A new topological approach to target the existence of solutions for nonlinear fractional impulsive wave equations

Svetlin G. Georgiev [a], Khaled Zennir[b]

[a] *Department of Differential Equations, Faculty of Mathematics and Informatics, University of Sofia, Sofia, Bulgaria.*

[b] *Department of Mathematics, College of Sciences and Arts, Qassim University, Ar-Rass, Saudi Arabia.*

E-mail: svetlingeorgiev1@gmail.com (S. Georgiev)

E-mail: khaledzennir4@gmail.com (Kh. Zennir)

Abstract

In the present paper and to improve the results in [8], we consider a class of fractional impulsive wave equations. Using a new topological approach, we prove the existence of classical solutions with a complex arguments caused by impulsive perturbations. To the best of our knowledge, There is a severe lack of results related to such impulsive equations.

Key words: Fractional impulsive wave equations, Classical solutions, Fixed point, Cone, Sum of operators.

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1 Introduction

The theory of nonlinear waves is still a young sciences, although research in this direction was carried out even in the 19^{th} century, mainly in connection with the problems of gas and hydrodynamics. For example, the works of J. Scott Russell [12] who was the first to observe solutions on the surfaces of a liquid, date back to 1830-1840. Nonlinear wave pgenomena have been the subject of research by such outstanding scientists as Poison, Stokes, Airy, Rayleigh, Boussinesq, Riemann. However, as a unified science, the theory of nonlinear waves developed in the late 1960s and early 1970s, which were the years of its rapid development.

This type of problem appears in several mathematical models which describe wave phenomena in areas such as fluid dynamics and electromagnetism. Many authors such as H. Brésiz, J. Mawhin, K. C. Chang and others, have developed topological tools, index theory and variational methods to obtain a classical existence results for the one-dimensional problem with various non-linearities. One can review the associated results in [2, 3, 4, 5] and the references therein.

A fractional derivative is a non-local characteristic of a function: it depends not only on the behavior of the function in the vicinity of the point x under consideration, but also on the values it takes over the entire interval (a, x). This non-locality means that the change in the particle flux density depends not only on its values in the vicinity of the point under consideration, but also on its values at distant points in space. We mention some related results on the impulsive equation in [8, 9] and these models have not been sufficiently studies, despite their versability and practical importance.

To beginwith, we consider the following problem

$$\begin{aligned}
{}^cD^{\beta}_{t,0+}u - \Delta u &= f(t, x, u, u_t, u_x), \quad t \in J = [0, 1], \quad t \neq t_k, \quad k \in \{1, \dots, n_1\}, \\
& \quad x \in \mathbb{R}^n, \\
u_t(t_k+, x) &= u_t(t_k-, x) + I_k(t_k, x, u(t_k, x)), \quad x \in \mathbb{R}^n, \quad k \in \{1, \dots, n_1\}, \\
u(t_k+, x) &= u(t_k-, x) + L_k(t_k, x, u(t_k, x)), \quad x \in \mathbb{R}^n, \quad k \in \{1, \dots, n_1\}, \\
u(0, x) &= h_1(x, u(0, x)), \quad u(1, x) = h_2(x, u(1, x)), \quad x \in \mathbb{R}^n,
\end{aligned} \tag{1.1}$$

where

(H1) ${}^cD^{\beta}_{t,0+}$ is the Caputo fractional derivative with respect to t, $\beta \in (1, 2]$, $0 = t_0 < t_1 < \ldots < t_{n_1} < t_{n_1+1} = 1$, $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, …, $J_{n_1} = (t_{n_1}, 1]$, $n_1 \in \mathbb{N}$.

(H2) $I_k, L_k \in \mathcal{C}([0, T] \times \mathbb{R}^{n+1})$,

$$\begin{aligned}
|I_k(t_k, x, u(t_k, x))| &\leq a_{1k}(t_k, x)|u(t_k, x)|^{s_{1k}}, \\
|L_k(t_k, x, u(t_k, x))| &\leq a_{2k}(t_k, x)|u(t_k, x)|^{s_{2k}}, \quad x \in \mathbb{R}^n, \quad k \in \{1, \dots, n_1\},
\end{aligned}$$

$a_{1k}, a_{2k} \in \mathcal{C}(J \times \mathbb{R}^n)$, $0 \leq a_{1k}, a_{2k} \leq B$ on $J \times \mathbb{R}^n$, $k \in \{1, \dots, n_1\}$, for some positive constant B, s_{1k}, $s_{2k} \geq 0$, $k \in \{1, \dots, n_1\}$.

(H3) $h_1, h_2 \in \mathcal{C}^2(\mathbb{R}^{n+1})$,

$$\begin{aligned} |h_1(x, u(0,x))| &\leq b_{11}(x)|u(0,x)|^{s_1}, \\ |h_2(x, u(1,x))| &\leq b_{12}(x)|u(1,x)|^{s_2}, \quad x \in \mathbb{R}^n, \end{aligned}$$

$b_{11},\ b_{12} \in \mathcal{C}(\mathbb{R}^n)$, $0 \leq b_{11}, b_{12} \leq B$ on $\mathbb{R}^n$, $s_1, s_2 \geq 0$.

(H4) $f \in \mathcal{C}(J \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n)$,

$$|f(t,x,u,v,w)| \leq \sum_{j=1}^{r} \left(a_j(t,x)|u|^{p_j} + b_j(t,x)|v|^{q_j} + \sum_{i=1}^{n} c_{ji}(t,x)|w_i|^{r_{ji}} \right),$$

$(t,x) \in J \times \mathbb{R}^n$, $u, v \in \mathbb{R}$, $w \in \mathbb{R}^n$, $a_j, b_j, c_{ji} \in \mathcal{C}(J \times \mathbb{R}^n)$, $0 \leq a_j, b_j, c_{ji} \leq B$ on $J \times \mathbb{R}^n$, $p_j, q_j, r_{ji} > 0$, $j \in \{1, \ldots, r\}$, $i \in \{1, \ldots, n\}$, $r \in \mathbb{N}$.

