PARAMETRIC RESONANCE IN NONLINEAR VIBRATIONS OF ELASTIC STRETCHED STRING UNDER HARMONIC HEATING

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Abstract

The existence of parametric resonance and transient phenomena in nonlinear systems under the action of an external force are important characteristics of dynamical systems. Nonlinear vibrations of a thin stretched string, with an alternating electric current passing through, in a non-uniform magnetic field are described by complicated equations of motion. The general mathematical model involves modes coupling by means of the intrinsic and improper nonlinearities; besides, the string suffers Joule heating. The purpose of the work is to study the combined effect of the intrinsic (geometrical) nonlinearity and Joule heating on the elastic string oscillation in the frame of a simplified model. We use a combined analytical-numerical approach in studying the dynamics of the proposed model. First, we solve our model analytically by iterations; then we solve it numerically. Both analytical and numerical results show a good agreement almost everywhere except small intervals near resonant frequencies of different modes. It was found that numerical solutions show instabilities near resonant frequencies in contrast to that of the approximate analytical solutions by iterations. We explain those instabilities using the theory of Mathieu equations.

Key words

Nonlinear string vibration; Parametric resonance; Instabilities; Mathieu equation.

1 Introduction

Elastic string is a basic and fundamental system in the theory of wave propagation. In particular, a string conducting electric current in a magnetic field show complex nonlinear oscillations [Kourmychev, 1998]. This system with space distributed parameters is frequently described by nonlinear models, even when the amplitude of string vibrations is small [Armstrong, 1982; Elliot, 1980; Tufillaro, 1989]. The current carrying string shows nonlinearities of the two types, the intrinsic and improper. The intrinsic nonlinearity is due to the variation of tension caused by elongation of the string at the transverse oscillating; it can be observed in any string. The improper nonlinearity is a specific characteristic of a current carrying string due to the interaction between electric current and magnetic field. Nonlinearities can produce coupling between transverse modes [Kourmychev, 1998].

2 Mathematical model

When transverse modes are decoupled of longitudinal ones, then following [Watzky, 1992] we can write nonlinear equation of small amplitude transverse vibrations on a thin string of longitude L, under the action of harmonic force $\tilde{f}(z)\cos(\Omega t)$ in the xz-plane as follows:

$$\ddot{x}+2\beta\dot{x}-\left[C_t^2+C_l^2\frac{1}{2L}\int_0^L (x')^2\,dz\right]x''=\tilde{f}(z)\cos(\Omega t) \quad (1)$$

where $x(t,z)$ is a transverse displacement of the string in the point z; $\dot{x}$ is the time derivative and x' is the derivative in z along the string; the terms under the integral describe intrinsic (geometric) nonlinearity; β is a damping coefficient for one of the transverse modes; Ω is an angular frequency of external force. C_t y C_l are the velocities of transverse and longitudinal waves, respectively; those depend on tension in the string.

2.1 Change of wave velocity at the heating of string

The interaction of an external magnetic field and an electric current in a string results in the Lorentz force that causes nonlinear oscillation of the string when the electric current is an alternating one. The intrinsic nonlinearity in oscillation of the string is the result of alternating increase of string tension due to the elongation of the string. On the other hand, the Joule heating by the electric current causes dilatation of the string, and consequently the alternating decrease of tension. To include the two opposite effects in the model we proceed the following way. At a constant magnetic field the driving force of oscillations is governed by the current, $I(t) = I_0 \cos(\Omega t)$ that causes Joule heating, $Q = I^2 R$. We think that temperature variations on a string can be expressed by the following semi phenomenological formula,

$$\Delta T(t) = \Delta T_0 + \delta \cos^2(\Omega t - \varphi) \quad (2)$$

where φ is a phase shift. Then, after some algebra using the Hooke's law we obtain time variations in the tension of the string:

$$F_T = F_{st} - \lambda\alpha\delta \cos^2(\Omega t - \varphi) \quad (3)$$

where $F_{st} = F - \lambda\alpha\Delta T_0$ is the stationary part of tension reduced with respect to the initial value F, and $\Delta F = \lambda\alpha\delta$ is the amplitude of time variation in the tension. From equation (3) we see the decrease of tension with the heating of the string.

