DISCONTINUOUS CONTROL DESIGN FOR TRACKING FEASIBLE TRAJECTORIES IN UNDERACTUATED MANIPULATORS

Ollin Penaloza-Mejia
Division de Ingenieria Mecatronica e Industrial
T E S E
Av. Tecnologico s/n, Valle de Anahuac
55220 Ecatepec, Estado de Mexico
openaloza@tese.edu.mx

Joaquin Alvarez Luis A. Marquez-Martinez
Departamento de Electronica y Telecomunicaciones
C I C E S E
Carr. Ensenada-Tijuana No. 3918
Zona Playitas, 22860 Ensenada, B.C
{jqalvar, lmarquez}@cicese.mx

Abstract
A discontinuous controller for tracking feasible trajectories in 2 degrees-of-freedom underactuated manipulators is proposed. The controller is designed as an extension to the underactuated case of the computed torque control approach with a PD-type controller. Provided some conditions were satisfied, the proposal ensures stability of the closed-loop system, and allows to track, simultaneously for all the system variables, feasible trajectories to reach static and non-static configurations under relatively large drifts in initial conditions.The effectiveness of the proposal is illustrated by means of simulations in two underactuated manipulators with different characteristics.

Key words
Discontinuous control, underactuated manipulators, control of mechanical systems.

1 Introduction

Underactuated mechanical systems (UMS) refer to those mechanical systems with less control inputs than degrees-of-freedom (DOF) [Jiang, 2011]. A general class of UMS does not satisfy the necessary Brockett's condition, which translates into the incapability of designing a continuous time-invariant feedback control law for stabilization or tracking [Reyhanoglu et al., 1999]. For this reason, alternative control approaches have been proposed for these systems, like time-varying, oscillatory, delayed, structural-variable, or adaptive control laws (see e.g. [Morin and Samson, 1997], [Hong, 2002], [Olgac and Cavdaroglu, 2011], [Ngo and Hong, 2012], and references therein).

Analysis and control of UMS are interesting and challenging problems due to the clear reduction of the control space, and the inherent constraints in their dynamics, which do not allow the system to track an arbitrary motion. The dynamic constraints are usually non-holonomic, and can depend on position and velocity terms. In this case, they are known as first-order, non-holonomic constraints (FONHC). They can also be a function of position, velocity, and acceleration, in which case are called second-order, non-holonomic constraints (SONHC) [Oriolo and Nakamura, 1991]. FONHC appear in kinematic models of wheeled mobile robots and wheeled vehicles, while SONHC appear in dynamic models of underwater vehicles, space robots, and underactuated manipulators.

Some important problems for UMS include stabilization around an equillibrium point or a manifold of equilibria (see e.g. [Oriolo and Nakamura, 1991], [Spong, 2002], [Fantoni and Lozano, 2002], [Olfati-Saber, 2001], [Grizzle *et al.*, 2005], and references therein), trajectory generation or motion planning (see. e.g. [Nagaragan *et al.*, 2009], and [Miranda, 2011]), and trajectory tracking, which has been mainly investigated for UMS with FONHC.

There are some works where the trajectory tracking problem for specific UMS with SONHC has been studied. In [Berkemeier and Fearing, 1999], the tracking control of fast inverted periodic motions for the Acrobot is considered. In [Begovich *et al.*, 2002], a fuzzy control scheme is proposed for tracking inverted trajectories in the Pendubot. In [Andary *et al.*, 2012], stable limit cycles are achieved for an inertia wheel pendulum by designing a family of parametrized periodic trajectories, and proposing a control scheme based on the model-free approach. Sliding mode control has been used for trajectory tracking in the inverted pendulum [Wang, 2012], and in an underactuated surface vessel [Yu *et al.*, 2012]. In [White *et al.*, 2009], a direct Lyapunov method is applied, and a sliding mode controller is proposed, which involves the solution of the so-called matching equations, as in [Liao and Hou, 2012], where a tracking controller is designed based on controlled lagrangians. In [Zilic *et al.*, 2012], discontinuous control is used for simultaneous stabilization and tracking considering actuator dynamics. In these works, the research has been mainly focused on underactuated systems with an specific application interest, and the proposed methodologies and control approaches cannot be applied to different underactuated devices. In addition, asymptotic stabilization of the tracking errors are not guaranteed in general, and the system is forced to start very close to (or on) the reference trajectory.

