STOCHASTIC OPTIMAL CONTROL FOR DISCRETE-TIME SYSTEMS W.R.T. THE PROBABILISTIC PERFORMANCE INDEX

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Abstract
The optimal control problem for discrete-time stochastic systems with probabilistic performance index is considered. New results of qualitative research based on the dynamic programming are presented.

Key words
the discrete-time stochastic systems, the optimal control, the probabilistic performance index, the dynamic programming.

1 Introduction

The problem in question of this paper is stochastic optimal control of discrete-time system w.r.t. the probabilistic performance index. Such models arise in the aerospace [Malyshev, 1987], economics [Kibzun, 1996] and robotics [Lagoa, 2013]. The existing numerical methods for solving such problems are ineffective because of the known curse of dimension.

The probabilistic performance index is defined as probability that a certain precision functional does not exceed a certain admissible level. Here the precision functional itself characterizes the accuracy of the control system but depends on the trajectory of the stochastic system. One example of such a precision functional is the terminal miss of a guidance system.

In this paper we present new results concerning of properties of the Bellman function on the basis of utilisation of the boundedness of the probability.

Using the dynamic programming and the properties of the Bellman function we find two-sided bounds on the Bellman function under general assumptions about the control system, the domain of feasible controls, the precision functional and the random noise distribution. It is proved that under certain conditions the solution of the original control problem coincides with one of the stochastic programming problem of a certain structure.

As an example, the optimal control problem of a portfolio of securities with one risk-free and a given number of risk assets is considered. we find a class of asymptotic optimal control and prove the asymptotic optimality of the risk strategy in the case of one risk asset [Bunto, 2003], [Grigor'ev, 2004].

2 Problem statement

Consider the discrete-time stochastic system

$$x_{k+1} = f_k\left(x_k, u_k, \xi_k\right), \quad k = \overline{0, N}, \tag{1}$$

$x_k \in \mathbb{R}^n$ – the state vector, $x_0 = X$ – random variable (in general case), $u_k \in \mathbb{R}^m$ – the control vector, $\xi_k \in \mathbb{R}^s$ – random variable, $f_k : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^n$ – continuous for all $k = \overline{0, N}$ function, $N \in \mathbb{N}$ – the control horizon.

Let $\xi^k = (\xi_0, ..., \xi_k)^{\mathrm{T}}$. Suppose that ξ^N not depend on X and ξ_{k+1} not depend on ξ^k. We denote $\mathcal{U}_k$ as the set of Borel functions $u_k = u_k(x)$ which values satisfy the constraints $u_k(x) \in U_k$.

Consider the probability performance index

$$P_\varphi\left(u\left(\cdot\right)\right) = \mathbf{P}\left(\Phi\left(x_{N+1}\left(u\left(\cdot\right), \zeta\right)\right) \le \varphi\right),$$

where $\mathbf{P}$ - the probability, $\Phi : \mathbb{R}^n \to \mathbb{R}$ – bounded from below continuous function, $\varphi \in \mathbb{R}$ – the scalar, $u(\cdot) = \left(u_0^T(\cdot), ..., u_N^T(\cdot)\right)^T$ - the control, $\zeta = \left(X^T, \xi_0^T, ..., \xi_N^T\right)^T$.

The problem of optimal control of discrete-time stochastic system w.r.t probability performance index has the form

$$P_\varphi\left(u\left(\cdot\right)\right) \to \max_{u(\cdot) \in \mathcal{U}}, \tag{2}$$

where $\mathcal{U} = \mathcal{U}_0 \times ... \times \mathcal{U}_N$.

