigma) = (I+\sigma A)a + \\ \sigma a' B^{(1)} a d^{(1)} + \sigma a' B^{(2)} a d^{(2)}, \\ Q^{(1)}(\sigma) = (p_1^{-1}+1)(I+\sigma R) k_1^2 (B^{(1)})^{-1} (I+\sigma R)' \\ +(p_1+1)\sigma^2 ||d^{(2)}||^2 k_1^4 \lambda_{12}^4 \cdot I, \\ Q^{(2)}(\sigma) = (p_2^{-1}+1)(I+\sigma R) k_2^2 (B^{(2)})^{-1} (I+\sigma R)' + \\ (p_2+1)\sigma^2 ||d^{(1)}||^2 k_2^4 \lambda_{21}^4 \cdot I, \\ R = A + 2d^{(1)} a' B^{(1)} + 2d^{(2)} a' B^{(2)} \end{gathered} \quad (14)$$

and p_1, p_2 *are the unique positive solutions of related algebraic equations*

$$\begin{gathered} \sum_{i=1}^{n} \frac{1}{p_1+\alpha_i} = \frac{n}{p_1(p_1+1)}, \\ \sum_{i=1}^{n} \frac{1}{p_2+\beta_i} = \frac{n}{p_2(p_2+1)} \end{gathered} \quad (15)$$

with $\alpha_i, \beta_i \geq 0$ $(i = 1, ..., n)$ *being the roots of the following equations*

$$\begin{gathered} \det((I+\sigma R) k_1^2 (B^{(1)})^{-1} (I+\sigma R)' - \\ \alpha \sigma^2 ||d^{(2)}||^2 k_1^4 \lambda_{12}^4 \cdot I) = 0, \\ \det((I+\sigma R) k_2^2 (B^{(2)})^{-1} (I+\sigma R)' - \\ \beta \sigma^2 ||d^{(1)}||^2 k_2^4 \lambda_{21}^4 \cdot I) = 0. \end{gathered} \quad (16)$$

4 Main results

Consider the general case

$$\begin{gathered} \dot{x} = A(t)x + x'Bx \cdot d + u(t), t_0 \leq t \leq T, \\ x_0 \in X_0 = E(a_0, Q_0), u(t) \in U = E(\hat{a}, \hat{Q}). \end{gathered} \quad (17)$$

We assume here that matrices B, $\hat{Q}$ and Q_0 are symmetric, $\hat{Q}$ and Q_0 are positive definite. This assumption produces more general case than in [Filippova, 2012] because we do not assume here the positive definiteness of the matrix B in the nonlinear term of the right-hand side of dynamic equations (17). This new setting generalizes previous results and is motivated also by applied problems (e.g., [Kuntsevich and Volosov, 2015]).

Using well-known diagonalization procedures of matrix analysis [Bellman, 1997] we can find the nondegenerate $n \times n$–matrix Z of transformation $z = Zx$ $(x, z \in R^n)$ of the state space R^n under which the system (17) will take the form

$$\begin{gathered} \dot{z} = A^*(t)z + z'B^*z \cdot d^* + w(t),\ t_0 \leq t \leq T, \\ z_0 \in Z_0 = E(a_0^*, Q_0^*), \\ w(t) \in W = E(\hat{a}^*, \hat{Q}^*), \end{gathered} \quad (18)$$

where $B^* = \mathrm{diag}\{b_1^*, \dots b_n^*\}$ with b_i^* $(i = 1, \dots, n)$ being the eigenvalues of the matrix B^*. We may assume without loss of generality that $b_i^* = \alpha_i^2$ $(i = 1, \dots, s)$ and $b_i^* = -\beta_i^2$ $(i = i+1, \dots, n)$.

Denote

$$\begin{gathered} f^{(1)}(z) = \sum_{i=1}^{s} \alpha_i^2 z_i^2,\ f^{(2)}(z) = \sum_{i=s+1}^{n} \beta_i^2 z_i^2, \\ d^{(1)} = d^*,\ d^{(2)} = -d^*, \end{gathered} \quad (19)$$

and rewrite the system (18) as

$$\begin{gathered} \dot{z} = A^*(t)z + f^{(1)}(z) \cdot d^{(1)} + f^{(2)}(z) \cdot d^{(2)} + w(t), \\ z_0 \in Z_0 = E(a_0^*, Q_0^*),\ t_0 \leq t \leq T, \\ w(t) \in W = E(\hat{a}^*, \hat{Q}^*). \end{gathered} \quad (20)$$

We see here that the quadratic functions $f^{(1)}(z)$ and $f^{(2)}(z)$ in (20) are of the same form as in (10) except the property of their positive definiteness because in general case both functions $f^{(i)}(z)$ $(i = 1, 2)$ are only positive semidefinite quadratic forms. To avoid this problem, we modify the system (20) as follows. Let $\lambda > 0$ be a small parameter and let

$$\begin{gathered} f_\lambda^{(1)}(z) = \sum_{i=1}^{s} \alpha_i^2 z_i^2 + \lambda^2 \cdot \sum_{i=s+1}^{n} z_i^2, \\ f_\lambda^{(2)}(z) = \lambda^2 \cdot \sum_{i=1}^{s} z_i^2 + \sum_{i=s+1}^{n} \beta_i^2 z_i^2. \end{gathered} \quad (21)$$