2.2 Equation of motion

Harmonic heating influences the transverse wave velocity in the following way,

$$(C_t^2)_T = \left(\frac{F}{\rho}\right)_T = \frac{F_{st} - \lambda\alpha\delta\cos^2(\Omega t - \varphi)}{\rho} \quad (4)$$

However, the longitudinal wave velocity remains the same,

$$(C_l^2)_T = \left(\frac{\lambda}{\rho}\right)_T \cong \frac{\lambda}{\rho} \quad (5)$$

Substituting these velocities in equation (1), we obtain the equation of transverse wave motion on a string in the xz-plane:

$$\ddot{x} + 2\beta\dot{x} - \left[\breve{C}_t^2 - \frac{\lambda\alpha\delta}{\rho}\cos^2(\Omega t - \varphi)\right] x'' - C_l^2 x'' \frac{1}{2L}\int_0^L (x')^2 \, dz = \tilde{f}(z)\cos(\Omega t) \quad (6)$$

Analyzing equation (6), we conclude that under the accepted approximations the variation of temperature in an elastic string is manifested through the variation of the transverse wave velocity, linearly; while the elongation of the string during its oscillation is expressed nonlinearly through the integral. The latter is a geometric, intrinsic nonlinearity in string oscillations.

3 The effect of heating on the vibrations of a string

In order to study the effect of heating on the vibrations of a string, we start from the simplified model neglecting temporally the proper nonlinearity in Eq. (6), and obtaining the following linear equation

$$\ddot{x} + 2\beta\dot{x} - \left[\breve{C}_t^2 - \frac{\lambda\alpha\delta}{\rho}\cos^2(\Omega t - \varphi)\right] x'' = \tilde{f}(z)\cos(\Omega t) \quad (7)$$

By separation of variables, the inhomogeneous wave equation (7) is reduced to the equation for the spatial part of the wave,

$$\frac{d^2 X}{dz^2} + k^2 X = 0 \quad ; \quad X_n(z) = \sin(k_n z) \quad (8)$$

which has solutions in normal modes, $X_n(z) = \sin(k_n z)$. Then, the equation for the temporal part of inhomogeneous equation (7), corresponding to the n-th normal mode, is as follows

$$\frac{d^2 T_n}{dt^2} + 2\beta\frac{dT_n}{dt} + \left[\breve{C}_t^2 - \frac{\lambda\alpha\delta}{\rho}\frac{1 + \cos(2\Omega t - 2\varphi)}{2}\right] k_n^2 T_n = f_n \cos(\Omega t) \quad (9)$$

where $\tilde{f}(z) = \Sigma_n f_n X_n(z)$. For the normal mode $n = 1$, Figure 1 shows three plots of numerical solution of Eq. (9) for different frequencies of driving force, (a) $\Omega = 30\ Hz$, (b) $\Omega = 30.5\ Hz$ and (c) $\Omega = \Omega_r = 30.67\ Hz$. We see that the amplitude of oscillations increases in infinity, oscillations of a string are diverged in small vicinity of resonant frequency $\Omega_r = 30.67 Hz$.

Solution by iterations. Equation (9) can be presented in the form such that its solutions are finding out by perturbations following the iteration process described in [Kourmychev, 2003]. This way the approximate solution at the i-th iteration is a solution of Eq. (10) for $n = 1, 2, ...$

$$\frac{d^2 T_n^{(i)}}{dt^2} + 2\beta\frac{dT_n^{(i)}}{dt} + \left[\breve{C}_t^2 - \frac{\lambda\alpha\delta}{2\rho}\right] k_n^2 T_n^{(i)} = f_n \cos(\Omega t) + \frac{\lambda\alpha\delta}{\rho}\frac{\cos(2\Omega t - 2\varphi)}{2} k_n^2 T_n^{(i-1)} \quad (10)$$

The initial iteration is a solution of the following equation:

$$\frac{d^2 T_n^{(0)}}{dt^2} + 2\beta\frac{dT_n^{(0)}}{dt} + \breve{C}_t^2 k_n^2 T_n^{(0)} = f_n \cos(\Omega t) \quad (11)$$

General solution of (11) is

$$TG_n^{(0)}(t) = Ce^{-\beta t}\cos(\tilde{\omega}_n t - \alpha_1) + \frac{f_n}{\sqrt{(\tilde{\omega}_n^2 - \Omega^2)^2 + 4\beta^2\Omega^2}}\cos\left(\Omega t - \tan^{-1}\frac{2\beta\Omega}{\tilde{\omega}_n^2 - \Omega^2}\right) \quad (12)$$