In this paper, the trajectory tracking problem for underactuated manipulators (*i.e.* UMS with SONHC [Oriolo and Nakamura, 1991]) is analysed. A discontinuous controller is proposed to track, simultaneously for all the system variables, feasible trajectories to reach static or non-static configurations. The key idea behind our approach is the introduction of a coupling matrix which permits to propose a control structure, and is relatively simple because it is based on an inverse dynamics approach plus a class of PD controller; moreover, it has been designed to obtain a stable closed-loop error dynamics. Provided some conditions were satisfied, the proposed controller ensures exponential convergence of the tracking errors to a small neighbourhood around zero, even under large drifts in initial conditions. Furthermore, provided that feasible trajectories to be tracked were available, the proposed control structure can be applied directly to underactuated manipulators either class-I or class-II (see the definition of these classes in section 2.1). In addition, since our controller does not require to solve on-line the so-called matching equations [White *et al.*, 2009], [Liao and Hou, 2012], the computational cost of the implementation is reduced. Simulation results are provided to show the effectiveness of the proposed control scheme.

2 Problem Formulation

2.1 Underactuated Dynamics and Properties

Let us consider underactuated manipulators with 2-DOF (one-degree of actuation, and one-degree of underactuation), described by the nonlinear matrix equation

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu, \tag{1}$$

where q is the generalized coordinate vector, $M(q)$ is the inertia matrix, $C(q,\dot{q})\dot{q}$ is the vector of Coriolis and centrifugal forces, $G(q)$ is the vector of gravitational forces obtained as the gradient of the potential energy, B is a distribution vector, and u is the control input. All matrices and vectors are defined with appropriate dimensions.

Two classes of underactuated manipulators are defined with respect to B [Zikmund and Moog, 2006]. It is said that the underactuated dynamics is class-I if $B = [0\ 1]^\top$, while it is class-II if $B = [1\ 0]^\top$. In both cases, the dynamics (1) has a SONHC denoted by $N(q,\dot{q},\ddot{q}) = 0$.

In addition, the underactuated dynamics (1) has the following properties [Spong, 2006].

Property 1. *The inertia matrix, and its inverse, are symmetric, and positive definite for all* q. *For some positive constants,* $\underline{\mu}_M \le \overline{\mu}_M$, *and* $\underline{\mu}_I \le \overline{\mu}_I$, *both matrices are lower and upper bounded for all* q *as* $\underline{\mu}_M \le ||M(q)|| \le \overline{\mu}_M$, *and* $\underline{\mu}_I \le ||\big(M(q)\big)^{-1}|| \le \overline{\mu}_I$.

Property 2. *The vector* $C(q,\dot{q})\dot{q}$ *satisfies* $||C(q,\dot{q})\dot{q}|| \le c_0||\dot{q}||^2$, *for all* q, $\dot{q}$, *and a positive constant* c_0.

Property 3. *For revolute joints, the vector* $G(q)$ *satisfies* $||G(q)|| \le g_0$, *for all* q, *and a positive constant* g_0.

2.2 Feasible Trajectories

A SONHC appears due to the reduction of the input space, and this translates into the incapability of the system to track an arbitrary motion. Then, it is required to calculate a set of 'feasible trajectories' to be tracked, that is, reference motions which satisfy $N(q,\dot{q},\ddot{q}) = 0$. So,

$$\mathcal{Q} = \{q_r(t) : N(q_r,\dot{q}_r,\ddot{q}_r) = 0\} \quad t \ge t_0 \tag{2}$$

will denote the set of all feasible trajectories of (1), which will be sufficiently smooth, and bounded for all t. For our purposes, it will be assumed that there is available a set of feasible trajectories to be tracked. Some algorithms for planning such feasible trajectories can be found in [Nagaragan *et al.*, 2009], [White *et al.*, 2009], and [Miranda, 2011].

2.3 Problem Statement

The problem analysed in this paper can be formulated as follows. Given the underactuated manipulator modelled by (1), which satisfies properties 1 to 3, design, if possible, a control law u to achieve the objective

$$\lim_{t\to\infty} ||q(t) - q_r(t)|| \le \epsilon, \tag{3}$$

for a sufficiently small non-negative constant $\epsilon \ge 0$, and a desired feasible trajectory $q_r \in \mathcal{Q}$.