It is established [Malyshev, 1987] that if there exists a strategy $u^{\varphi}(\cdot) \in \mathcal{U}$ that satisfy the following recurrence relations of dynamic programming, then it is optimal in the problem (2):

$$u_k^{\varphi}(x_k) = \\ = \arg\max_{u_k \in U_k} \mathbf{M}\left[\mathsf{B}_{k+1}^{\varphi}\left(f_k\left(x_k, u_k, \xi_k\right)\right) \middle| x_k\right],$$

$$\mathsf{B}_k^{\varphi}(x) = \\ = \max_{u_k \in U_k} \mathbf{M}\left[\mathsf{B}_{k+1}^{\varphi}\left(f_k\left(x_k, u_k, \xi_k\right)\right) \middle| x_k = x\right],$$

with the boundary condition

$$\mathsf{B}_{N+1}^{\varphi}(x) = \mathbf{I}_{(-\infty,\varphi]}\left(\Phi(x)\right), \tag{3}$$

where $\mathbf{M}$ – the expected value, $\mathbf{I}_{\mathcal{A}}(x)$ – the indicator function of set $\mathcal{A}$, $\mathsf{B}_k^{\varphi}(x)$ – the Bellman function in problem (2)

$$\mathsf{B}_k^{\varphi}(x) = \\ = \sup_{u_k(\cdot) \in \mathcal{U}_k, \ldots, u_N(\cdot) \in \mathcal{U}_N} \mathbf{P}\left(\Phi\left(x_{N+1}\right) \leq \varphi \middle| x_k = x\right).$$

In [Azanov, 2017] using the surfaces of level 1 and 0 of the Bellman function, called Isabella 1 and 0, managed to obtain a properties of Bellman equation and Bellman function. Based on this properties in [Azanov, 2017] was obtained a solutions of complex problems in aerospace focus. The following section describes the main statements of the [Azanov, 2017] that used further to solve the problem of optimal control of a portfolio of securities.

3 The definition of Isabella and modification of the Bellman equation

The main idea in [Azanov, 2017] is to consider the Bellman equation in different regions of the state space, namely, where the Bellman function is equal to one (Isabella of level 1) and zero (Isabella of level 0). Consider the sets

$$\mathcal{I}_k^{\varphi} = \left\{x \in \mathbb{R}^n : \quad \mathsf{B}_k^{\varphi}(x) = 1\right\},$$
$$\mathcal{O}_k^{\varphi} = \left\{x \in \mathbb{R}^n : \quad \mathsf{B}_k^{\varphi}(x) = 0\right\}.$$

Consider also the set where the Bellman function is not equal neither 1, nor 0

$$\mathcal{B}_k^{\varphi} = \mathbb{R}^n \setminus \left\{\mathcal{I}_k^{\varphi} \cup \mathcal{O}_k^{\varphi}\right\}.$$

From the definition of the sets $\mathcal{I}_k^{\varphi}$, $\mathcal{B}_k^{\varphi}$, $\mathcal{O}_k^{\varphi}$ it follows that

$$\mathcal{I}_k^{\varphi} \cup \mathcal{B}_k^{\varphi} \cup \mathcal{O}_k^{\varphi} = \mathbb{R}^n, \quad \begin{cases} \mathsf{B}_k^{\varphi}(x) = 1, & x \in \mathcal{I}_k^{\varphi}, \\ \mathsf{B}_k^{\varphi}(x) \in (0,1), & x \in \mathcal{B}_k^{\varphi}, \\ \mathsf{B}_k^{\varphi}(x) = 0, & x \in \mathcal{O}_k^{\varphi}. \end{cases}$$

In [Azanov, 2017] proved the following Lemma that Isabella satisfy the recurrent relations that does not depend on the Bellman function.

Lemma 1. *The following are true:*

1. *The set $\mathcal{I}_k^{\varphi}$ satisfy the recurrent relations in reverse time $k = \overline{0, N}$*

$$\mathcal{I}_k^{\varphi} = \Big\{x \in \mathbb{R}^n : \quad \exists u_k \in U_k : \\ \mathbf{P}\left(f_k\left(x, u_k, \xi_k\right) \in \mathcal{I}_{k+1}^{\varphi}\right) = 1\Big\}$$

with the boundary condition

$$\mathcal{I}_{N+1}^{\varphi} = \left\{x \in \mathbb{R}^n : \quad \Phi(x) \leq \varphi\right\};$$

2. *The set $\mathcal{O}_k^{\varphi}$ satisfy the recurrent relations in reverse time $k = \overline{0, N}$*

$$\mathcal{O}_k^{\varphi} = \Big\{x \in \mathbb{R}^n : \quad \forall u_k \in U_k : \\ \mathbf{P}\left(f_k\left(x, u_k, \xi_k\right) \in \mathcal{O}_{k+1}^{\varphi}\right) = 1\Big\},$$

with the boundary condition

$$\mathcal{O}_{N+1}^{\varphi} = \left\{x \in \mathbb{R}^n : \quad \Phi(x) > \varphi\right\};$$

