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LIMIT CYCLE BIFURCATIONS OF A PIECEWISE LINEAR DYNAMICAL SYSTEM

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Abstract
In this paper, we consider a planar dynamical system with a piecewise linear function containing an arbitrary (but finite) number of dropping sections and approximating some continuous nonlinear function. Studying all possible local and global bifurcations of its limit cycles, we give a sketch of the proof of the theorem stating that such a piecewise linear dynamical system with k dropping sections and $2k+1$ singular points can have at most $k+2$ limit cycles, $k+1$ of which surround the foci one by one and the last, $(k+2)$-th, limit cycle surrounds all of the singular points of this system.

Key words
piecewise linear dynamical system, limit cycle

1 Introduction

The paper is based on the applications of Bifurcation Theory originated by Andronov, Arnold, Thom, Whitney, Zeeman and can be used for modeling problems, where system parameters play a certain role in various bifurcations. The theoretical studies of bifurcations deal with so-called universal problems. This means that sufficiently many parameters are available for universality of generic families of dynamical systems in the context at hand, under a relevant equivalence relation. This has led to the classification of generic, local bifurcations. In many applications, models have a given number of parameters. Moreover, the bifurcation analysis, taking place in the product of phase space and parameter space, is not restricted to local features only. On the contrary, often the interest is the global organization of the parameter space regarding bifurcations which can be both local and global.

This paper deals with so-called sewed dynamical systems, i.e., with systems for which the domain of definition is divided into sub-domains where different analytic systems are defined. The trajectories of these partial systems are sewed in one way or another on the boundaries of the sub-domains. Such systems have some typical features, namely: 1) the system sewing is immediate from the physical meaning of the problem under consideration; 2) the system is piecewise linear, i.e., the partial systems from which it is sewed are linear systems; 3) on the line of sewing, a point map (a first return function) is defined, what allows to determine the character of the system under consideration.

Piecewise linear dynamical systems always contain some parameters and, under the variation of these parameters, the qualitative behavior of the systems can obviously change. We will consider the simplest bifurcations possible in the sewed systems when the sewing lines are unchanged under the parameter variations. It is natural to consider the following bifurcations which are similar to the simplest bifurcations of continuous dynamical systems: 1) the bifurcation of a singular point of focus type; 2) the bifurcation of an immovable point of focus type, a quasi-focus; 3) the bifurcation of a sewed limit cycle; 4) the bifurcation of a sewed separatrix going from a saddle to another saddle (the saddles can be both sewed and unsewed); 5) the bifurcation of a separatrix of a saddle-shaped singular point (sewed or unsewed) going from a saddle-shaped singular point to another such point or to a saddle (sewed or unsewed); 6) the bifurcation of a sewed saddle-node; 7) the bifurcation of a sewed separatrix of a saddle-node (sewed or unsewed) going out of the saddle-node and going back to it. Besides, some specific bifurcations can occur in sewed systems. Since in such systems, for example, arches of attraction or repulsion composed of immovable points can be similar to singular points, some bifurcations which are similar to the generation of a limit cycle from a focus can occur in the corresponding constructions.

Piecewise linear systems have many applications in science and engineering. Special cases of such systems provide mathematical models for mechanical systems with Coulomb friction, for valve oscillators with a discontinuous characteristic, for direct control systems with a two-point relay mechanism, for planar dynamical systems modeling neural activity, etc. Despite their simple structure and relevance to the applications, there is, to the best of our knowledge, no complete study of their dynamical properties. In most existing papers, which deal with planar dynamical systems with piecewise linear right-hand sides, either the systems considered are continuous or only particular cases are investigated. The first analytical results on such systems go back to Andronov, Vitt, and Khaikin in the 1930s (see [Andronov, Vitt and Khaikin, 1987]). The existence and non-existence of an asymptotically stable periodic solution (limit cycle) of a piecewise linear system can be comparatively easily proved. However, for example, periodic solutions with sliding motion are of great importance to the applications. In particular, they describe the so-called stick-slip oscillations which appear in mechanical systems with dry friction.