3 Control Scheme

Define the tracking errors

$$\tilde{q} = q - q_r, \qquad \dot{\tilde{q}} = \dot{q} - \dot{q}_r, \qquad \ddot{\tilde{q}} = \ddot{q} - \ddot{q}_r, \tag{4}$$

and consider the control law

$$Bu = M(q)\big(v + \ddot{q}_r + \big(M(q)\big)^{-1}C(q,\dot{q})\dot{q} + G(q)\big), \tag{5}$$

where v is an auxiliary control input yet to be defined. Then, putting (5) in (1), the following ideal error dynamics is obtained

$$\ddot{\tilde{q}} = v, \tag{6}$$

where v can be designed to drive $\tilde{q}$ to the origin, solving the tracking problem. However, the control law cannot be calculated from (5) because B is not a square invertible matrix.

At this point, let us define the coupling matrix

$$D(q) = \big(M(q)\big)^{-1} B\overline{B}M(q), \tag{7}$$

which will play an important role in the definition of the control scheme. It should be noted that this matrix, for any q, is lower and upper bounded, because of Property 1. Then, for some positive constants $\underline{\mu}_D \leq \overline{\mu}_D$, we have that $\underline{\mu}_D \leq ||D(q)|| \leq \overline{\mu}_D$.

After solving explicitly for u in (5), one has

$$u = \overline{B}M(q)\big[v + \ddot{q}_r + \big(M(q)\big)^{-1}\big(C(q,\dot{q})\dot{q} + G(q)\big)\big], \tag{8}$$

where $\overline{B}$ is the Moore-Penrose pseudo-inverse of B, given by

$$\overline{B} = (B^\top B)^{-1}B^\top. \tag{9}$$

Following with (8) and (1), and after some algebraic manipulations, one obtains the error dynamics

$$\begin{aligned}\ddot{\tilde{q}} = D(q)v + [D(q) - I]\big[\ddot{q}_r + \big(M(q)\big)^{-1} \times \\ \big(C(q,\dot{q})\dot{q} + G(q)\big)\big],\end{aligned} \tag{10}$$

that contains, in the second term of the right-hand side, various residual terms due to the model simplification made so far, and it can be seen as a disturbance of the ideal relation (6). Note that the second term of the right-hand side satisfies the matching condition [Khalil, 2002], and this opens the possibility to cancel its (undesirable) effects in the system through v.

Remark 1. *For underactuated devices, $D(q)$ can be seen as a matrix which couples the residual non-linear dynamics that contains the SONHC with the error configuration vector $\ddot{\tilde{q}}$. This observation can be used to extend some control schemes designed for completely actuated systems, to the underactuated case. Note that, if the device were fully-actuated, the matrix $D(q) \equiv I$, and the ideal relation (6) is recovered from (10).*

Now the tracking problem can be solved if it is possible to stabilize (10). To this end, we establish the next result.

Theorem 1. *The dynamics (10), obtained after substituting (8) in (1), can be stabilized around the origin with the control law*

$$v = \phi + \omega, \tag{11}$$

where

$$\phi = -(\gamma + \alpha)\dot{\tilde{q}} - \alpha\gamma\tilde{q} = -\gamma\dot{\tilde{q}} - \alpha s, \tag{12}$$

$$\omega = \begin{cases} -\eta\frac{z}{||z||} & \text{if} \quad ||z|| \neq 0 \\ 0 & \text{if} \quad ||z|| = 0 \end{cases} \tag{13}$$

$$s(\tilde{q}, \dot{\tilde{q}}) = \dot{\tilde{q}} + \gamma\tilde{q}, \tag{14}$$

$$z = D^\top s, \tag{15}$$

for some positive constants γ, α, and η satisfying

$$\eta \geq \frac{||\delta||}{\underline{\mu}_D}, \tag{16}$$

with

$$\begin{aligned}\delta = \big[D(q) - I\big]\big[\ddot{q}_r + \big(M(q)\big)^{-1}\big(C(q,\dot{q})\dot{q} + G(q)\big) \\ -(\gamma + \alpha)\dot{\tilde{q}} - \alpha\gamma\tilde{q}\big].\end{aligned} \tag{17}$$