3. *The Bellman equation in the range $\mathcal{B}_k^{\varphi}$ has the form*

$$\mathsf{B}_k^{\varphi}(x) = \max_{u_k \in U_k} \Big\{\mathbf{P}\left(f_k\left(x, u_k, \xi_k\right) \in \mathcal{I}_{k+1}^{\varphi}\right) + \\ + \mathbf{P}\left(f_k\left(x, u_k, \xi_k\right) \in \mathcal{B}_{k+1}^{\varphi}\right) \times \\ \times \mathbf{M}\left[\mathsf{B}_{k+1}^{\varphi}\left(f_k\left(x, u_k, \xi_k\right)\right) \middle| f_k\left(x, u_k, \xi_k\right) \in \mathcal{B}_{k+1}^{\varphi}\right]\Big\}$$

From Lemma 1 implies the following important result, proved in [Azanov, 2017]. We give it below in a formulation that convenient for this article.

Corollary 1. *The following are true:*

1. *For $x_k \in \mathcal{I}_k^{\varphi}$ the optimal control at step k is an any element from the set $U_k^{\mathcal{I}}(x_k)$*

$$U_k^{\mathcal{I}}(x_k) = \\ = \left\{u \in U_k : \ \mathbf{P}\left(f_k\left(x_k, u, \xi_k\right) \in \mathcal{I}_{k+1}^{\varphi}\right) = 1\right\}.$$

2. *For $x_k \in \mathcal{O}_k^{\varphi}$ the optimal control at step k is an any element from the set U_k.*

3. *$\forall x \in \mathcal{B}_k^{\varphi}$, $u_k(x) \in U_k$ we have a system of inequalities*

$$\mathbf{P}\left(f_k\left(x, u_k, \xi_k\right) \in \mathcal{I}_{k+1}^{\varphi}\right) \leq \\ \leq \mathbf{M}\left[\mathsf{B}_{k+1}^{\varphi}\left(f_k\left(x, u_k, \xi_k\right)\right)\right] \leq \\ \leq 1 - \mathbf{P}\left(f_k\left(x, u_k, \xi_k\right) \in \mathcal{O}_{k+1}^{\varphi}\right). \tag{4}$$

4. *For $k = N$ Bellman function satisfy the equality*

$$\mathsf{B}_N^{\varphi}(x) = \max_{u_N \in U_N} \mathbf{P}\left(f_N(x, u_N, \xi_N) \in \mathcal{I}_{N+1}^{\varphi}\right).$$

Using Lemma 1 and Corollary 1 allows, first, to find Isabella, secondly, to find the optimal controls for $x \in \mathcal{I}_k^{\varphi}$ without affecting the Bellman equations and related problems of stochastic programming of complex structure. In this case, from claim 2 we conclude that the definition Isabella level 0 automatically closes the issue of optimal control in $x \in \mathcal{O}_k^{\varphi}$. Thus, the relevant issue is the determination of the optimal controls for $x \in \mathcal{B}_k^{\varphi}$.

Using the inequality in item 3 from Corollary 2, we can get the overall result of the qualitative character, namely, two-way evaluation of the Bellman function in $\mathcal{B}_k^{\varphi}$. This result is presented in the next section.

4 Bilateral evaluation of the Bellman function

Using inequality (4), we formulate the statement about bilateral evaluation of the Bellman function in $\mathcal{B}_k^{\varphi}$.

Theorem 1. *For $x \in \mathcal{B}_k^{\varphi}$ the Bellman function satisfies the inequalities*

$$\underline{\mathsf{B}}_k^{\varphi}(x) \leq \mathsf{B}_k^{\varphi}(x) \leq \overline{\mathsf{B}}_k^{\varphi}(x), \tag{5}$$

where

$$\underline{\mathsf{B}}_k^{\varphi}(x) = \sup_{u_k \in U_k} \mathbf{P}\left(f_k(x, u_k, \xi_k) \in \mathcal{I}_{k+1}^{\varphi}\right),$$
$$\overline{\mathsf{B}}_k^{\varphi}(x) = \sup_{u_k \in U_k} \left\{1 - \mathbf{P}\left(f_k(x, u_k, \xi_k) \in \mathcal{O}_{k+1}^{\varphi}\right)\right\},$$

The proof of the theorem 1 is obtained by calculating supremum in the left and right parts of the inequalities (4).