Here $u_x = (u_{x_1}, \ldots, u_{x_n})$, $u_t(t_k-, x) = \lim_{t \to t_k-} u_t(t,x)$, $u_t(t_k+, x) = \lim_{t \to t_k+} u_t(t,x)$, $u(t_k-, x) = \lim_{t \to t_k-} u(t,x)$, $u(t_k+, x) = \lim_{t \to t_k+} u(t,x)$, $x \in \mathbb{R}^n$, $k \in \{1, \ldots, n_1\}$. For $l, s \in \mathbb{N} \cup \{0\}$, define

$$\begin{aligned} PC(J) &= PC^0(J) \\ &= \{g : J \to \mathbb{R}, \quad g \in \mathcal{C}(J \backslash \{t_j\}_{j=1}^{m-1}), \\ &\quad \exists g(t_j+), \quad g(t_j-) \quad \text{and} \quad g(t_j-) = g(t_j), \\ &\quad j \in \{1, \ldots, n_1\}\}, \\ PC^l(J) &= \{g : J \to \mathbb{R}, \quad g \in PC^{l-1}(J), \quad g \in \mathcal{C}^l(J \backslash \{t_j\}_{j=1}^{n_1}), \\ &\quad \exists g^{(l)}(t_j-), g^{(l)}(t_j+) \quad \text{and} \quad g^{(l)}(t_j-) = g^{(l)}(t_j), \\ &\quad j \in \{1, \ldots, n_1\}\} \end{aligned}$$

and

$$\begin{aligned} PC^l(J, \mathcal{C}^s(\mathbb{R}^n)) &= \{u : u(\cdot, x) \in PC^l(J), \quad x \in \mathbb{R}^n, \\ &\quad u(t, \cdot) \in \mathcal{C}^s(\mathbb{R}^n), \quad t \in J\}. \end{aligned}$$

In $PC^2(J,\mathcal{C}^2(\mathbb{R}^n))$, we define the norm

$$\begin{aligned}
\|u\| &= \max\{\max_{j\in\{0,1,\ldots,n_1\}}\sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n}|u(t,x)|,\\
&\max_{j\in\{0,1,\ldots,n_1\}}\sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n}|u_t(t,x)|,\\
&\max_{j\in\{0,1,\ldots,n_1\}}\sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n}|u_{tt}(t,x)|,\\
&\max_{j\in\{0,1,\ldots,n_1\}}\sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n}|u_{x_i}(t,x)|,\\
&\max_{j\in\{0,1,\ldots,n_1\}}\sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n}|u_{x_ix_i}(t,x)|, \quad i\in\{1,\ldots,n\}\},
\end{aligned}$$

as long as it exists. Here $PC^2(J,\mathcal{C}^2(\mathbb{R}^n))$ is a Banach space.
We state now our main results.

Theorem 1.1. *Suppose* $(H1)$-$(H4)$. *Then the problem* (1.1) *has a solution in* $PC^2(J,\mathcal{C}^2(\mathbb{R}^n))$.

Theorem 1.2. *Suppose* $(H1)$-$(H4)$. *Then the problem* (1.1) *has at least two solutions in* $PC^2(J,\mathcal{C}^2(\mathbb{R}^n))$.

2 Preliminary

Here, we introduce some preliminary results which will be used to prove the main results. The following fixed point theorem for sum of two operators will be used to prove the existence of at least one solution to the problem (1.1).

Theorem 2.1. *Let* $\epsilon\in(0,1)$, $B>0$, E *be a Banach space and* $X=\{x\in E:\|x\|\le B\}$. *Let also,* $Tx=-\epsilon x$, $x\in X$, $S:X\to E$ *is continuous,* $(I-S)(X)$ *resides in a compact subset of* E *and*

$$\{x\in E: x=\lambda(I-S)x, \quad \|x\|=B\}=\emptyset \tag{2.1}$$

for any $\lambda\in\left(0,\frac{1}{\epsilon}\right)$. *Then there exists a* $x^*\in X$ *so that*

$$Tx^*+Sx^*=x^*.$$

Here $\mu X=\{\mu x: x\in X\}$ *for any* $\mu\in\mathbb{R}$.

Proof. Define

$$r\left(-\frac{1}{\epsilon}x\right)=\begin{cases}-\frac{1}{\epsilon}x & \text{if} \quad \|x\|\le B\epsilon\\ \\ \frac{Bx}{\|x\|} & \text{if} \quad \|x\|>B\epsilon.\end{cases}$$

Then $r\left(-\frac{1}{\epsilon}(I-S)\right):X\to X$ is continuous and compact. Then, owing to the Schauder fixed point theorem, there exists $x^*\in X$ such that

$$r\left(-\frac{1}{\epsilon}(I-S)x^*\right)=x^*,$$

where $-\frac{1}{\epsilon}(I-S)x^*\notin X$. Thus

$$\Big\|(I-S)x^*\Big\|>B\epsilon, \quad \frac{B}{\|(I-S)x^*\|}<\frac{1}{\epsilon}$$

and

$$x^*=\frac{B}{\|(I-S)x^*\|}(I-S)x^*=r\left(-\frac{1}{\epsilon}(I-S)x^*\right)$$

and hence, $\|x^*\|=B$. This contradicts with (2.1). Therefore $-\frac{1}{\epsilon}(I-S)x^*\in X$ and

$$x^*=r\left(-\frac{1}{\epsilon}(I-S)x^*\right)=-\frac{1}{\epsilon}(I-S)x^*$$

or

$$-\epsilon x^*+Sx^*=x^*,$$

or

$$Tx^*+Sx^*=x^*.$$

The proof is now completed. □

Let X be a real Banach space.

Definition 2.2. *A mapping $K:X\to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept of contraction of the set l is linked to that of the Kuratowski measure of non-compactness which we recall for completeness.

Definition 2.3. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha:\Omega_X\to[0,\infty)$ is defined by*

$$\alpha(Y)=\inf\left\{\delta>0:Y=\bigcup_{j=1}^{m}Y_j \quad \text{and} \quad diam(Y_j)\le\delta, \quad j\in\{1,\ldots,m\}\right\},$$

where $diam(Y_j)=\sup\{\|x-y\|_X : x,y\in Y_j\}$ is the diameter of Y_j, $j\in\{1,\ldots,m\}$.

For more detail on the properties for measure of noncompactness, we refer to [1].

Definition 2.4. *A mapping $K : X \to X$ is said to be l-set contraction if it is continuous, bounded and there exists a constant $l \geq 0$ such that*

$$\alpha(K(Y)) \leq l\alpha(Y),$$

for any bounded set $Y \subset X$. The mapping K is said to be a strict set contraction if $l < 1$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction(see [7]).