Proof. Putting the control (11), (12), and (13), in (10), the closed-loop error dynamics is described by

$$\dot{s} = -\alpha s + D\omega + \delta, \tag{18}$$

where δ is given in (17), and represents the vector of residual dynamics, which satisfies

$$\begin{aligned}||\delta|| \leq \overline{\mu}_D\big(||\ddot{q}_r|| + \overline{\mu}_I(c_0||\dot{\tilde{q}} + \dot{q}_r||^2 + g_0) \\ +(\gamma + \alpha)||\dot{\tilde{q}}|| + \alpha\gamma||\tilde{q}||\big).\end{aligned} \tag{19}$$

To drive (18) to the origin, ω has been designed to cancel the effects of δ out. System (18) is considered as a perturbation of the nominal system

$$\dot{s} = -\alpha s, \tag{20}$$

being $s = 0$ its globally exponentially stable equilibrium point. This is concluded from Theorem 4.10 in

[Khalil, 2002], after taking the time derivative of the Lyapunov function

$$V(s) = \frac{1}{2}s^\top s, \tag{21}$$

along the trajectories of (20).

For the perturbed system, the derivative of (21) along (18) is

$$\dot{V}(s) = -\alpha||s||^2 + s^\top D\omega + s^\top \delta, \tag{22}$$

which, after taking ω as in (13), with (15), turns into

$$\begin{aligned}\dot{V}(s) &= -\alpha||s||^2 + z^\top\left(-\eta\frac{z}{||z||}\right) + s^\top\delta \\ &= -\alpha||s||^2 - \eta||z|| + s^\top\delta \\ &\leq -\alpha||s||^2 - \left(\eta\underline{\mu}_D - ||\delta||\right)||s||. \end{aligned} \tag{23}$$

With the gain η chosen as in (16), it follows that $\dot{V}(s)$ is negative definite whenever $z \neq 0$. Therefore, $\forall\, t \geq t_0$

$$s(t) = s(t_0)e^{-\alpha(t-t_0)}, \tag{24}$$

and

$$\tilde{q}(t) = \tilde{q}(t_0)e^{-\gamma(t-t_0)} + \frac{s(t_0)}{\gamma-\alpha}e^{-\alpha(t-t_0)}, \tag{25}$$

satisfying (3).

Remark 2. *The previous result allows exponential stabilization of the tracking errors whenever $||z|| \neq 0$. However, there might be some circumstances for which $||z|| = 0$, and as a consequence, the auxiliary control ω will not be available to cancel δ out. Under this scenario, two different cases are possible: either s is exactly zero (and then $\tilde{q} = 0$ is ES), or $s \neq 0$, in which case it can be proved that $||s||$ is uniformly ultimately bounded, and so the tracking errors. These cases are described below.*

- In case $||z|| = 0$, with $||s|| = 0$, one has that $s \equiv 0$, and from (14), one has

$$\tilde{q}(t) = \tilde{q}(t_0)e^{-\gamma(t-t_0)} \quad \forall \quad t \geq t_0, \tag{26}$$

which proves that (3) is satisfied.

- In case $||z|| = 0$, with $||s|| \neq 0$, one has that $\dot{V}(s)$ along the trajectories of (18) is

$$\dot{V}(s) \leq -\alpha||s||^2 + ||\delta||\cdot||s||, \tag{27}$$

which is negative for all s, such that

$$||s|| > \frac{||\delta||}{\alpha} := b. \tag{28}$$

That is, if $||s|| > b$, then $\dot{V}(s)$ is negative definite, and s decreases until the bound relation $||s|| \leq b$ is satisfied for a finite time T. This states that s is uniformly ultimately bounded. After using Lemma 9.2 in [Khalil, 2002], the solution of the perturbed system satisfies

$$||s(t)|| \leq b \quad \forall \quad t \geq t_0 + T, \tag{29}$$

which for the tracking errors is written as in (25), for $t_0 \leq t \leq t_0 + T$, and

$$||\tilde{q}(t)|| \leq \epsilon \quad \forall \quad t \geq t_0 + T. \tag{30}$$

Then (3) is satisfied.