So, the first approximation is obtained by substituting (12) in (11) and, finally, is the solution of the following equation

$$\frac{d^2 T_n^{(1)}}{dt^2} + 2\beta\frac{dT_n^{(1)}}{dt} + \tilde{\omega}_n^2 T_n^{(1)} = A\cos(\Omega t - \chi_1) + B\cos(3\Omega t - 2\varphi - \chi) + D\cdot Ce^{-\beta t}\cos[(2\Omega + \tilde{\omega}_n)t - \theta_1] + D\cdot Ce^{-\beta t}\cos[(2\Omega - \tilde{\omega}_n)t - \theta_2] \quad (13)$$

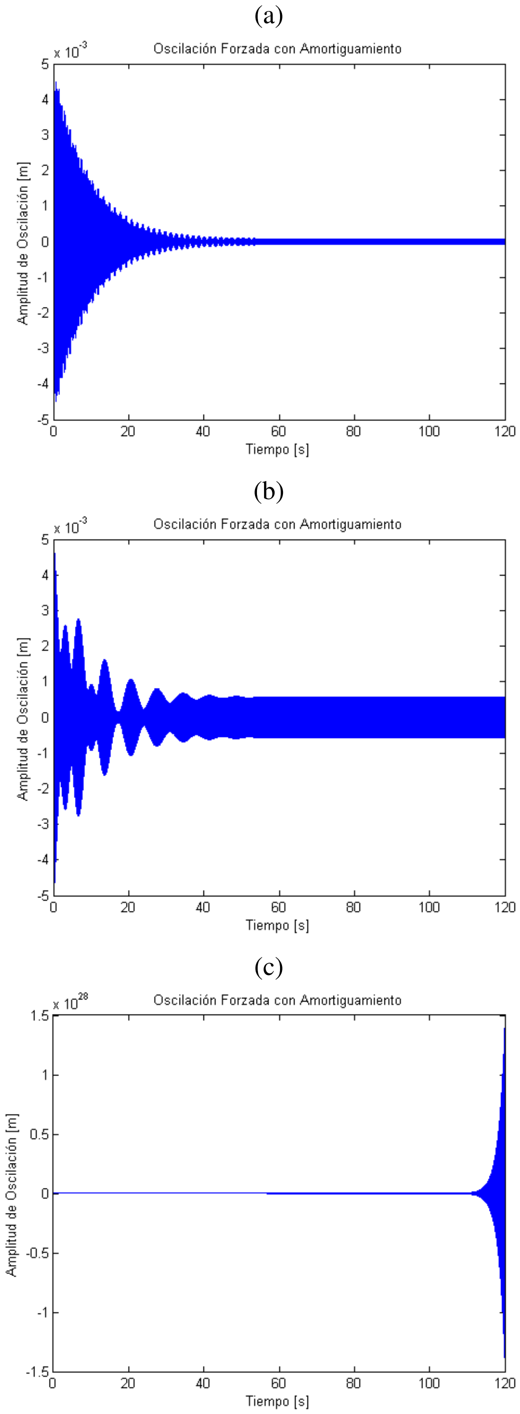

Figure 1. Numerical solution of the nonlinear inhomogeneous equation (9), for the first normal mode $n = 1$ at $\Delta T_0 = 5°C$, $\delta = 0.5°C$, $\varphi = \pi/48$ and $f_n = 0.1$: (a) $\Omega = 30\ Hz$, (b) $\Omega = 30.5\ Hz$, (c) $\Omega = \Omega_r$

The solution of (13) is of the following form:

$$T_n^{(1)}(t) = e^{-\beta t} a_0 \cos(\hat{\omega} t - \delta_0) + e^{-\beta t} a_3 \cos[(2\Omega + \tilde{\omega}_n)t - \theta_1 - \delta_3] + e^{-\beta t} a_4 \cos[(2\Omega - \tilde{\omega}_n)t - \theta_2 - \delta_4] + a_1 \cos(\Omega t - \chi_1 - \delta_1) + a_2 \cos(3\Omega t - 2\varphi - \chi - \delta_2) \quad (14)$$

Analytical solution for the first iteration, Eq. (14) is plotted for different driving frequencies in Figure 2.