Definition 2.5. *Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Definition 2.6. *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$.

Lemma 2.7. *Let X be a closed convex subset of a Banach space E and $U \subset X$ a bounded open subset with $0 \in U$. Assume there exists $\varepsilon > 0$ small enough and that $K : \overline{U} \to X$ is a strict k-set contraction that satisfies the boundary condition:*

$$Kx \notin \{x,\ \lambda x\} \ \text{ for all } \ x \in \partial U \ \text{ and } \lambda \geq 1 + \varepsilon.$$

Then $i\,(K, U, X) = 1$.

Proof. Consider the homotopic deformation $H : [0, 1] \times \overline{U} \to X$ defined by

$$H(t, x) = \frac{1}{\varepsilon + 1}\, tKx.$$

The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t, .)$ is a strict set contraction for each $t \in [0, 1]$.In addition, $H(t, .)$ has no fixed point on ∂U. On the contrary,

• If $t=0$, there exists some $x_0 \in \partial U$ such that $x_0 = 0$, contradicting $x_0 \in U$.
• If $t \in (0,1]$, there exists some $x_0 \in \mathcal{P} \cap \partial U$ such that $\frac{1}{\varepsilon+1}\, tKx_0 = x_0$; then $Kx_0 = \frac{1+\varepsilon}{t} x_0$ with $\frac{1+\varepsilon}{t} \geq 1+\varepsilon$, contradicting the assumption. From the invariance under homotopy and the normalization properties of the index, we deduce

$$i\,(\frac{1}{\varepsilon+1}\,K, U, X) = i\,(0, U, X) = 1.$$

New, we show that

$$i\,(K, U, X) = i\,(\frac{1}{\varepsilon+1}\,K, U, X).$$

We have

$$\frac{1}{\varepsilon+1} Kx \neq x,\ \forall\, x \in \partial U. \tag{2.2}$$

Then there exists $\gamma > 0$ such that

$$\|x - \frac{1}{\varepsilon+1}\,Kx\| \geq \gamma,\ \forall\, x \in \partial U.$$

In other hand, we have $\frac{1}{\epsilon+1} Kx \to Kx$ as $\epsilon \to 0$, for $x \in \overline{U}$. So for ε small enough

$$\|Kx - \frac{1}{\varepsilon+1}\,Kx\| < \frac{\gamma}{2},\ \forall\, x \in \partial U.$$

Define the convex deformation $G : [0,1] \times \overline{U} \to X$ by

$$G(t,x) = tKx + (1-t)\frac{1}{\varepsilon+1} Kx.$$

The operator G is continuous and uniformly continuous in t for each x, and the mapping $G(t,.)$ is a strict set contraction for each $t \in [0,1]$ (since $t + \frac{1}{\varepsilon+1}(1-t) < t+1-t = 1$).In addition, $G(t,.)$ has no fixed point on ∂U. In fact, for all $x \in \partial U$, we have

$$\begin{array}{rcl} \|x - G(t,x)\| & = & \|x - tKx - (1-t)\frac{1}{\varepsilon+1} Kx\| \\ & \geq & \|x - \frac{1}{\varepsilon+1} Kx\| - t\|Kx - \frac{1}{\varepsilon+1} Kx\| \\ & > & \gamma - \frac{\gamma}{2} > \frac{\gamma}{2}. \end{array}$$

Then our claim follows from the invariance property by homotopy of the index.

□

Proposition 2.8. *Let $\mathcal{P}$ be a cone in a Banach space E. Let also, U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $S : \overline{U} \to E$ is a l-set contraction with $0 \leq l < h - 1$, and $S(\overline{U}) \subset (I - T)(\Omega)$. If there exists $\varepsilon \geq 0$ such that*

$$Sx \neq \{(I - T)(x), \quad (I - T)(\lambda x)\} \text{ for all } x \in \partial U \cap \Omega \text{ and } \lambda \geq 1 + \varepsilon,$$

then the fixed point index $i_* (T + S, U \cap \Omega, \mathcal{P}) = 1$.

Proof. The mapping $(I - T)^{-1} S : \overline{U} \to \mathcal{P}$ is a strict set contraction and it is readily seen that the following condition is satisfied

$$(I - T)^{-1} Sx \notin \{x, \lambda x\} \quad \text{for} \quad \text{all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1 + \epsilon.$$

Our claim then follows from the definition of i_* and the following Lemma 2.7. □

The following result will be used to prove existence of at least two non-negative solutions to the problem (1.1).

Theorem 2.9. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \leq k < h - 1$ and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\epsilon \geq 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If $Sx = (I - T)x$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T + S$. Suppose that $Sx \neq (I - T)x$ for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_1 \cap$

Ω and $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$, otherwise the conclusion has been proved. By [6, Proposition 2.11 and Proposition 2.16] and Proposition 2.8, we have

$$i_*\left(T+S, U_1\cap\Omega, \mathcal{P}\right) = i_*\left(T+S, U_3\cap\Omega, \mathcal{P}\right) = 0 \text{ and } i_*\left(T+S, U_2\cap\Omega, \mathcal{P}\right) = 1.$$

The additivity property of the index yields

$$i_*\left(T+S, (U_2 \setminus \overline{U}_1)\cap\Omega, \mathcal{P}\right) = 1 \text{ and } i_*\left(T+S, (U_3 \setminus \overline{U}_2)\cap\Omega, \mathcal{P}\right) = -1.$$

Consequently, by the existence property of the index, $T+S$ has at least two fixed points $x_1 \in (U_2 \setminus U_1) \cap \Omega$ and $x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. $\square$