Remark 3. *As described, whenever ω is not available to compensate δ, exponential stabilization of the tracking errors is not guaranteed. In this situation, as proved, tracking is achieved with bounded errors, which will depend on initial conditions. Then the smaller drift in initial conditions, the better tracking performance.*

Remark 4. *If some terms of unmodeled dynamics and parametric uncertainties can be put into δ, the proposed controller would only need an estimation of their maximal bounds to compensate their undesirable effects in the system. This could include some robustness characteristics to the proposal.*

4 Simulations

The aim of this section is to show the effectiveness of the proposed result by means of simulations in two different underactuated manipulators, a class-I and a class-II.

4.1 The Acrobot

First, we have considered the Acrobot, an underactuated manipulator class-I, shown in Figure 1. Its dynamics can be written in the form of (1) as follows

$$\begin{aligned}&\underbrace{\begin{bmatrix} a_1 + a_2 - 2a_3C_2 & a_2 - a_3C_2 \\ a_2 - a_3C_2 & a_2 \end{bmatrix}}_{M(q)}\underbrace{\begin{bmatrix}\ddot{q}_1 \\ \ddot{q}_2\end{bmatrix}}_{\ddot{q}} \\ &+ \underbrace{\begin{bmatrix} 2a_3\dot{q}_1\dot{q}_2S_2 + a_3\dot{q}_2^2S_2 \\ -a_3\dot{q}_1^2S_2 \end{bmatrix}}_{C(q,\dot{q})\dot{q}} + \underbrace{\begin{bmatrix} -a_4S_1 + a_5S_{12} \\ a_5S_{12}\end{bmatrix}}_{G(q)} \\ &= \underbrace{\begin{bmatrix}0 \\ 1\end{bmatrix}}_{B}u,\end{aligned} \tag{31}$$

where q_1 denotes the non-actuated link, while q_2 denotes the actuated one. Also, the following notations are used $C_2 = \cos(q_2)$, $S_1 = \sin(q_1)$, $S_2 = \sin(q_2)$, and $S_{12} = \sin(q_1 + q_2)$.

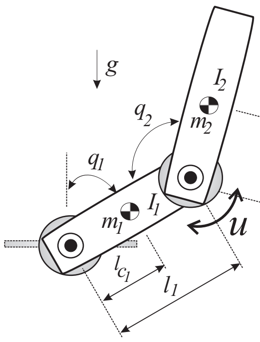

Figure 1. Diagram of the Acrobot (class-I).

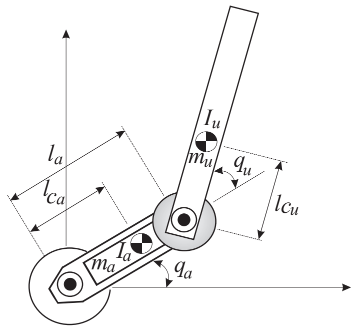

Figure 2. Diagram of the underactuated manipulator (class-II).

The system parameters are lumped in a_i, $i \in \{1, \ldots, 5\}$, with values taken from [Berkemeier and Fearing, 1999]

$$\left.\begin{array}{l} a_1 = m_1 l_{c1}^2 + m_2 l_1^2 + I_1 = 0.0043 \text{ kg·m}^2 \\ a_2 = m_2 l_{c2}^2 + I_2 = 0.00506 \text{ kg·m}^2 \\ a_3 = m_2 l_1 l_{c2} = 0.0338 \text{ kg·m}^2 \\ a_4 = (m_1 l_{c1} + m_2 l_1) g = 0.0493 \text{ N·m} \\ a_5 = m_2 g l_{c2} = 0.0379 \text{ N·m} \end{array}\right\} \quad (32)$$

4.1.1 Feasible Trajectories The desired trajectories to be tracked involve inverted periodic motions. These trajectories, denoted by q_r, were proposed in [Berkemeier and Fearing, 1999], and are solutions of the Acrobot's equation of motion

$$M(q_r)\ddot{q}_r + C(q_r, \dot{q}_r)\dot{q}_r + G(q_r) = B\overline{u}. \quad (33)$$

In this case, the input

$$\overline{u} = \big(E\big(M(q_r)\big)^{-1}B\big)^{-1}E\big(M(q_r)\big)^{-1}\times \\ \big(C(q_r, \dot{q}_r)\dot{q}_r + G(q_r)\big), \quad (34)$$

with $E = -[\,2\ 1\,]$, makes the virtual output

$$y_r(t) = 2q_{r_1}(t) + q_{r_2}(t) - \phi \quad (35)$$

to remain at zero for all $t \geq t_0$, and with the initial conditions $y_r(t_0) = \dot{y}_r(t_0) = 0$. The constant ϕ parametrizes the equilibrium manifold of the Acrobot, and the inverted periodic motions occur when $\phi = \pi$.