The main objective of the present paper is to provide a complete analysis of the dynamical properties of piecewise linear systems, their dependence on the system parameters studying, first of all, their limit cycle bifurcations. There are several ways to investigate the qualitative dynamics of such systems [Filippov, 1988]. There are also numerous methods and good results on studying limit cycles. However, the most important impulse to their studying was given by the introduction of ideas coming from Bifurcation Theory [Andronov, Leontovich, Gordon and Maier, 1971; Bautin and Leontovich, 1990; Gaiko, 2003; Guckenheimer and Holms, 1990]. We know three principal limit cycle bifurcations which we will study [Gaiko, 2003]: 1) the Andronov-Hopf bifurcation (from a singular point of center or focus type); 2) the separatrix cycle bifurcation (from a homoclinic or heteroclinic orbit); 3) the multiple limit cycle bifurcation. All of these local bifurcations will be globally connected and will be applied to the qualitative analysis of piecewise linear dynamical systems.

2 Preliminaries

In this paper, geometric aspects of Bifurcation Theory are used and developed. It gives a global approach to the qualitative analysis and helps to combine all other approaches, their methods and results. First of all, the two-isocline method which was developed by Erugin is used [Gaiko, 2003]. An isocline portrait is the most natural construction for a polynomial equation. It is sufficient to have only two isoclines (of zero and infinity) to obtain principal information on the original polynomial system, because these two isoclines are right-hand sides of the system. Geometric properties of isoclines (conics, cubics, quadrics, etc.) are well-known, and all isoclines portraits can be easily constructed. By means of them, all topologically different qualitative pictures of integral curves to within a number of limit cycles and distinguishing center and focus can be obtained. Thus, it is possible to carry out a rough topological classification of the phase portraits for the polynomial dynamical systems and for the corresponding piecewise linear systems. It is the first application of Erugin's method. After studying contact and rotation properties of isoclines, the simplest (canonical) systems containing limit cycles can be also constructed. Two groups of the parameters can be distinguished in such systems: static and dynamic. Static parameters determine the behavior of phase trajectories in principle, since they control the number, position and character of singular points in a finite part of the plane (finite singularities). The parameters from the first group determine also a possible behavior of separatrices and singular points at infinity (infinite singularities) under the variation of the parameters from the second group. The dynamic parameters are rotation parameters. They do not change the number, position and index of the finite singularities and involve the vector field into directional rotation. The rotation parameters allow to control the infinite singularities, the behavior of limit cycles and separatrices. The cyclicity of singular points and separatrix cycles, the behavior of semi-stable and other multiple limit cycles are controlled by these parameters as well. Therefore, by means of the rotation parameters, it is possible to control all limit cycle bifurcations and to solve the most complicated problems of the qualitative theory of dynamical systems (both continuous and piecewise linear).

In [Gaiko, 2003] some complete results on continuous quadratic systems have been presented and some preliminary results on generalizing geometric ideas and bifurcation methods for cubic dynamical systems have been obtained. So, in [Gaiko and van Horssen, 2004], a canonical cubic dynamical system of Kukles-type was constructed and the global qualitative analysis of its special case corresponding to a generalized Liénard equation was given. In particular, it was proved that the foci of such a Liénard system could be at most of second order and that the system could have at least three limit cycles in the whole phase plane. Moreover, unlike all previous works on the Kukles-type systems, by means of arbitrary (including as large as possible) field rotation parameters of the canonical system, the global bifurcations of its limit and separatrix cycles were studied. As a result, the classification of all possible types of separatrix cycles was obtained and all possible distributions of limit cycles were found for the generalized Liénard system. In [Botelho and Gaiko, 2006], the global qualitative analysis of centrally symmetric cubic systems which are used as learning models of planar neural networks was established. All of these results can be generalized to the corresponding piecewise linear Liénard-type dynamical systems.