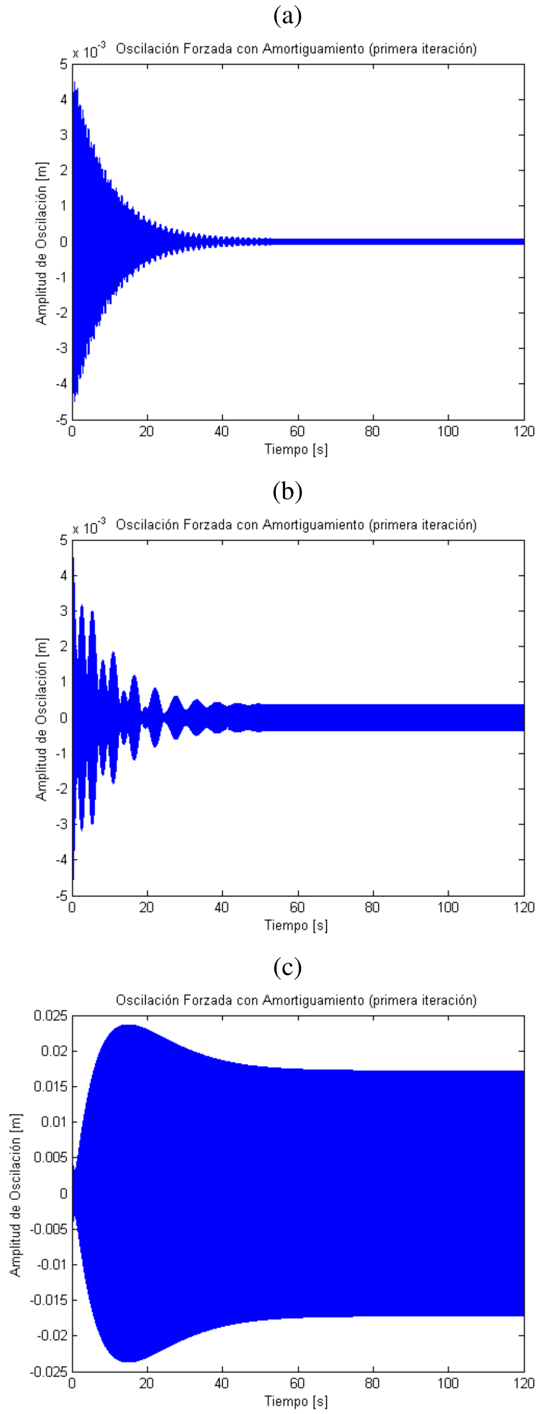

Figure 2. Plots of analytical solution for the first iteration, Eq. (14) for the first normal mode $n = 1$ at $\Delta T_0 = 5°C$, $\delta = 0.5°C$, $\varphi = \pi/48$ and $f_n = 0.1$: (a) $\Omega = 30\ Hz$, (b) $\Omega = 30.5\ Hz$, (c) $\Omega = \Omega_r$

When the frequency of driving force is close to the resonant frequency of the m-th normal mode, then the first approximation solution of Eq. (9) is of following form,

$$x(z,t)=\sum_{i=1}^{\infty}\sin\left(k_i z\right)\cdot T_i^{(1)}(t)\ \approx\ \sin\left(k_m z\right)\cdot T_m^{(1)}(t) \quad (15)$$

4 Heating and intrinsic nonlinearity of a string

In order to take into account the heating of the string, we substitute in the integral of Eq. (6) the approximate analytical solution $x(z,t) = T_m^{(1)}(t)\sin\left(k_m z\right)$, Eq. (15), so that $x' = T_m^{(1)}(t)\cdot k_m\cos\left(k_m z\right)$. Then, integrating with respect to z in Eq. (16),

$$\ddot{x}+2\beta\dot{x}-\left[\breve{C}_t^2-\frac{\lambda\alpha\delta}{\rho}\cos^2\left(\Omega t-\varphi\right)\right]x''- C_l^2 x''\frac{1}{2L}\int_0^L\left[T_m^{(1)}(t)\cdot k_m\cos\left(k_m z\right)\right]^2 dz = \tilde{f}(z)\cos(\Omega t) \quad (16)$$

and separating variables in normal modes $X_n(z) = \sin(k_n z)$ and harmonics $T_n(t)$, we get the equation