Then, (33) and (34) generate the exact periodic trajectories given by

$$\ddot{q}_{r_1} = \frac{a_4 \sin(q_{r_1}) + a_5 \sin(\phi - q_{r_1})}{a_1 - a_2}, \quad (36)$$

which can be interpreted as the zero dynamics of the system (33) with respect to the output (35) [Berkemeier and Fearing, 1999].

4.1.2 Results Numerical simulations were carried out in Matlab/Simulink using the Euler integration method with fixed-step size of 1ms. Drifts in initial conditions were considered for the Acrobot's positions with respect to the feasible trajectories; these were of 0.2 rad ($\approx$ 11.5 deg) for the non-actuated link, and 0.4 rad ($\approx$ 23 deg) for the actuated link. Also, the control parameters were set to $\gamma = 30$, and $\alpha = 15$.

The simulation results are shown in Figure 3, where it can be seen that the Acrobot's positions simultaneously track the desired trajectories with good performance. In the graphics, the tracking errors exponentially converge to a small neighbourhood around the origin after 0.5s. At that time, chattering in the control signal is displayed due to the switching activity of the discontinuous term w, which do try to keep the error signals to not leave the neighbourhood. Then, the proposed controller allows to satisfy (3).

4.2 An Underactuated Planar Manipulator

The 2-DOF underactuated manipulator shown in Figure 2 is the other system considered. This corresponds to the class-II, and it can be viewed as the Pendubot [Spong, 1995] without gravity. The absence of gravity terms allows the system to have many equilibrium configurations for position control. Its dynamics is described in the form of (1) as

$$\underbrace{\begin{bmatrix} a_1 + 2a_2\cos(q_u) & a_3 + a_2\cos(q_u) \\ a_3 + a_2\cos(q_u) & a_3 \end{bmatrix}}_{M(q)} \underbrace{\begin{bmatrix} \ddot{q}_a \\ \ddot{q}_u \end{bmatrix}}_{\ddot{q}} + \\ \underbrace{\begin{bmatrix} -a_2\sin(q_u)(2\dot{q}_a\dot{q}_u + \dot{q}_u^2) \\ a_2\dot{q}_a^2 S_u \end{bmatrix}}_{C(q,\dot{q})\dot{q}} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u, \quad (37)$$

where q_a denotes the actuated link, and q_u the non-actuated link. The system parameters are lumped in the coefficients a_1, a_2, and a_3. Their values correspond to those of the physical device in [Spong, 1995], and are $a_1 = 0.05653$ kg· m^2, $a_2 = 0.01081$ kg· m^2, and $a_3 = 0.01341$ kg· m^2.

4.2.1 Feasible Trajectories Following [Nagaragan *et al.*, 2009], some feasible trajectories to reach

Table 1. Parameters used to generate the feasible trajectories.

Case	(q_{a_0}, q_{u_0})	(q_{a_f}, q_{u_f})	α_1 [rad]	α_2 [rad]
T_1	$(-\frac{\pi}{6}, \frac{\pi}{6})$	$(0, 0)$	-0.3189	-0.3775
T_2	$(0, 0)$	$(0, \frac{\pi}{2})$	-0.6204	0.9552

desired static configurations in finite time t_f are proposed from the set of equations

$$q_a(t) = q_{a_0} + \tfrac{1}{2}(q_{a_f} - q_{a_0})(1 + \tanh(a_4 t - a_4 a_5)) + \tfrac{\alpha_1}{\cosh(a_6 t - a_7)} - \tfrac{\alpha_2}{\cosh(a_8 t - a_9)}, \tag{38}$$

$$\dot{q}_a(t) = \frac{\frac{1}{2}a_4(q_{a_f} - q_{a_0})}{\cosh^2(a_4 t - a_4 a_5)} - \frac{a_6\alpha_1 \sinh(a_6 t - a_7)}{\cosh^2(a_6 t - a_7)} + \frac{a_8\alpha_2 \sinh(a_8 t - a_9)}{\cosh^2(a_8 t - a_9)}, \tag{39}$$