Such systems have been considered in several paper. For example, Bautin [Bautin, 1974] studied a dynamical system which was used in radio-engineering for describing tunnel diode circuits, where the nonlinear function was approximated by a piecewise linear function composed of three linear pieces. Giannakopoulos and Pliete [Guckenheimer and Holms, 1990] generalized that results for a piecewise linear system with a line of discontinuity modeling stick-slip oscillations which appeared in mechanical systems with dry friction. In this paper, the obtained results are generalized to an arbitrary piecewise linear Liénard-type system, where the polynomial is approximated by a piecewise linear function composed of an arbitrary (but finite) number of linear pieces. In particular, the following bifurcations are studied for such a system: 1) the bifurcation of singular points (division of the parameter space according to the number and character of singular points, stability of singular points lying on the lines of sewing, generation of limit cycles from singular points of focus type under transferring the points through the lines of sewing, generation of limit cycles (hyperbolic and semi-stable) from the boundaries of the domains filled by closed trajectories); 2) the separatrix bifurcation (location of the bifurcation curves for the separatrix loops, stability of the separatrix loops); 3) the bifurcation of multiple limit cycles (location of the bifurcation curves for the limit cycles of various multiplicity, qualitative structure of the division in the phase plane). As was shown in [Bautin and Leontovich, 1990], the return map constructed in a neighborhood of a multiple limit cycle of a piecewise linear dynamical system will be an analytic function; therefore, all statements formulated for multiple limit cycles of an analytic (or polynomial) system will be also valid for sewed multiple limit cycles which are considered in this paper. In particular, we will apply the Wintner–Perko termination principle (see [Gaiko, 2003; Perko, 2002]) for studying global bifurcations of sewed multiple limit cycles of a piecewise linear Liénard-type dynamical system.

3 The Main Result

Consider the system

$$\dot{x} = y - \varphi(x),\ \dot{y} = \beta - \alpha x - y,\ \alpha > 0,\ \beta > 0,\ \ (1)$$

where $\varphi(x)$ is a piecewise linear function containing k dropping sections and approximating some continuous nonlinear function. The line $\beta - \alpha x - y = 0$ and the curve $y = \varphi(x)$ can be considered as the isoclines of zero and infinity, respectively, for the corresponding equation. Such systems and equations may occur, for example, when tunnel diode circuits and some other problems are studied (see [Andronov, Vitt and Khaikin, 1987; Bautin, 1974; Bautin and Leontovich, 1990; Filippov, 1988; Guckenheimer and Holms, 1990]).

Suppose that the ascending sections of system (1) have an inclination $k_1 > 0$ and the descending (dropping) sections have an inclination $k_2 < 0$. Then the phase plane of (1) can be divided onto $2k + 1$ parts in every of which (1) is a linear system: the ascending sections are in $k + 1$ strip regions $(I, III, V, \dots, 2K + 1)$ and the descending sections are in other k such regions $(II, IV, VI, \dots, 2K)$. The parameters k_1, k_2, and also α can be considered as rotation parameters for the sewed vector field of (1) (see [Bautin and Leontovich, 1990; Gaiko, 2003]).

System (1) can have an odd number of simple singular points: $1, 3, 5, \dots, 2k + 1$. If (1) has the only singular point, this point will be always an antisaddle (center, focus or node). A focus (node) will be always stable in odd regions and unstable in even regions if $k_2 > 1$. If system (1) has $2k + 1$ singularities, then k of them are saddles (they are in even regions) and $k+1$ others are antisaddles (foci or nodes) which are always stable (they are in odd regions). The pieces of the straight lines $\beta = x_{2i-1}\alpha + y_{2i-1}$ and $\beta = x_{2i}\alpha + y_{2i}$ $(i = 1, 2, \dots, k)$, where (x_{2i-1}, y_{2i-1}) and (x_{2i}, y_{2i}) are the coordinates of the upper and lower corner points of the curve $\varphi(x)$, respectively, form a discriminant curve separating the domains in the plane (α, β), where $\alpha \le k_2$, with different numbers of singular points. The points of the discriminant curve correspond to the sewed singularities of saddle-focus or saddle-node type $(\alpha < k_2)$ and its corner points correspond to the unstable equilibrium segments $(\alpha = k_2)$ which coincide with the dropping sections of the curve $y = \varphi(x)$.

In the case when $k_2 < 1$, closed trajectories cannot exist and only bifurcations of singular points are possible in system (1). Therefore, we will consider further only the case when $k_2 > 1$ and $(k_1 - 1)^2 < 4k_2$ giving various bifurcations and, first of all, the bifurcations of limit cycles. In the next section, studying all such bifurcations (local and global), we will give a sketch of the proof of the following theorem.

Theorem 3.1. *System (1) with k dropping sections and $2k + 1$ singular points can have at most $k + 2$ limit cycles, $k + 1$ of which surround the foci one by one and the last, $(k + 2)$-th, limit cycle surrounds all of the singular points of (1).*

4 Limit Cycle Bifurcations

Proof. To prove the theorem, we will study both local and global bifurcations of limit cycles. The limit cycle of system (1) will be called *small* if it belongs to at most two adjoining regions; the cycle will be called *big* if it belongs to at least three adjoining regions.