In [13], it is proved that the problem

$$\begin{aligned}
{}^cD^{\beta}_{t,0+}u(t) &= f_1(t), \quad t \in J, \quad t \neq t_k, \quad k \in \{1,\dots,n_1\},\\
u'(t_k+) &= u'(t_k-) + \widetilde{I}_k(u(t_k)), \quad t_k \in (0,1), \quad k \in \{1,\dots,n_1\},\\
u(t_k+) &= u(t_k-) + \widetilde{L}_k(u(t_k)), \quad t_k \in (0,1), \quad k \in \{1,\dots,n_1\},\\
u(0) &= h_1(u(0)), \quad u(1) = h_2(u(1)),
\end{aligned}$$

where $f_1 \in \mathcal{C}(J)$, $h_1, h_2 \in \mathcal{C}(\mathbb{R})$, has a solution of the form

$$u(t) = \begin{cases} c_1(t,u(t))t + h_1(u(t)) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f_1(s)ds, \quad t \in J_0,\\[2mm]
c_1(t,u(t))t + h_1(u(t)) + \frac{1}{\Gamma(\beta)}\int_{t_k}^t (t-s)^{\beta-1} f_1(s)ds\\[2mm]
+\sum\limits_{j=1}^{k} \frac{1}{\Gamma(\beta)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\beta-1}f_1(s)ds + \sum\limits_{j=1}^{k}(t-t_j)\widetilde{I}_j(u(t_j))\\[2mm]
+\sum\limits_{j=1}^{k} \frac{t-t_j}{\Gamma(\beta-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\beta-2}f_1(s)ds + \sum\limits_{j=1}^{k}\widetilde{L}_j(u(t_j)), \quad t \in J_k, \quad k \in \{1,\dots,n_1\},
\end{cases}$$

where

$$
\begin{aligned}
c_1(t,u(t)) \;\; = \;\; & h_2(u(t)) - h_1(u(t)) - \sum_{j=1}^{n_1+1} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} f_1(s) ds \\
& - \sum_{j=1}^{n_1} \widetilde{L}_j(u(t_j)) - \sum_{j=1}^{n_1} \frac{1-t_j}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-2} f_1(s) ds \\
& - \sum_{j=1}^{n_1} (1-t_j) I_j(u(t_j)), \quad t \in J.
\end{aligned}
$$

3 Proof of Theorem 1.1

For convenience, we set $X = PC^2(J, \mathcal{C}^2(\mathbb{R}^n))$. For $u \in X$, define the operator

$$
S_1 u(t,x) = \begin{cases}
-u(t,x) + c(t,x,u(t,x)) + h_1(x,u(0,x)) \\
+\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds, \\
(t,x) \in J \times \mathbb{R}^n, \\
-u(t,x) + c(t,x,u(t,x)) + h_1(x,u(0,x)) \\
+\frac{1}{\Gamma(\beta)} \int_{t_k}^t (t-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds \\
+ \sum\limits_{j=1}^{k} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds \\
+ \sum\limits_{j=1}^{k} (t-t_j) I_j(t_j,x,u(t_j,x)) \\
+ \sum\limits_{j=1}^{k} \frac{t-t_j}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds \\
+ \sum\limits_{j=1}^{k} L_j(t_j,x,u(t_j,x)), \quad t \in J_k, \quad k \in \{1,\dots,n_1\},
\end{cases}
$$

where

$$
\begin{aligned}
c(t,x,u(t,x)) &= h_2(x,u(1,x)) - h_1(x,u(0,x)) \\
&\quad - \sum_{j=1}^{n_1+1} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds \\
&\quad - \sum_{j=1}^{n_1} L_j(t_j,x,u(t_j,x)) \\
&\quad - \sum_{j=1}^{n_1} \frac{1-t_j}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds
\end{aligned}
$$

$$
-\sum_{j=1}^{n_1} (1-t_j) I_j(t_j,x,u(t_j,x)), \quad (t,x) \in J \times \mathbb{R}^n.
$$

Note, that if $u \in X$ satisfies the equation

$$
S_1 u(t,x) = 0, \quad (t,x) \in J \times \mathbb{R}^n,
$$

then u is a solution to the problem (1.1). Set

$$
\begin{aligned}
B_1 &= B + 2B^{1+s_1} + 2B^{1+s_2} + 2\sum_{j=1}^{m-1} \left(B^{1+s_{1j}} + B^{1+s_{2j}} \right) \\
&\quad + \left(\frac{n_1+3}{\Gamma(\beta+1)} + \frac{n_1+1}{\Gamma(\beta)} \right) \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1} \right) + nB \right).
\end{aligned}
$$

Lemma 3.1. *Suppose* $(H1)$-$(H4)$. *For* $u \in X$, $\|u\| \leq B$, *we have*

$$
|S_1 u(t,x)| \leq B_1, \quad (t,x) \in J \times \mathbb{R}^n.
$$

Proof. We have

$$
\begin{aligned}
|\Delta u(t,x)| &= \left| \sum_{j=1}^{n} u_{x_j x_j}(t,x) \right| \\
&\leq \sum_{j=1}^{n} |u_{x_j x_j}(t,x)| \\
&\leq nB, \quad (t,x) \in J \times \mathbb{R}^n,
\end{aligned}
$$

and

$$\begin{aligned}|f(t,x,u(t,x),u_t(t,x),u_x(t,x))| &\leq \sum_{j=1}^{n}\bigg(a_j(t,x)|u(t,x)|^{p_j}+b_j(t,x)|u(t,x)|^{q_j}\\&\quad+\sum_{i=1}^{n}c_{ji}(t,x)|u_{x_i}(t,x)|^{r_{ji}}\bigg)\\&\leq \sum_{j=1}^{r}\bigg(B^{p_j+1}+B^{q_j+1}+\sum_{i=1}^{n}B^{r_{ji}+1}\bigg),\end{aligned}$$

$(t,x)\in J\times\mathbb{R}^n$, and

$$\begin{aligned}|I_k(t_k,x,u(t_k,x))| &\leq a_{1k}(t_k,x)|u(t_k,x)|^{s_{1k}}\\&\leq B^{1+s_{1k}},\quad x\in\mathbb{R}^n,\quad k\in\{1,\ldots,n_1\},\end{aligned}$$

and

$$\begin{aligned}|L_k(t_k,x,u(t_k,x))| &\leq a_{2k}(t_k,x)|u(t_k,x)|^{s_{2k}}\\&\leq B^{1+s_{2k}},\quad x\in\mathbb{R}^n,\quad k\in\{1,\ldots,n_1\},\end{aligned}$$

and

$$\begin{aligned}|h_1(x,u(0,x))| &\leq b_{11}(x)|u(0,x)|^{s_1}\\&\leq B^{1+s_1},\quad x\in\mathbb{R}^n,\end{aligned}$$

and

$$\begin{aligned}|h_2(x,u(1,x))| &\leq b_{12}(x)|u(1,x)|^{s_2}\\&\leq B^{1+s_2},\quad x\in\mathbb{R}^n,\end{aligned}$$