Thus, the Bellman function in $\mathcal{B}_k^{\varphi}$ is bounded from below by the maximum probability that the trajectory of the control system belongs to Isabella of level 1 in the next step and bounded from top of the maximum probability of missing the trajectory of the control system on Isabella of level 0 in the next step.

From theorem 1 it is possible to obtain two-sided estimate for the optimal value of probabilistic performance index. Let $X \in \mathbb{R}^n$ is a deterministic vector. Then the optimal value of probabilistic performance index [Kan, 2001] has the form

$$F(\varphi, N, X) = \mathsf{B}_0^{\varphi}(X) = P_{\varphi}(u^{\varphi}(\cdot)).$$

We introduce

$$\underline{F}(\varphi, N, x) = \underline{\mathsf{B}}_0^{\varphi}(x), \quad \overline{F}(\varphi, N, x) = \overline{\mathsf{B}}_0^{\varphi}(x).$$

Corollary 2. *Let $X \in \mathbb{R}^n$ is a deterministic vector. Then for any $\varphi \in \mathbb{R}$, $N \in \mathbb{N}$ and $X \in \mathcal{B}_0^{\varphi}$ the following inequality holds*

$$\underline{F}(\varphi, N, X) \leq F(\varphi, N, X) \leq \overline{F}(\varphi, N, X).$$

Corollary 2 is the key in the study of the asymptotic properties of strategies in the problem of optimal portfolio control of securities. We show this by example. Let $u^{\infty}(\cdot) \in \mathcal{U}$ – is some strategy and

$$F^{\infty}(\varphi, N, X) = P_{\varphi}(u^{\infty}(\cdot)).$$

Suppose that for any $\varphi \in \mathbb{R}$, $N \in \mathbb{N}$, $X \in \mathcal{B}_0^{\varphi}$ we have

$$\underline{F}(\varphi, N, X) \leq F^{\infty}(\varphi, N, X). \tag{6}$$

If

$$\lim_{N \to \infty} \underline{F}(\varphi, N, X) = \lim_{N \to \infty} \overline{F}(\varphi, N, X), \tag{7}$$

then the strategy $u^{\infty}(\cdot) \in \mathcal{U}$ is as asymptotic optimal, because of the equality

$$\lim_{N \to \infty} F^{\infty}(\varphi, N, X) = \lim_{N \to \infty} F(\varphi, N, X).$$

At made assumptions the last revenue is performed in view of the fact that for any $\varphi \in \mathbb{R}$, $N \in \mathbb{N}$, $X \in \mathcal{B}_0^{\varphi}$ fair system of inequalities

$$\underline{F}(\varphi, N, X) \leq F^{\infty}(\varphi, N, X) \leq$$
$$\leq F(\varphi, N, X) \leq \overline{F}(\varphi, N, X).$$

Using the results of sections 3 and 4, let us consider the problem of optimal control of a portfolio of securities.

5 Portfolio selection

Consider the scalar control system

$$\begin{cases} x_{k+1} = x_k \left(1 + u_k^1 b + \sum\limits_{i=2}^{m} u_k^i \xi_k^i\right), \\ x_0 = X, \end{cases} \tag{8}$$

where $n = 1$ is the dimension of the state vector x_k, $m > 1$ - dimension of control vector u_k, $m - 1$ - dimension of random vector ξ_k, $X > 0$, $b > -1$ - deterministic scalars. It is assumed the independence of the vectors ξ_{k+1} from ξ^k (see section 2). Let the carrier distribution of a random vector ξ_k has the form

$$\Xi = \left\{\zeta \in \mathbb{R}^{m-1} : \quad \underline{b}^i \leq \zeta^i \leq \overline{b}^i, \quad i = \overline{2, m}\right\},$$

where $\overline{b}^i > b > \underline{b}^i \geq -1$, for all i and k. That inequalities have an important economic meaning. The inequality $\overline{b}^i > b$ means that the maximum yield of risk asset larger than yield of non risk assets (otherwise the optimal strategy will be investing in non risk asset). The inequality $b > \underline{b}^i$ means that minimum yield of risky assets smaller than yield of non risk asset (otherwise the optimal value of component u_k^1 of control vector will be equal to zero). The case $\underline{b}^i = -1$ (for some i) means that if we will invest all capital into this asset there are some non zero probability of complete ruin.