4.1 Local Bifurcations

Following [Bautin, 1974], we will study first stability of the singular points on the line of sewing. Suppose that the straight line $\beta - \alpha x - y = 0$ passes through the corner point (x_1, y_1) of the curve $y = \varphi(x)$ on the boundary of regions I, II and that $\alpha > (k_2 + 1)^2/4$. Then the region I (II) will be filled by the pieces of trajectories of the stable (unstable) focus.

Introduce positive coordinates S_0 (lower (x_1, y_1)) and S_1 (upper (x_1, y_1)) on the line of sewing of regions I and II; S_2 (lower (x_2, y_2)) and S_3 (upper (x_2, y_2)) on the line of sewing of regions II and III, etc. The maps $S_0 \to S_1$ along the trajectories of region I and $S_1 \to S_0$ along the trajectories of region II are written as follows:

$$S_1 = S_0 e^{\pi\sigma_1/\omega_1}, \quad \bar{S}_0 = S_1 e^{\pi\sigma_2/\omega_2}, \tag{2}$$

where σ_i, ω_i $(i = 1, 2)$ are the real and imaginary parts of the roots of the characteristic equation for a singular point of regions I, II, respectively.

The singular point (x_1, y_1) will be a sewed center $(\bar{S}_0 = S_0)$ iff $\sigma_1/\omega_1 + \sigma_2/\omega_2 = 0$, i.e., when $\alpha = \alpha^* \equiv (1 - k_1/k_2)/(k_2 - k_1 + 2)$. The sewed focus (x_1, y_1) will be stable $(\bar{S}_0 < S_0)$ when $\alpha > \alpha^*$ and unstable $(\bar{S}_0 > S_0)$ when $\alpha < \alpha^*$.

Consider the return map $S_0 \to \bar{S}_0$ along the trajectories of regions I and II. For region I, we will have

$$\begin{aligned} S_0 &= \frac{\delta_0}{\sin\omega_1\tau_1}(\omega_1\cos\omega_1\tau_1 - \sigma_1\sin\omega_1\tau_1 - \omega_1 e^{-\sigma_1\tau_1}) \\ &\equiv \delta_0\zeta(\tau_1), \\ S_1 &= \frac{\delta_0}{\sin\omega_1\tau_1}(\omega_1\cos\omega_1\tau_1 + \sigma_1\sin\omega_1\tau_1 - \omega_1 e^{\sigma_1\tau_1}) \\ &\equiv \delta_0\chi(\tau_1), \end{aligned} \tag{3}$$

where δ_0 is the distance from the boundary of regions I, II to the singular point; ζ and χ are monotonic functions. The return map along the trajectories of region II has a similar form.

Calculation of the first derivative for the return map gives

$$\frac{\mathrm{d}\bar{S}_0}{\mathrm{d}S_0} = \frac{S_0}{\bar{S}_0}\, e^{2(\sigma_1\tau_1 + \sigma_2\tau_2)}, \tag{4}$$

where τ_i $(i = 1, 2)$ is motion time along the trajectories of regions I, II, respectively; $\sigma_i = (1 + k_i)/2$ $(i = 1, 2)$.

Studying the return map $S_0 \to \bar{S}_0$ by means of (4), we prove that at most one limit cycle can exist in regions I and II (see also [Bautin, 1974]). The same result can be obtained for regions III and $IV, \ldots, 2K - 1$ and $2K$.

Consider now the map $\bar{S}_0 = f(S_0)$ sewed of two pieces: $\bar{S}_0 = \xi(S_0)$ along the trajectories in regions I, $II, \ldots, 2K$ and $\bar{S}_0 = \psi(S_0)$ along the trajectories in all regions, $I, II, \ldots, 2K, 2K+1$. The map $S_0 \to S_1$ in region I is given by (3). The maps $S_1 \to S_3$, $S_3 \to S_5, \ldots, S_{2k-1} \to S_{2k-2}$ $(S_{2k-1} \to S_{2k+1}$, $S_{2k+1} \to S_{2k}$, $S_{2k} \to S_{2k-2})$, $S_{2k-2} \to S_{2k-4}, \ldots$, $S_2 \to S_0$ have similar forms.