$$\frac{d^2T_{nm}}{dt^2}+2\beta\frac{dT_{nm}}{dt}+ \left\{\breve{C}_t^2-\frac{\lambda\alpha\delta}{\rho}\cos^2\left(\Omega t-\varphi\right)+C_l^2\frac{k_m^3}{4}\left[T_m^{(1)}(t)\right]^2\right\}k_n^2T_{nm} = f_n\cos(\Omega t) \quad (17)$$

where the second subindex m in $T_{nm}(t)$ show the dependence of each n-th harmonic on the harmonic m, because $T_m^{(1)}(t)$ was chosen to contribute into the intrinsic nonlinearity. Equation (17) is the linearization of Eq. (6), which describes driven oscillations of a string subject to the harmonic heating. Plots of numerical solutions of (17) at three frequencies of driven force are presented in Figure 3; in small vicinity of resonant frequency of m-th harmonic, the huge (unphysical) increase of amplitude is observed, similar to that of the model without intrinsic nonlinearity, Eq. (9).

5 Stability of oscilations

We found that oscillations of a string described by Eqs. (9) and (17) diverge in time, see Figures 1(c) and 3(c). To find the origin of divergence we consider the homogeneous equation corresponding to the Eq. (9). Note that substitutions $T_n(t) = e^{-\beta t}u(t)$ and $z = t-\varphi$ transform the homogeneous equation into the canonic Mathieu equation,

$$\widetilde{u}''(z)+\left[a-2q\cos(2z)\right]\widetilde{u}(z)=0 \quad (18)$$

where $a = \frac{\hat{a}}{\Omega^2}$ and $q = \frac{\hat{q}}{\Omega^2}$, $\hat{a} = k_n^2\left(c_t^2-\frac{\lambda\alpha\delta}{2\rho}\right)-\beta^2$ and $\hat{q} = k_n^2\frac{\lambda\alpha\delta}{4\rho}$. The second order homogeneous equation (18) has harmonic modulation of coefficient, where a and q are characteristic value and parameter respectively, which determine the properties of the system. The canonic form of the Mathieu equation permits to know the stability of its solution without their knowing explicitly.

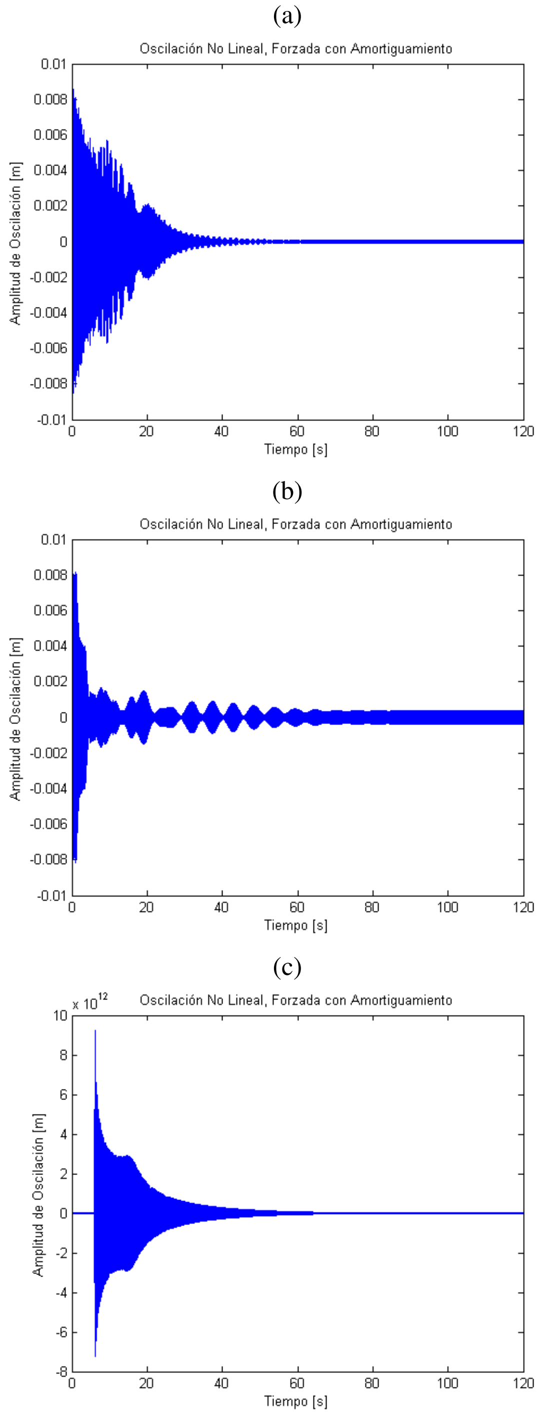

Figure 3. Numerical solution of the linearized Eq. (17), for the first normal mode $n = 1$ and first iteration $T_1^{(1)}(t)$ at $\Delta T_0 = 5°C$, $\delta = 0.5°C$, $\varphi = \pi/48$ and $f_n = 0.1$: (a) $\Omega = 30\,Hz$, (b) $\Omega = 30.5\,Hz$, (c) $\Omega = \Omega_r$.