$$\ddot{q}_a(t) = -\frac{a_4^2(q_{a_f} - q_{a_0})\sinh(a_4 t - a_4 a_5)}{\cosh^3(a_4 t - a_4 a_5)} - \frac{a_6^2\alpha_1}{\cosh^3(a_6 t - a_7)} + \frac{a_6^2\alpha_1 \sinh^2(a_6 t - a_7)}{\cosh^3(a_6 t - a_7)} + \frac{a_8^2\alpha_2}{\cosh^3(a_8 t - a_9)} - \frac{a_8^2\alpha_2 \sinh^2(a_8 t - a_9)}{\cosh^3(a_8 t - a_9)}, \tag{40}$$

$$\ddot{q}_u(t) = -\frac{1}{a_3}\big((a_3 + a_2\cos(q_u))\ddot{q}_a + a_2\dot{q}_a^2\sin(q_u)\big), \tag{41}$$

$$a_4 = \tfrac{8}{t_f - t_0}, \quad a_5 = \tfrac{t_0 + t_f}{2}, \quad a_6 = \tfrac{18}{t_m - t_0}, \quad a_7 = \tfrac{9(t_0 + t_m)}{t_m - t_0}, \quad a_8 = \tfrac{18}{t_f - t_m}, \quad a_9 = \tfrac{9(t_m + t_f)}{t_f - t_m},$$

for $t \in [t_0, t_f]$.

Two feasible trajectories were generated to reach two different static configurations. These are named T_1 and T_2, and their parameters are given in Table 1.

4.2.2 Results Simulations were carried out in Matlab/Simulink, using an Euler integration method with fixed-step size of 1ms. A feasible trajectory was used with the set of parameters T_1, from 0s to 5s, and with T_2 from 5s to 10s. This was the desired reference to be tracked by the manipulator under the action of the proposed control scheme. The selected control parameters were $\gamma = 7$, and $\alpha = 25$. To show the robustness of the proposed controller, viscous friction is included in the manipulator joints $(0.005\dot{q}_a, 0.005\dot{q}_u)$, while the controller is not fed with this information. In this case, friction is considered as a perturbation, whose undesirable effects must be compensated by the controller (and in particular, by ω). The simulation results are shown in Figure 4, where it can be seen that the tracking objective (3) is satisfied with small bounded errors, even in the presence of drifts in initial conditions. However, it should be noted that this is achieved with a major effort, which is manifested in the plot of the control signal, through the appearing of high frequency switching.

5 Conclusions

A discontinuous control scheme was proposed to track feasible trajectories for 2-DOF underactuated manipulators, class-I and class-II. The proposal ensures ultimate boundedness of the solution of the closed-loop system dynamics, and allows to track feasible trajectories under drifts in initial conditions. Numerical simulations in two different underactuated manipulators verified the effectiveness of the proposed controller.

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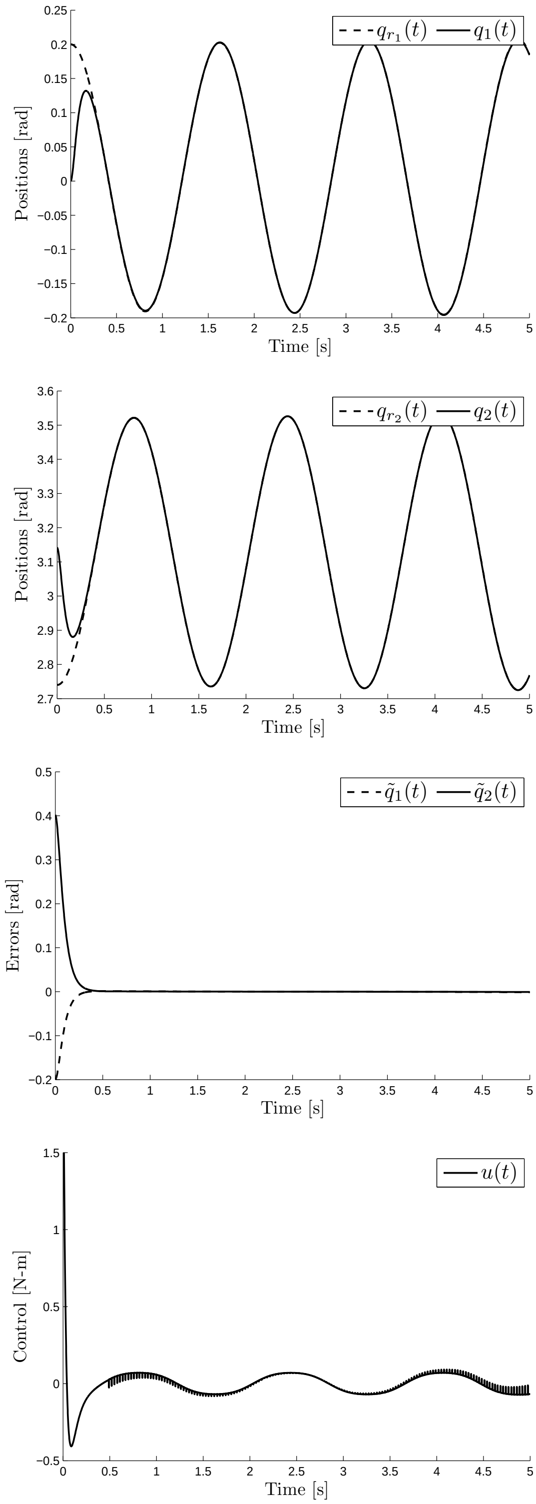