The derivatives for the functions $\xi(S_0)$, $\psi(S_0)$ are given by the following expressions, respectively:

$$\begin{aligned} \frac{\mathrm{d}\bar{S}_0}{\mathrm{d}S_0} &= \frac{S_0}{\bar{S}_0}\, e^{2\sigma_1(\tau_1 + \tau_3^+ + \tau_3^- + \ldots + \tau_{2k-1})} \\ &\quad \times e^{2\sigma_2(\tau_2^+ + \tau_2^- + \ldots + \tau_{2k-2}^+ + \tau_{2k-2}^-)}, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{\mathrm{d}\bar{S}_0}{\mathrm{d}S_0} &= \frac{S_0}{\bar{S}_0}\, e^{2\sigma_1(\tau_1 + \tau_3^+ + \tau_3^- + \ldots + \tau_{2k+1})} \\ &\quad \times e^{2\sigma_2(\tau_2^+ + \tau_2^- + \ldots + \tau_{2k}^+ + \tau_{2k}^-)}, \end{aligned} \tag{6}$$

where τ_1, τ_{2k-1}, τ_{2k+1} are motion times in regions I, $2K - 1$, $2K + 1$ and τ_{2i}^+ (τ_{2i}^-), τ_{2i+1}^+ (τ_{2i+1}^-), $i = 1, 2, \ldots, k$, are motion times in the upper (lower) parts of regions $II, III, \ldots, 2K$, respectively.

Studying the return map $\bar{S}_0 = f(S_0)$ by means of (5) and (6), we prove that at most two limit cycles can be generated by the boundary of the domain filled by closed trajectories of (1) and that these two limit cycles can be only outside the boundary.

Suppose that a part of the straight line $\beta - \alpha x - y = 0$ coincides with a dropping section of (1), for example, with the first one $(\alpha = k_2)$. The dropping section of (1) will be an unstable equilibrium segment and regions I, II (because of the condition $(k_1 - 1)^2 < 4k_2$) will be filled by trajectories of the stable foci. It is easy to obtain an explicit expression for the map of the half-line S_0 into itself:

$$\bar{S}_0 = S_0\, e^{2\pi\sigma_1/\omega_1} + \delta(k_2 - 1)(1 + e^{\pi\sigma_1/\omega_1}), \tag{7}$$

where δ is the width of regions II.

This map has the only stable fixed point, and we can show that two stable foci surrounded by unstable limit cycles (one by one) are generated from the ends of the equilibrium segment under the rotation of the line $\beta - \alpha x - y = 0$ (see also [Bautin, 1974]).

The simplest type of separatrix cycles of (1) is a so-called eight-loop formed by two ordinary saddle loops. In the case of $2k+1$ simple singular points, a separatrix cycle can contain $k + 1$ saddle loops, the first and the last of which are ordinary loops with one rough saddle on each and the $k - 1$ others are separatrix digons with two rough saddles on each. Such a separatrix cycle will be called *nondegenerate.* In the cases when the straight line $\beta - \alpha x - y = 0$ passes through the corner points of the curve $y = \varphi(x)$, we will have *degenerate* separatrix cycles of lips-type containing one or two sewed saddle-nodes. It is clear that the bifurcations of separatrix cycles do not depend on the parameter β (see [Bautin, 1974]). The separatrix cycles can be formed or destroyed only under a variation of the parameter α. The character of their stability will be determined by the sign of the saddle quantities which are always positive in our case, when the saddles are inside or on the boundary of even regions $II, IV, \ldots$, $2K$ and $k_2 > 1$ (Theorems 44 and 47 from [Andronov, Leontovich, Gordon and Maier, 1971] are valid for the piecewise linear dynamical systems as well). It follows that the separatrix cycles of (1) are always unstable (inside and outside) and, under a variation of α, a nondegenerate separatrix cycle can generate at most $k + 1$ small unstable limit cycles inside its loops (digons) or the only big unstable limit cycle outside it.