Set U_k has the following form

$$\left\{u \in \mathbb{R}^m : \quad \sum_{i=1}^{m} u^i = 1, \quad u^i \geq 0, \quad i = \overline{1,m}\right\},$$

which means that we invest all of our capital and we cant do the "short sale" operation.

We consider the problem

$$\mathbf{P}\left(-x_{N+1} \leq \varphi\right) \to \max_{u(\cdot)\in\mathcal{U}}. \tag{9}$$

In economics interpretation we have that X is the size of the initial capital, x_k - the amount of capital at the beginning of the k-th year, u_k^1 - the fraction of capital x_k that invested in risk-free instrument (e.g., in reliable Bank), with a yield of b, u_k^i - the share of capital x_k that invested in risky assets, characterized by yields ξ_k^i $i = \overline{2,m}$. The problem (9) is to maximize the probability of achieving by the value of capital level $-\varphi$ at a given point in time $N+1$ by investing in certain assets.

Note that in earlier works (see for example [Kibzun, 1996]), which dealt with the case of only one risky asset, much attention was paid to the asymptotic properties of strategies, which is suboptimal in problem (9). The interest in these properties manifests because of the so-called 'exchange paradox" arising because of the using the strategy that is optimal with respect to criterion of "average return" [Kibzun, 2001], i.e. $\mathbf{M}\left[x_{N+1}\right]$. This paradox is given below: when $N \to \infty$, the system synthesized control, optimal by the criterion of 'average return", behaves in such a way that the average value of capital tends to infinity, and the probability of ruin to one. Due to the fact that to this day has not found the optimal probabilistic strategy (even for a particular case) in the 'multi-stage" task, the test of its asymptotic properties is difficult. In this section, using theorem 1, there is a class of asymptotic optimal strategy in problem (9), we study its new properties, are characteristic findings.

In introduced in this article, the designations have

$$f_k\left(x_k, u_k, \xi_k\right) = x_k\left(1 + u_k^1 b + \sum_{i=2}^{m} u_k^i \xi_k^i\right),$$
$$\Phi(x) = -x, \quad \varphi < 0.$$

Using Lemma 1 we find the sets $\mathcal{I}_k^\varphi$, $\mathcal{B}_k^\varphi$, $\mathcal{O}_k^\varphi$.

Statement 1. *Sets $\mathcal{I}_k^\varphi$, $\mathcal{B}_k^\varphi$, $\mathcal{O}_k^\varphi$ are determined by the expressions*

$$\mathcal{I}_k^\varphi = \left[\varphi_k^{\mathcal{I}}, \infty\right),$$
$$\mathcal{B}_k^\varphi = \left(\varphi_k^{\mathcal{O}}, \varphi_k^{\mathcal{I}}\right),$$
$$\mathcal{O}_k^\varphi = \left(-\infty, \varphi_k^{\mathcal{O}}\right],$$

where $\varphi_k^{\mathcal{I}}$, $\varphi_k^{\mathcal{O}}$ determine

$$\varphi_k^{\mathcal{I}} = -\varphi\left(1+b\right)^{k-N-1},$$
$$\varphi_k^{\mathcal{O}} = -\varphi\left(1 + \max_{j=\overline{2,m}} \overline{b}^j\right)^{k-N-1},$$

With respect to paragraph 1 of corollary 1, any set of vectors $u_k(x)$ of one-parameter families of sets

$$U_k^{\mathcal{I}}(x) = \left\{u \in U : \mathbf{P}\left(x\left(1 + u^1 b + \sum_{i=2}^{m} u^i \xi_k^i\right) \geq \varphi_{k+1}^{\mathcal{I}}\right) = 1\right\},$$

is the optimal control at step k for $x \geq \varphi_k^{\mathcal{I}}$. Such control, for example, is $u_k(x) = (1, 0, ..., 0)^{\mathrm{T}}$. In accordance with paragraph 2 of the corollary, any set of vectors $u_k(x) \in U$ is the optimal control at step k for $x \leq \varphi_k^{\mathcal{O}}$.