and

$$
\begin{aligned}
|c(t,x,u(t,x))| \quad = \quad & \Bigg| h_2(x,u(1,x)) - h_1(x,u(0,x)) \\
& - \sum_{j=1}^{n_1+1} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds \\
& - \sum_{j=1}^{n_1} L_j(t_j, x, u(t_j, x)) \\
& - \sum_{j=1}^{n_1} \frac{1-t_j}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds \\
& - \sum_{j=1}^{n_1} (1-t_j) I_j(t_j, x, u(t_j, x)) \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & |h_2(x,u(1,x))| + |h_1(x,u(0,x))| \\
& + \sum_{j=1}^{n_1+1} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} \left(|f(s,x,u(s,x),u_t(s,x),u_x(s,x))| + |\Delta u(s,x)| \right) ds \\
& + \sum_{j=1}^{n_1} |L_j(t_j, x, u(t_j, x))| \\
& + \sum_{j=1}^{n_1} \frac{1-t_j}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} \left(|f(s,x,u(s,x),u_t(s,x),u_x(s,x))| + |\Delta u(s,x)| \right) ds \\
& + \sum_{j=1}^{n_1} (1-t_j) |I_j(t_j, x, u(t_j, x))| \\
\leq \quad & B^{1+s_1} + B^{1+s_2} + \sum_{j=1}^{m-1} \left(B^{1+s_{1j}} + B^{1+s_{2j}} \right) \\
& + \left(\frac{n_1+1}{\Gamma(\beta+1)} + \frac{n_1}{\Gamma(\beta)} \right) \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1} \right) + nB \right),
\end{aligned}
$$

$(t,x) \in J \times \mathbb{R}^n$. Hence,

$$
\begin{aligned}
|S_1 u(t,x)| &= \Bigg| - u(t,x) + c(t,x,u(t,x)) + h_1(x,u(0,x)) \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x) \right) ds \Bigg| \\
&\leq |u(t,x)| + |c(t,x,u(t,x))| + |h_1(x,u(0,x))| \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(|f(s,x,u(s,x),u_t(s,x),u_x(s,x))| + |\Delta u(s,x)| \right) ds \\
&\leq B + \left(\frac{n_1+1}{\Gamma(\beta+1)} + \frac{n_1}{\Gamma(\beta)} \right) \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1} \right) + nB \right) \\
&\quad + B^{1+s_1} + B^{1+s_2} + \sum_{j=1}^{n_1} \left(B^{1+s_{1j}} + B^{1+s_{2j}} \right) + B^{1+s_1}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{1}{\Gamma(\beta+1)} \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1} \right) + nB \right) \\
&= B + 2B^{1+s_1} + B^{1+s_2} + \sum_{j=1}^{m-1} \left(B^{1+s_{1j}} + B^{1+s_{2j}} \right) \\
&\quad + \left(\frac{n_1+2}{\Gamma(\beta+1)} + \frac{n_1}{\Gamma(\beta)} \right) \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1} \right) + nB \right) \\
&\leq B_1,
\end{aligned}
$$

$(t,x) \in J_0 \times \mathbb{R}^n$, and

$$
\begin{aligned}
|S_1 u(t,x)| &= \Bigg| -u(t,x) + c(t,x,u(t,x)) + h_1(x,u(0,x)) \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x)\right) ds \\
&\quad + \sum_{j=1}^{k} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x)\right) ds \\
&\quad + \sum_{j=1}^{k} (t-t_j) I_j(t_j,x,u(t_j,x)) \\
&\quad + \sum_{j=1}^{k} \frac{t-t_j}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \left(f(s,x,u(s,x),u_t(s,x),u_x(s,x)) + \Delta u(s,x)\right) ds \\
&\quad + \sum_{j=1}^{k} L_j(t_j,x,u(t_j,x)) \Bigg| \\
&\le |u(t,x)| + |c(t,x,u(t,x))| + |h_1(x,u(0,x))| \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} \left(|f(s,x,u(s,x),u_t(s,x),u_x(s,x))| + |\Delta u(s,x)|\right) ds \\
&\quad + \sum_{j=1}^{k} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \left(|f(s,x,u(s,x),u_t(s,x),u_x(s,x))| + |\Delta u(s,x)|\right) ds \\
&\quad + \sum_{j=1}^{k} (t-t_j) |I_j(t_j,x,u(t_j,x))| \\
&\quad + \sum_{j=1}^{k} \frac{t-t_j}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \left(|f(s,x,u(s,x),u_t(s,x),u_x(s,x))| + |\Delta u(s,x)|\right) ds \\
&\quad + \sum_{j=1}^{k} |L_j(t_j,x,u(t_j,x))| \\
&\le B + 2B^{1+s_1} + B^{1+s_2} + \sum_{j=1}^{m-1} \left(B^{1+s_{1j}} + B^{1+s_{2j}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& + \left(\frac{n_1+2}{\Gamma(\beta+1)} + \frac{n_1}{\Gamma(\beta)}\right) \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1}\right) + nB\right) \\
& + \frac{1}{\Gamma(\beta+1)} \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1}\right) + nB\right) \\
& + \frac{1}{\Gamma(\beta)} \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1}\right) + nB\right) \\
& + \sum_{j=1}^{k} B^{1+s_{1j}} + \sum_{j=1}^{k} B^{1+s_{2j}} \\
= \quad & B + 2B^{1+s_1} + 2B^{1+s_2} + 2\sum_{j=1}^{m-1} \left(B^{1+s_{1j}} + B^{1+s_{2j}}\right) \\
& + \left(\frac{n_1+3}{\Gamma(\beta+1)} + \frac{n_1+1}{\Gamma(\beta)}\right) \left(\sum_{j=1}^{r} \left(B^{p_j+1} + B^{q_j+1} + \sum_{i=1}^{n} B^{r_{ji}+1}\right) + nB\right) \\
= \quad & B_1, \quad (t,x) \in J_k \times \mathbb{R}^n,
\end{aligned}
$$

$k \in \{1, \ldots, n_1\}$. The proof is now completed. □

Let us suppose that $A \in \mathbb{R}_*^+$ and g to be continuous function on $\mathbb{R}^n$, where

(H5) $g > 0$ on $\mathbb{R}^n \backslash \{\bigcup_{i=1}^n \{x_i = 0\}\}$,

$$g(0, x_2, \ldots, x_n) = \ldots = g(x_1, \ldots, x_{n-1}, 0) = 0, \quad x_j \in \mathbb{R}, \quad j \in \{1, \ldots, n\},$$

and

$$2 \cdot 8^n \prod_{j=1}^{n} \left(1 + |x_j| + x_j^2\right) \left|\int_0^x g(y)dy\right| \le A,$$

$x \in \mathbb{R}^n$, where

$$\int_0^x = \int_0^{x_1} \ldots \int_0^{x_n}, \quad dy = dy_n \ldots dy_1.$$

We define for $u \in X$, the operator

$$S_2u(t,x) = \int_0^t (t-s)^2 \int_0^x \prod_{j=1}^n (x_j - y_j)^2 g(y) S_1u(s,y)dyds,$$

$(t,x) \in J \times \mathbb{R}^n$.