We can assume that all parameters α_i^2 $(i = 1, \dots, s)$ and β_i^2 $(i = s+1, \dots, n)$ are positive, otherwise instead of related zeros we may add small positive terms in the same way as before. So instead of the system (20) we have

$$\begin{gathered} \dot{z} = A^*(t)z + f_\lambda^{(1)}(z) \cdot d^{(1)} + f_\lambda^{(2)}(z) \cdot d^{(2)} + w(t), \\ z_0 \in Z_0 = E(a_0^*, Q_0^*), \\ w(t) \in W = E(\hat{a}^*, \hat{Q}^*),\ t_0 \leq t \leq T, \end{gathered} \quad (22)$$

where functions $f_\lambda^{(i)}(z)$ $(i = 1, 2)$ are positive definite quadratic forms.

Using results [Filippova, 2014], we may conclude that if we find the external ellipsoidal estimates for the modified system (22) then they will be close to external ellipsoidal estimate of the original system (18) (and therefore (17)) in the Hausdorff metric for small $\lambda > 0$.

We may formulate now the following scheme that produces the external estimate of trajectory tube $X(t)$ of the system (22) with given accuracy.

Algorithm. Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ ($i = 1, \dots, m$), $h = (T - t_0)/m$, $t_m = T$.

1. Given $Z_0 = E(a_0^*, Q_0^*)$, take $\sigma = h$ and define ellipsoids $E(a^{(1)}(\sigma), Q^{(1)}(\sigma))$ and $E(a^{(2)}(\sigma), Q^{(2)}(\sigma))$ from Theorem 3.
2. Find the smallest (with respect to some criterion [Chernousko, 1994; Kurzhanski and Valyi, 1997]) ellipsoid $E(a^*, Q^*)$ which contains the intersection

$$E(a^*, Q^*) \supseteq$$
$$E(a^{(1)}(\sigma), Q^{(1)}(\sigma)) \bigcap E(a^{(2)}(\sigma), Q^{(2)}(\sigma)).$$

3. From Theorem 1 find the ellipsoid $E(a_1, Q_1)$ which is the upper estimate of the sum [Chernousko, 1994; Kurzhanski and Valyi, 1997] of two ellipsoids, $E(a^*, Q^*)$ and $\sigma E(g, G)$:

$$E(a^*, Q^*) + \sigma E(g, G) \subseteq E(a_1, Q_1).$$

4. Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, Q_1)$ as the initial ellipsoid at instant t_1.
5. Next steps continue iterations 1-3. At the end of the process we will get the external estimate $E(a(t), Q(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.

The following example illustrates the main procedure of the above algorithm.

Example 2. Consider the following control system with two quadratic forms in its dynamical equations:

$$\begin{cases} \dot{x}_1 = 1.5x_1 + x_1^2 + 2x_2^2 + u_1, \\ \dot{x}_2 = 1.5x_2 + 2x_1^2 + x_2^2 + u_2, \end{cases} \quad (23)$$
$$x_0 \in X_0, \qquad t_0 \le t \le T.$$

Here we take $t_0 = 0$, $T = 0.3$, $X_0 = B(0, 1)$ and $U = B(0, 0.1)$. Steps of the Algorithm of constructing the external ellipsoidal estimate $E(a(t), Q(t))$ of the reachable set $X(t)$ are shown in Fig. 2.

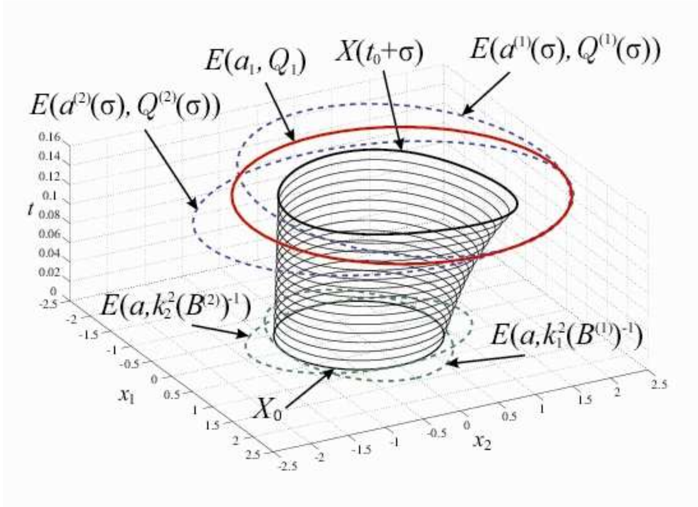

Figure 2. Algorithms of ellipsoidal estimating of the trajectory tube X(t): several steps.

5 Conclusions

The paper deals with the problems of state estimation for uncertain dynamical control systems for which we assume that the initial system state is unknown but bounded with given constraints.

Basing on the results of ellipsoidal calculus developed earlier for linear uncertain systems we present the modified state estimation approaches which use the special quadratic structure of nonlinearity of the control system and allow to construct the external ellipsoidal estimates of reachable sets. The special case when the quadratic form in the equations of dynamics of the controlled system may be not positive definite is studied.

Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented. The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory.

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