Finally, we obtain that for $x \notin \mathcal{B}_k^\varphi$ the optimal control has the form

$$u_k^\varphi(x) = \begin{cases} \text{any element } U_k^{\mathcal{I}}(x), & x \in \left[\varphi_k^{\mathcal{I}}, +\infty\right), \\ \text{any element } U, & x \in \left(-\infty, \varphi_k^{\mathcal{O}}\right]. \end{cases}$$

Now let $x \in \mathcal{B}_k^\varphi$. Using theorem 1 and statement 1, we conclude that the the lower and upper bounds of Bellman function satisfy the equalities

$$\underline{\mathsf{B}}_k^\varphi(x) = \max_{u_k \in U} \mathbf{P}\left(x\left(1 + u_k^1 b + \sum_{i=2}^{m} u_k^i \xi_k^i\right) \geq \varphi_{k+1}^{\mathcal{I}}\right),$$
$$\overline{\mathsf{B}}_k^\varphi(x) = \max_{u_k \in U} \mathbf{P}\left(x\left(1 + u_k^1 b + \sum_{i=2}^{m} u_k^i \xi_k^i\right) \geq \varphi_{k+1}^{\mathcal{O}}\right).$$

Let start investigating the lower and the upper bounds of the Bellman function. Note, that by virtue of point 4 of corollary 1 these functions with a precision up to

$\varphi_{k+1}^{\mathcal{I}}$, $\varphi_{k+1}^{\mathcal{O}}$, φ and distributions ξ_k, ξ_N coincides with the Bellman function at $k = N$, i.e.

$$\mathsf{B}_N^{\varphi}(x) = \\ = \max_{u_N \in U} \mathbf{P}\left(x\left(1 + u_N^1 b + \sum_{i=2}^{m} u_N^i \xi_N^i\right) \geq -\varphi\right).$$

We conclude that the solution of the corresponding problems of stochastic programming in (10) exists (this condition is proved in [Kibzun, 1996]).

Let us use the result of section 4 to find a class of asymptotically optimal strategies. With the approval of 1 and (11), we find the lower $\underline{F}$ and upper $\overline{F}$ estimates for functions of the optimal probability

$$\underline{F}(\varphi, N, X) = \\ = \max_{u_0 \in U} \mathbf{P}\left(X\left(1 + u_0^1 b + \sum_{i=2}^{m} u_0^i \xi_0^i\right) \geq \\ \geq -\varphi(1+b)^{-N}\right),$$

$$\overline{F}(\varphi, N, X) = \\ = \max_{u_0 \in U} \mathbf{P}\left(X\left(1 + u_0^1 b + \sum_{i=2}^{m} u_0^i \xi_0^i\right) \geq \\ \geq -\varphi\left(1 + \max_{j=\overline{2,m}} \overline{b}^j\right)^{-N}\right).$$

Based on the properties of continuous probability functions [Kibzun, 1996], we obtain the equality (7), i.e.

$$\lim_{N\to\infty} \underline{F}(\varphi, N, X) = \lim_{N\to\infty} \overline{F}(\varphi, N, X).$$

Thus, to build a class asymptotic optimal strategies, using corollary 2, it is enough to specify such strategies that satisfied inequality (6). Les us show that this property has a one-parameter class of strategy $u^{\beta}(\cdot)$, whose definition is given below

$$u_k^{\beta}(x) = \\ = \arg\max_{u_k \in U} \mathbf{P}\left(x\left(1 + u_k^1 b + \sum_{i=2}^{m} u_k^i \xi_k^i\right) \geq \\ \geq -\varphi(1+\beta)^{k-N}\right), \quad (10)$$

where β is a numeric parameter, chosen from the interval $\left[b, \max_{j=\overline{2,m}} \overline{b}^j\right]$. We denote

$$\mathsf{B}_k^{\beta}(x) = \\ = \max_{u_k \in U} \mathbf{P}\left(x\left(1 + u_k^1 b + \sum_{i=2}^{m} u_k^i \xi_k^i\right) \geq \\ \geq -\varphi(1+\beta)^{k-N}\right). \quad (11)$$