Lemma 3.2. *Suppose* $(H1)$*-*$(H5)$*. If* $u \in X$ *satisfies the equation*

$$S_2u(t,x) = 0, \quad (t,x) \in J \times \mathbb{R}^n, \tag{3.1}$$

then u *satisfies the problem* (1.1)*.*

Proof. Differentiating three times in t and three times in x_1, …, x_n the equation (3.1), we get

$$g(x)S_1u(t,x) = 0, \quad (t,x) \in J \times \left(\mathbb{R}^n \backslash \left\{\bigcup_{i=1}^n \{x_i = 0\}\right\}\right),$$

whereupon

$$S_1u(t,x) = 0, \quad (t,x) \in J \times \left(\mathbb{R}^n \backslash \left\{\bigcup_{i=1}^n \{x_i = 0\}\right\}\right).$$

Since $S_1u \in \mathcal{C}(J \times \mathbb{R}^n)$, we have

$$\begin{aligned} 0 &= S_1u(t,0,x_2,\ldots,x_n) \\ &= \lim_{x_1 \to 0} S_1u(t,x_1,x_2,\ldots,x_n), \\ &\ldots \\ 0 &= S_1u(t,x_1,x_2,\ldots,0) \\ &= \lim_{x_n \to 0} S_1u(t,x_1,x_2,\ldots,x_n), \quad x_1,\ldots,x_n \in \mathbb{R}, \quad t \in J. \end{aligned}$$

Therefore

$$S_1u(t,x) = 0, \quad (t,x) \in J \times \mathbb{R}^n.$$

Hence, we then conclude that u satisfies (1.1). The proof is now completed. □

Lemma 3.3. *Suppose* $(H1)$*-*$(H5)$*. If* $u \in X$ *and* $\|u\| \leq B$*, then*

$$\|S_2 u\| \leq AB_1.$$

Proof. We have

$$\begin{aligned}
|S_2u(t,x)| &= \left| \int_0^t \int_0^x \prod_{j=1}^n (t-s)^2 (x_j - s_j)^2 g(t_1,s) S_1 u(t_1,s) ds dt_1 \right| \\
&\leq \int_0^t \left| \int_0^x \prod_{j=1}^n (t-s)^2 (x_j - s_j)^2 g(t_1,s) |S_1 u(t_1,s)| ds \right| dt_1 \\
&\leq B_1 \int_0^t \left| \int_0^x \prod_{j=1}^n (x_j - s_j)^2 g(t_1,s) ds \right| dt_1 \\
&\leq B_1 4^n \prod_{j=1}^n x_j^2 \int_0^t \left| \int_0^x g(t_1,s) ds \right| dt_1 \\
&\leq 2B_1 8^n \prod_{j=1}^n \left(1 + |x_j| + x_j^2\right) \int_0^t \left| \int_0^x g(t_1,s) ds \right| dt_1 \\
&\leq AB_1, \quad (t,x) \in J \times \mathbb{R}^n,
\end{aligned}$$

and

$$\begin{aligned}
\left| \frac{\partial}{\partial t} S_2 u(t,x) \right| &= \left| 2 \int_0^t \int_0^x \prod_{j=1}^n (t-s)(x_j - s_j)^2 g(t_1,s) S_1 u(t_1,s) ds dt_1 \right| \\
&\leq 2 \int_0^t \left| \int_0^x \prod_{j=1}^n (t-s)(x_j - s_j)^2 g(t_1,s) |S_1 u(t_1,s)| ds \right| dt_1 \\
&\leq 2B_1 \int_0^t \left| \int_0^x \prod_{j=1}^n (x_j - s_j)^2 g(t_1,s) ds \right| dt_1 \\
&\leq 2B_1 4^n \prod_{j=1}^n x_j^2 \int_0^t \left| \int_0^x g(t_1,s) ds \right| dt_1 \\
&\leq 2B_1 8^n \prod_{j=1}^n \left(1 + |x_j| + x_j^2\right) \int_0^t \left| \int_0^x g(t_1,s) ds \right| dt_1 \\
&\leq AB_1, \quad (t,x) \in J \times \mathbb{R}^n,
\end{aligned}$$