4.2 Global Bifurcations

Now we are able to consider the global bifurcations of limit cycles. Suppose again that the zero isocline $\beta-\alpha x-y=0$ passes through the corner point (x_1, y_1) of the infinite isocline $y=\varphi(x)$ and that $\alpha>\alpha^*$. In this case, the only singular point in the phase plane is a sewed stable focus and all trajectories of (1) tend to it when $t\to+\infty$. For decreasing α $(k_2<\alpha<\alpha^*)$, the sewed focus becomes unstable and a stable limit cycle is generated from the boundary curve of the domain filled by closed trajectories (immediately after passing the value α^* by the parameter α).

For $\alpha=k_2$, the first dropping section of (1) will coincide with a part of the straight line $\beta-\alpha x-y=0$ and an unstable equilibrium segment will appear inside the stable limit cycle. If we rotate the line $\beta-\alpha x-y=0$ around an interior point of the segment (changing both of the parameters, α and β), two unstable limit cycles surrounding stable foci (one by one) will be generated from the ends (x_1, y_1) and (x_2, y_2) of the equilibrium segment. Under the further rotation of the line $\beta-\alpha x-y=0$, it will pass first through the next corner point, (x_4, y_4), and then, successively, through the points $(x_6, y_6), \ldots, (x_{2k}, y_{2k})$. Every time, the corner point becomes a sewed saddle-node generating an unstable limit cycle surrounding a stable focus. So, we will get a piecewise linear system with $2k+1$ singular points having at least $k+1$ small unstable limit cycles surrounding the stable foci (one by one) inside a big stable limit cycle, $k+2$, surrounding all of the singular points.

Under the further rotation of the zero isocline, all $k+1$ small limit cycles simultaneously disappear in a separatrix cycle consisting of $k+1$ loops (digons), this separatrix cycle generates a big (unstable) limit cycle which combines with another big (stable) limit cycle of (1) forming a semi-stable (double) limit cycle which finally disappears in a so-called trajectory condensation.

Let us prove that system (1) cannot have more than $k+2$ limit cycles. The proof is carried out by contradiction by means of the Wintner–Perko termination principle [Bautin and Leontovich, 1990; Gaiko, 2003; Perko, 2002]. Since a small limit cycle is always unique in the corresponding strip regions, suppose that system (1) with three field rotation parameters, k_1, k_2, and α, has three big limit cycles. Then we get into some domain in the space of these parameters which is bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles [Gaiko, 2003; Perko, 2002].

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by a field rotation parameter, for example, by the parameter α, we will obtain a monotonic curve which, by the Wintner–Perko termination principle, terminates either at the boundary curve of the domain filled by closed trajectories of (1) or on some degenerate separatrix cycle of (1) [Gaiko, 2003; Perko, 2002].

Since we know at least the cyclicity of the boundary curve which is equal to two, we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the end bifurcation points in which they terminate [Gaiko, 2003; Perko, 2002].

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same principle, this again contradicts with the cyclicity result for the boundary curve not admitting the multiplicity of limit cycles to be higher than two. Moreover, it also follows from the termination principle that the degenerate separatrix cycles of (1) cannot have the multiplicity (cyclicity) higher than two. Therefore, according to the same principle, there are no more than two big limit cycles in the exterior domain outside the boundary curve of (1).

The same results can be obtained by means of the new geometric methods developed in [Gaiko, 2007, NA; Gaiko, 2007, IJMM]. The phase portraits and bifurcation diagrams for system (1) will be similar to that which were constructed in [Bautin, 1974, Bautin and Leontovich, 1990]. Thus, system (1) with $2k+1$ singular points cannot have more than $k+2$ limit cycles, i. e., $k+2$ is the maximum number of limit cycles of such system and the obtained distribution $(k+1$ small limit cycles plus a big limit cycle) is the only possibility for their distribution. The theorem is proved.

5 Conclusion

Thus, generalizing the results of [Bautin, 1974; Giannakopoulos and Pliete, 2001], where a planar dynamical system with a piecewise linear function containing the only dropping section and approximating some continuous nonlinear function was considered, we have studied a system with an arbitrary number of dropping sections and, by means of the bifurcation methods developed in [Botelho and Gaiko, 2006; Gaiko, 2003 – Gaiko and van Horssen, 2004], have given a sketch of the proof of the theorem stating that such a piecewise linear dynamical system with k dropping sections and $2k+1$ singular points can have at most $k+2$ limit cycles, $k+1$ of which surround the foci one by one and the last, $(k+2)$-th, limit cycle surrounds all of its singular points.

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