Figure 4 shows the regions of stability and instability of Mathieu functions, separated by characteristic curves. The behavior of Mathieu functios dependes on the value of parametric point (q, a). According to [McLachlan, 1947], solution (see Figure 4):

- is stable if (q, a) is in the regin between the curves a_m and b_{m+1}.
- is unstable if (q, a) is between the curves b_m and a_m.

- shows pulsations when the parametric point (q,a) is on the one of the curves separating stable from unstable regions.

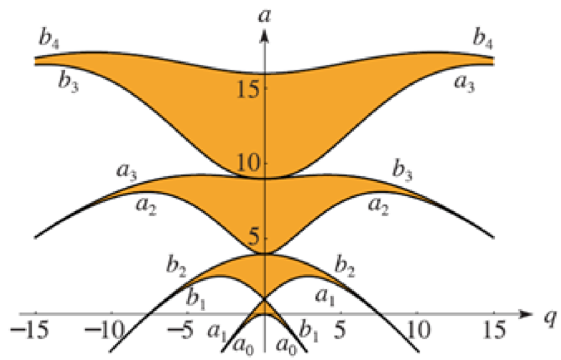

Figure 4. Stability regions for the Mathieu functions, reproduced from [NIST, 2010]

When $q = 0$, then Eq. (18) turns into the harmonic oscillator; in addition, if $a = m^2$ the equation takes the form,

$$\widetilde{u}''(z) + m^2 \widetilde{u}(z) = 0 \tag{19}$$

where $m \geq 0$ is the order of Mathieu equation. In general, when the parametric point (q,a) is between the curves a_m and b_{m+1} of Figure 4, the order of Mathieu function is $\nu = (m + \hat{\beta})$. If ν is an integer, then the Mathieu function is periodic one with period π or 2π. If $\nu = s/p$ is a rational number, then the function is periodic in $2\pi p$. If ν is an irrational number, then the function is nonperiodic.
In our case, parameters of the problem are: $\beta = 0.1$, $\alpha = 1710^{-6} 1/°C$, $\lambda = 2,545N$, $\rho = 210^{-4} kg/m$, $F = 0.98N$, $L = 1m$, $\Delta T_0 = 5, 10°C$, $\delta = 0.5, 1°C$. For each normal mode studied in this work, $n = 1,2,3$ all the parameters given above are fixed, except the frequency Ω of the external force, which is considered as a control parameter. In addition, $k_n = n\pi/L$ and $c_t^2 = \frac{F - \lambda\alpha\Delta T_0}{\rho}$. The frequency Ω was varied in an interval close to the resonant frequency of each normal mode. So, the values of (q,a) corresponding to the modes $n = 1,2,3$ were varied in the intervals shown in Tables 1 and 2: As an example, we present here

	$n=1$ $30 \leq \Omega \leq 31$	$n=2$ $60 \leq \Omega \leq 62$	$n=3$ $90 \leq \Omega \leq 93$
q	$0.0075 \geq q \geq 0.0070$		
a	$1.0456 \geq a \geq 0.9793$		
$\hat{\beta}$	$\hat{\beta} = 0.0222786$; $\hat{\beta} = 0.9902205$		

Table 1. Values of parametric point (q,a) at $\Delta T_0 = 5°C$

	$n=1$ $25 \leq \Omega \leq 26$	$n=2$ $50 \leq \Omega \leq 52$	$n=3$ $75 \leq \Omega \leq 78$
q	$0.0216 \geq q \geq 0.0199$		
a	$1.0515 \geq a \geq 0.9722$		
$\hat{\beta}$	$\hat{\beta} = 0.0231118$; $\hat{\beta} = 0.9905777$		