The meaning of the function $\mathsf{B}_k^{\beta}(x)$ is: when $\beta = b$ it coincides with the lower, and at $\beta = \max_{j=\overline{2,m}} \overline{b}^j$ with the upper border of the Bellman function. Otherwise, she, as the Bellman function, satisfies the following inequality (by the properties of probability)

$$\underline{\mathsf{B}}_k^{\varphi}(x) \leq \mathsf{B}_k^{\beta}(x) \leq \overline{\mathsf{B}}_k^{\varphi}(x).$$

Consider the function

$$F^{\beta}(\varphi, N, X) = P_{\varphi}\left(u^{\beta}(\cdot)\right).$$

It is easily to show that for all $\varphi < 0$, $X \in \mathcal{B}_0^{\varphi}$, $N \in \mathbb{N}$ the next inequalities are true

$$\underline{F}(\varphi, N, X) \leq F^{\beta}(\varphi, N, X) \leq \overline{F}(\varphi, N, X).$$

From the last inequalities and (12) we finally have that

$$\lim_{N\to\infty} F^{\beta}(\varphi, N, X) = \lim_{N\to\infty} F(\varphi, N, X). \quad (12)$$

The equality (12) is the evidence of the asymptotic optimality of the one-parametric class of policies $u^{\beta}(\cdot)$ with parameter β.

Finally, using the results [Kibzun, 1996], were was found the analytically solution of stochastic programming problem type (10), we can find $u_k^{\beta}(x)$ and $\mathsf{B}_k^{\beta}(x)$ analytically in the case of one risk asset $m = 2$.

Statement 2. *Let $m = 2$, $\underline{b}^m = \underline{b}$ and $\overline{b}^m = \overline{b}$, then the control system determines*

$$f_k(x_k, u_k, \xi_k) = x_k\left(1 + u_k^1 b + u_k^2 \xi_k\right),$$

function $u_k^{\beta}(x)$ has the form

$$\begin{cases} (1,\ 0)^{\mathrm{T}}, & x \geq -\varphi(1+\beta)^{k-N}(1+b)^{-1}, \\ (0,\ 1)^{\mathrm{T}}, & x < -\varphi(1+\beta)^{k-N}(1+b)^{-1}. \end{cases}$$

Function $\mathsf{B}_k^{\beta}(x)$ *has the form*

$$\begin{cases} 1, & x > \psi_k(\beta, b), \\ 0, & x < \psi_k(\beta, a), \\ 1 - F_\xi\left(\frac{-\varphi}{x(1+\beta)^{N-k}} - 1\right), & \textit{else}, \end{cases}$$

where $\psi_k(x, y) = -\varphi(1+x)^{k-N}(1+y)^{-1}$ *and* F_ξ *is the distribution function of a random variable* ξ_k.

The strategy $u^{\beta}(\cdot) = \left(u_0^{\beta}(\cdot), ..., u_N^{\beta}(\cdot)\right)$ from statement 2 in case of $\beta = b$ called in [Kibzun, 1996] as "the risk strategy". It is interesting that in [Kibzun, 1996] the risk strategy was found using some heuristic considerations based on economic sense. This economic sense is this: until we reach the level of capital $\psi_k(b, b)$ (or $\varphi_k^{\mathcal{I}}$, which is the same), we will invest in the risky asset, and after reaching this level we will invest in non risky asset. Using point 4 of corollary 1 we conclude that risk strategy is optimal control for $k = N$.

6 Conclusion

The new properties of the Bellman equation in the problem of optimal control for discrete-time stochastic systems with probabilistic performance index are researched. On their basis we found a class of asymptotic optimal strategies in the problem of optimal investment subject to risk.

Acknowledgements

The research was supported by the Russian Science Foundation grant 16-11-00062.

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