Figure 3. Simulation results for the Acrobot: tracking inverted periodic motions. From top to bottom: graphics of positions for non-actuated and actuated joints, graphic of position errors, and graphic of control signal.

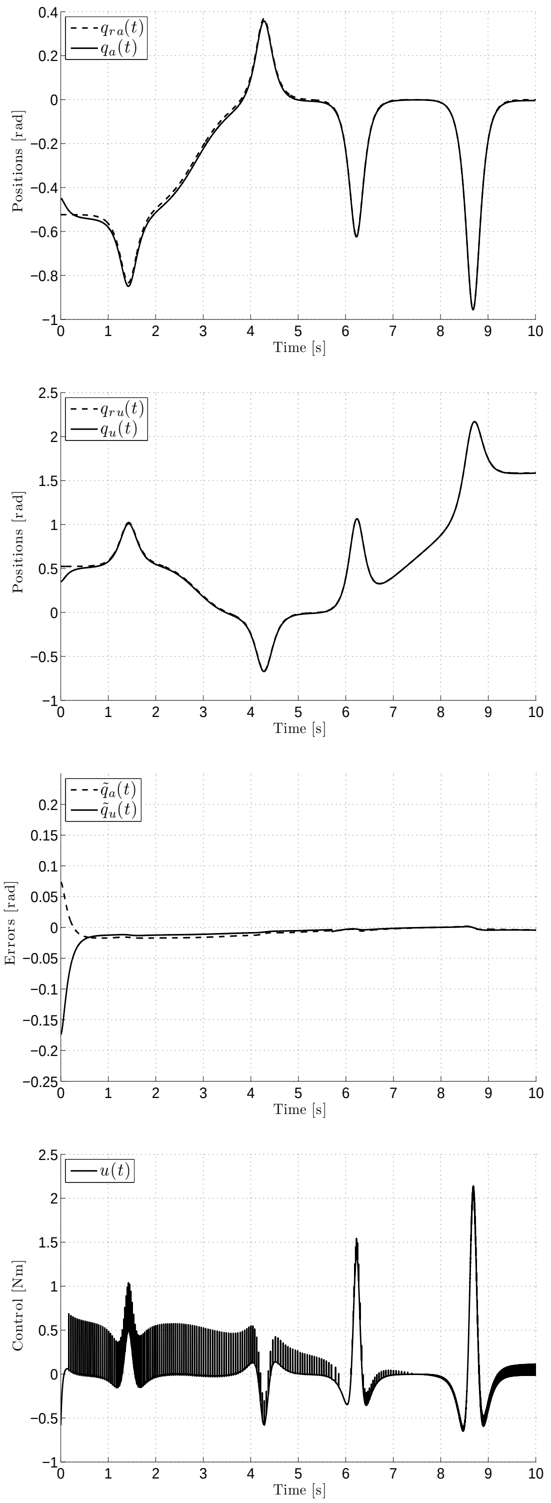

Figure 4. Simulation results for the 2-DOF underactuated manipulator with viscous friction. From top to bottom: graphics of positions for actuated and non-actuated joints, graphic of position errors, and graphic of control signal.