Table 2. Values of parametric point (q,a) at $\Delta T_0 = 10°C$

the analysis of stability of solutions in the case of the first normal mode, $n = 1$ and $\Delta T_0 = 5°C$. In case of other modes and/or $\Delta T_0 = 10°C$ the analysis is similar. When Ω is varied in the interval of frequencies shown in Table 1, at $\Omega = 30\ Hz$ the parametric point (q,a), denotated as **A** in Figure 5, is located between the characteristic curves a_1 and b_2; in this case the Mathieu function is of order $\nu = 1.0222786$ and is a periodic one. Similar behavior is observed for the point **D** that corresponds to the periodic solution of order $\nu = 0.9902205$. At $\Omega = 30.5\ Hz$, the parametric point **B** in Figure 5 is localized near to the curve that separates the regions of stability and instability. In accordance with [McLachlan, 1947] our solution, either numerical or approximated analytical, shows pulsations (see Figures 1(b) and 2(b)). When $\Omega = \Omega_r = 30.67\ Hz$, the parametric point **C** in Figure 5, is localized in the region of instability between the characteristic curves b_1 and a_1. This instability is clear seen in our numerical solution, Figure 1(c); approximated analytical solution shows significant increase of amplitude but not the divergence, see Figure 2(c).

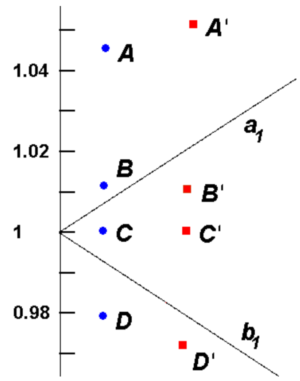

Figure 5. Position of the parametric point (q,a) for the first normal mode, $n = 1$, equation (18). Points **A**, **B**, **C** and **D** correspond to the $\Delta T_0 = 5°C$, and the points **A'**, **B'**, **C'** y **D'** corresponds to the $\Delta T_0 = 10°C$.

6 Conclusion

We proposed a semiheuristic model of driven oscillations of a string subject to the harmonic heating that, in addition, takes into account intrinsic, geometric nonlineariry of oscillations. This model, described by a set of nonlinear integral-differential equations is considered as a previous step to study driven oscillations of a string conducting an electric current in a magnetic field.

In order to study the dynamics of oscillations we proposed and used a combined analytical-numerical technique. The technique allows us to separate, to a certain degree, the intrinsic nonlinearity from the Joule heating of a string by an electric current, using an iteration scheme.

We start first the study of influence of Joule harmonic heating on the string oscillations, neglecting the effect of intrinsic geometric nonlinearity. The resulting equation was solved analytically by perturbations up to the first approximation. Then, in order to analyze additionally the effect of intrinsic nonlinearity, we substitute the first order approximation into the integral-differential equation reducing the latter to the linear equation of the Mathieu type with the time modulated frequency. The latter equation was solved numerically.

In addition to the approximate first order analytical solution of lineal equation (9), which represents the effect of Joule heating on the oscillations of a string neglecting intrinsic nonlinearity, we studied Eq. (9) numerically for the normal modes $n = 1, 2, 3$ at temperatures $\Delta T_0 = 5$ and $10°C$ in the range of frequencies close to the resonant frequencies of corresponding modes. We found the divergence of solutions in an interval of frequencies close to the resonant frequency; this was not observed in the case of approximate analytical solutions. Comparing numerical and approximate analytical solutions permitted us to establish the range of validity for the approximate analytical solutions.

The instability observed in the numerical solutions of the linear model, Eq. (9), was found to be of the Mathieu type. Reducing the original equation (9) to the Mathieu equation, we found the range of parameters at which the solutions are diverged. In addition, we found that the range of near resonant frequencies, $\Omega \approx \Omega_r$, where solutions are unstable, increases with the increase of temperature ΔT_0.

Because the heating and intrinsic nonlinearity are opposite acting effects, at the beginning we expected that the intrinsic nonlinearity will suppress the divergence caused by the parametric resonance of the harmonic heating. Nevertheless, this was not confirmed: numerical solution of the linearized equation (17) also shows the divergence at the frequencies nearby the resonant one of each normal mode. So, in the frame of the model proposed and studied in this work, the intrinsic nonlinearity does reach to stabilize oscillations of the string and suppress the divergence of solutions caused by harmonic heating at frequencies nearby to the resonant frequencies of normal modes.

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