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial t^2}S_2u(t,x)\right| &= \left|2\int_0^t\int_0^x\prod_{j=1}^n (x_j-s_j)^2 g(t_1,s)S_1u(t_1,s)dsdt_1\right| \\
&\le 2\int_0^t\left|\int_0^x\prod_{j=1}^n (x_j-s_j)^2 g(t_1,s)|S_1u(t_1,s)|ds\right|dt_1 \\
&\le 2B_1\int_0^t\left|\int_0^x\prod_{j=1}^n (x_j-s_j)^2 g(t_1,s)ds\right|dt_1 \\
&\le 2B_1 4^n\prod_{j=1}^n x_j^2\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le 2B_1 8^n\prod_{j=1}^n \left(1+|x_j|+x_j^2\right)\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le AB_1, \quad (t,x)\in J\times\mathbb{R}^n,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial}{\partial x_k}S_2u(t,x)\right| &= 2\left|\int_0^t\int_0^x\prod_{j=1,j\neq k}^n (t-s)^2(x_j-s_j)^2(x_k-s_k)g(t_1,s)S_1u(t_1,s)dsdt_1\right| \\
&\le 2\int_0^t\left|\int_0^x\prod_{j=1,j\neq k}^n (t-s)^2(x_j-s_j)^2|x_k-s_k|g(t_1,s)|S_1u(t_1,s)|ds\right|dt_1 \\
&\le 2B_1\int_0^t\left|\int_0^x\prod_{j=1,j\neq k}^n (x_j-s_j)^2|x_k-s_k|g(t_1,s)ds\right|dt_1 \\
&\le B_1 4^n\prod_{j=1}^n x_j^2|x_k|\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le B_1 8^n\prod_{j=1}^n \left(1+|x_j|+x_j^2\right)\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le AB_1, \quad (t,x)\in J\times\mathbb{R}^n, \quad k\in\{1,\dots,n\},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x_k^2} S_2 u(t,x)\right| &= 2\left|\int_0^t \int_0^x \prod_{j=1, j\neq k}^n (t-s)^2 (x_j - s_j)^2 g(t_1, s) S_1 u(t_1, s)\, ds\, dt_1\right| \\
&\le 2\int_0^t \left|\int_0^x \prod_{j=1, j\neq k}^n (t-s)^2 (x_j - s_j)^2 g(t_1, s) |S_1 u(t_1, s)|\, ds\right| dt_1 \\
&\le 2B_1 \int_0^t \left|\int_0^x \prod_{j=1, j\neq k}^n (x_j - s_j)^2 g(t_1, s)\, ds\right| dt_1 \\
&\le B_1 4^{n-1} \prod_{j=1}^n x_j^2 \int_0^t \left|\int_0^x g(t_1, s)\, ds\right| dt_1 \\
&\le B_1 8^n \prod_{j=1}^n \left(1 + |x_j| + x_j^2\right) \int_0^t \left|\int_0^x g(t_1, s)\, ds\right| dt_1 \\
&\le AB_1, \quad (t,x) \in J \times \mathbb{R}^n, \quad k \in \{1, \ldots, n\}.
\end{aligned}
$$

Thus,

$$
\|S_2 u\| \le AB_1.
$$

The proof is now completed. $\square$

Below, suppose

(H6) $\epsilon \in (0,1)$, A and B satisfy the inequalities $\epsilon B_1(1+A) < 1$ and $AB_1 < 1$.

Let $\widetilde{\widetilde{\widetilde{Y}}}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $\widetilde{\widetilde{Y}} = \overline{\widetilde{\widetilde{\widetilde{Y}}}}$ be the closure of $\widetilde{\widetilde{\widetilde{Y}}}$, $\widetilde{Y} = \widetilde{\widetilde{Y}} \cup \{h_1, h_2\}$,

$$
Y = \{u \in \widetilde{Y} : \|u\| \le B\}.
$$

Note that Y is a compact set in X. For $u \in X$, define the operators

$$
\begin{aligned}
Tu(t,x) &= -\epsilon u(t,x), \\
Su(t,x) &= u(t,x) + \epsilon u(t,x) + \epsilon S_2 u(t,x), \quad (t,x) \in J \times \mathbb{R}^n.
\end{aligned}
$$

For $u \in Y$, using Lemma 3.3, we have

$$\begin{aligned}
\|(I-S)u\| &= \|\epsilon u - \epsilon S_2 u\| \\
&\leq \epsilon\|u\| + \epsilon\|S_2 u\| \\
&\leq \epsilon B_1 + \epsilon A B_1 \\
&= \epsilon B_1(1+A) \\
&< B
\end{aligned}$$

Thus, $S : Y \to E$ is continuous and $(I-S)(Y)$ resides in a compact subset of E. Now, suppose that there is a $u \in E$ so that $\|u\| = B$ and

$$u = \lambda(I-S)u$$

or

$$\frac{1}{\lambda}u = (I-S)u = -\epsilon u - \epsilon S_2 u,$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right) u = -\epsilon S_2 u$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2 u\| \leq AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right) B = \left(\frac{1}{\lambda} + \epsilon\right) \|u\| = \epsilon\|S_2 u\| < \epsilon B,$$

which is a contradiction. Hence and Theorem 2.1, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned}
u^*(t,x) &= Tu^*(t,x) + Su^*(t,x) \\
&= -\epsilon u^*(t,x) + u^*(t,x) + \epsilon u^*(t,x) + \epsilon S_2 u^*(t,x), \quad (t,x) \in J \times \mathbb{R}^n,
\end{aligned}$$

whereupon

$$0 = S_2 u^*(t,x), \quad (t,x) \in J \times \mathbb{R}^n.$$

From here and from Lemma 3.2, it follows that u is a solution to the BVP (1.1). The proof is now completed.

4 Proof of Theorem 1.2

Let X be the space used in the previous section. Suppose

(H7) Let $m > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

$$AB_1 < \frac{L}{5}.$$

Let

$$\widetilde{P} = \{u \in X : u \geq 0 \quad \text{on} \quad J \times \mathbb{R}^n\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t) &= (1 + m\epsilon)v(t) - \epsilon \frac{L}{10}, \\ S_3 v(t) &= -\epsilon S_2 v(t) - m\epsilon v(t) - \epsilon \frac{L}{10}, \end{aligned}$$

$t \in [0, \infty)$. Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the BVP (1.1). Define

$$\begin{aligned} U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\ U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}, \\ R_2 &= R_1 + \frac{A}{m} B_1 + \frac{L}{5m}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \leq R_2\}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

   $$\|T_1 v_1 - T_1 v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|,$$

   whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get

$$\begin{aligned} \|S_3 v\| &\leq \varepsilon \|S_2 v\| + m\varepsilon \|v\| + \varepsilon \frac{L}{10} \\ &\leq \varepsilon \left(AB_1 + mR_1 + \frac{L}{10} \right). \end{aligned}$$

   Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.

3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m} S_2 v_1 + \frac{L}{5m}.$$

   Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $J \times \mathbb{R}^n$. We have $v_2 \geq 0$ on $J \times \mathbb{R}^n$ and

$$\begin{aligned} \|v_2\| &\leq \|v_1\| + \frac{1}{m} \|S_2 v_1\| + \frac{L}{5m} \\ &\leq R_1 + \frac{A}{m} B_1 + \frac{L}{5m} \\ &= R_2. \end{aligned}$$

   Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2 v_1 - \varepsilon \frac{L}{10} - \varepsilon \frac{L}{10}$$

   or

$$\begin{aligned} (I - T_1) v_2 &= -\varepsilon m v_2 + \varepsilon \frac{L}{10} \\ &= S_3 v_1. \end{aligned}$$

   Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $x \in \partial \mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $x \in \partial \mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

$$S_3 x = (I - T_1)(x - \lambda u_0).$$

   Then

$$-\epsilon S_2 x - m\epsilon x - \epsilon \frac{L}{10} = -m\epsilon (x - \lambda u_0) + \epsilon \frac{L}{10}$$

or

$$-S_2 x = \lambda m u_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 x\| = \left\|\lambda m u_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

This is a contradiction.

5. Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $x_1 \in \partial \mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ and

$$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{4.1}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $x_1 \in \partial \mathcal{P}_L$, $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1 + \epsilon_1$ and (4.1) holds. Since $x_1 \in \partial \mathcal{P}_L$ and $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, it follows that

$$\left(\frac{2}{5m} + 1\right) L < \lambda_1 L = \lambda_1 \|x_1\| \leq R_1.$$

Moreover,

$$-\epsilon S_2 x_1 - m\epsilon x_1 - \epsilon \frac{L}{10} = -\lambda_1 m \epsilon x_1 + \epsilon \frac{L}{10},$$

or

$$S_2 x_1 + \frac{L}{5} = (\lambda_1 - 1) m x_1.$$

From here,

$$2\frac{L}{5} \geq \left\|S_2 x_1 + \frac{L}{5}\right\| = (\lambda_1 - 1) m \|x_1\| = (\lambda_1 - 1) m L$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 1.2 hold. Hence, the BVP (1.1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

5 An Example

Below, we will illustrate our main results. Let $m = 2$, $n = 1$,

$$s_1 = s_2 = 0, \quad s_{1k} = s_{2k} = 2, \quad k \in \{1, 2\}, \quad p_1 = 3, \quad q_1 = 0, \quad r_{11} = 0,$$

$t_1 = \frac{1}{4}$, $t_2 = \frac{1}{2}$ and

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \frac{1}{10B_1}, \quad \epsilon = \frac{1}{5B_1(1+A)}.$$

Then

$$AB_1 = \frac{1}{10} < B, \quad \epsilon B_1(1+A) < 1,$$

i.e., $(H6)$ holds. Next,

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L, \quad AB_1 < \frac{L}{5}.$$

i.e., $(H7)$ holds. Take

$$h(s) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s \to \pm\infty} (1 + s + s^2) h(s) < \infty, \\ -\infty &< \lim_{s \to \pm\infty} (1 + s + s^2) l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$(1 + s + s^2)^3 \left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}} \right) \leq C_1,$$

$s \in \mathbb{R}$. Note that $\lim\limits_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [11] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(x) = Q(x_1)\dots Q(x_n), \quad x \in \mathbb{R}^n.$$

Then there exists a constant $C > 0$ such that

$$2 \cdot 8^n \prod_{j=1}^{n} \left(1+|x_j|+x_j^2\right) \left| \int_0^x g_1(y)dy \right| \le C, \quad (t,x) \in J \times \mathbb{R}^n.$$

Let

$$g(x) = \frac{A}{C} g_1(x), \quad x \in \mathbb{R}^n.$$

Then

$$2 \cdot 8^n \prod_{j=1}^{n} \left(1+|x_j|+x_j^2\right) \left| \int_0^x g(y)dy \right| \le A, \quad x \in \mathbb{R}^n,$$

i.e., $(H7)$ holds. Therefore for the problem

$$\begin{array}{lll}
{}^cD^{\frac{5}{3}}_{t,0+}u - u_{xx} & = & \frac{u^3}{1+x^4}, \quad t \in [0,1], \quad x \in \mathbb{R}, \\
u(t_1^+, x) & = & u(t_1,x) + \frac{(u(t_1,x))^2}{1+x^{10}}, \quad x \in \mathbb{R}, \\
u(t_2^+, x) & = & u(t_2,x) + \frac{(u(t_2,x))^2}{1+x^{18}}, \quad x \in \mathbb{R}, \\
u_t(t_1^+, x) & = & u_t(t_1,x) + \frac{(u(t_1,x))^2}{10+20x^{30}}, \quad x \in \mathbb{R}, \\
u_t(t_2^+, x) & = & u_t(t_2,x) + \frac{(u(t_2,x))^2}{1+4x^{20}}, \quad x \in \mathbb{R}, \\
u(0,x) & = & \frac{1}{1+x^4}, \quad x \in \mathbb{R}, \\
u(1,x) & = & \frac{1}{1+x^6}, \quad x \in \mathbb{R},
\end{array}$$

are fulfilled all conditions of Theorem 1.1 and Theorem 1.2.

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References

[1] J. Banas, K. Goebel, Measures of noncompactness in Banach spaces, Lecture Notes in Pure and Applied Mathematics, **60**. Marcel Dekker, Inc., New York, 1980.

[2] H. Brézis, Periodic solutions of nonlinear vibrating strings and duality principles. Bull. Am. Math. So(N.S.) 8, 409426 (1983).

[3] H. Brézis, J. M. Coron and L. Nirenberg, Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz. Commun. Pure Appl. Math. 33, 667689 (1980).

[4] K. C. Chang, Solutions of asymptotically linear operator equations via Morse theory. Commun. Pure Appl. Math. 34, 693712 (1981).

[5] K. C. Chang, S. P. Wu and S. J. Li, Multiple periodic solutions for an asymptotically linear wave equation. Indiana Univ. Math. J. 31(5), 721731 (1982).

[6] S. Djebali, K. Mebarki, *Fixed point index theory for perturbation of expansive mappingsby of k-set contractions*, Top. Meth. Nonli. Anal., Vol 54, No 2 (2019), 613–640.

[7] P. Drabek, J. Milota, Methods in nonlinear analysis, applications to differential equations, Birkhäuser, 2007.

[8] S. Georgiev, Kh. Zennir, Existence of solutions for a class of nonlinear impulsive wave equations, Ricerche mat (2021). https://doi.org/10.1007/s11587-021-00649-2.

[9] S. Georgiev, Kh. Zennir, Boundary Value Problems on time Scales, Volume II. Chapman and Hall/CRC. 2021, 457 pages.

[10] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, Boston, Mass, USA, vol. 5, (1988).

[11] A. Polyanin and A. Manzhirov, Hoandbook of integral equations, CRC Press, 1998.

[12] S. J. Russell, Report on waves, Rept. 14th meetings of the British Assoc. for the Advancement of science. London: John Murray, 1844, P 311-390.

[13] S. Yang and S. Zhang, Boundary value problems for impulsive fractyional differential equations in Banach spaces, Filomat 31:18(2017), 5603-5616.