VARATIONAL APPROACH AND SPLINE TECHNIQUE TO OPTIMIZATION OF CONTROLLED BEAM MOTIONS

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Abstract

An variational approach to modelling and optimization of controlled dynamical systems with distributed elastic and inertial parameters is considered. The general integrodifferential method for solving a wide class of initial-boundary value problems is developed and criteria of solution quality are proposed. A numerical algorithm based on the variational approach to spline approximation of controlled beam motions is worked out and applied to design optimal control laws minimizing the total mechanical energy at the end of the process. The polynomial control of plane motions of a homogeneous cantilever beam is investigated. The numerical results obtained are analyzed and discussed.

Key words

Optimal control, vatiational principle, spline approximation.

1 Introduction

The elastic properties of structural elements have significant influence on their dynamical behavior. For a large number of mechanical structures their elements have a special geometrical feature that one of characteristic dimensions is much larger than the other two. Therefore, the beam theories occupy a special place among approximate approaches in mechanics.

The investigation of beam systems leads to a wide class of initial-boundary value problems for which many approaches are developed. The regular perturbation method (the small parameter method) for analysis of nonuniform rod dynamics with arbitrary distributed bending stiffness and linear density and various boundary conditions is proposed in [**?**]. Based on the classical Rayleigh–Ritz approach, a numerical-analytic method of fast convergence that allows one to obtain eigenvalues and eigenfunctions of nonuniform beams with given accuracy [**?**]. In elastic system modeling to reduce an initial-boundary value problem for partial differential equations to a system of ordinary differential equations the methods based on finite-dimensional approximation of unknown functions, for example, the decomposition method and the regularization method, are developed. It is worth noting the method of separation of variables which is widely used to solve beam motion problems [**?**]. In [**?**] the comparison analysis was performed for the method of integrodifferential relations and the Fourier approach. The direct discretization methods in optimal control problems are well known (see, e.g. [**?**], [**?**]).

In this paper the method of integrodifferential relations (MIDR) proposed in [**?**], [**?**]–[**?**], [**?**], [**?**], [**?**] is applied to finding the boundary optimal control for the movement of elastic beams under linear boundary conditions. In Section 2 and 3 a family of variational principles for the initial-boundary value problems is proposed and grounded. The relations of these principles and Hamilton's principles are stated and discussed. In the next section an optimization algorithm for motion of uniform beams is constructed based on the MIDR and spline technique. In Section 5 analysis of the numerical results obtained by using this method for a polynomial control and a quadratic cost functional is performed.

2 Statement of the problem

Consider plane controlled motions of a homogeneous rectilinear elastic beam. One end of the beam is free, and the other is clamped on a truck that can move along a horizontal line. The location of the clamped beam point at an arbitrary instant t is specified by the coordinate x in a stationary coordinate system $O'XY$ (see Fig. 1). In the undeformed state, the beam is fixed in the vertical position, and the beam midline coincides with the Y-axis. The control action on the beam is the horizontal displacement u of the truck. Initially, the shape of the beam lateral deflection (displacement) w

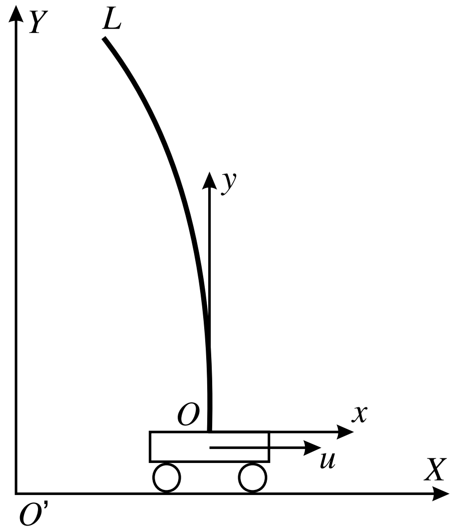

Figure 1. Clamped beam on a truck.

and its relative linear momentum density p are given in a coordinate system $O'XY$. Without loss of generality, it can be assumed that the coordinate and velocity of the truck are initially equal to zero.

Taking into account the assumption about smallness of elastic deformations the controlled motion of the beam can be described by the following system of partial differential equations [**?**]:

$$\dot{p} + m'' = 0, \quad y \in (0, L), \tag{1}$$
$$p = \rho\dot{w}, \quad m = EIw'', \quad t \in (0, T). \tag{2}$$

Here p is the linear momentum density; m is the bending moment in the beam cross section; w are the lateral displacements (deflections); L and ρ are the length and linear density of the beam, respectively; EI is its flexural rigidity; and T is the terminal instant of the control process. The dotted symbols denote the partial derivatives with respect to the time t, and the primed symbols stand for the partial derivatives with respect to the beam coordinate y.

We confine ourselves to the case when the boundary conditions at the beam ends $y = 0, L$ may be written in the following linear form:

$$\begin{aligned} &w(t, 0) = u, \\ &w'(t, 0) = m(t, L) = m'(t, L) = 0. \end{aligned} \tag{3}$$

The shape of the beam lateral displacements w and its relative linear momentum density p are given at the initial time $t = 0$

$$w(0, y) = f(y), \quad p(0, y) = g(y). \tag{4}$$

It is worth noting that the initial conditions (4) and boundary conditions (3) should be compatible. Due to the absence of dissipative forces in the system in the case of the fixed truck position ($u(t) = 0$) the beam will vibrate with the constant total mechanical energy defined by the initial functions given in Eq. (4). This energy can be presented in the form

$$\begin{aligned} W(t) &= \int_0^L \psi_{pm} dy, \\ &\qquad \psi_{pw}(t, y) = \frac{p^2}{2\rho} + \frac{EI(w'')^2}{2}, \\ W(0) &= \int_0^L \left(\frac{g(y)^2}{2\rho} + \frac{EI(f''(y))^2}{2} \right) dy. \end{aligned} \tag{5}$$

The problem is to find an optimal control $u(t)$ that moves the truck from its initial state to a terminal position

$$u(0) = \dot{u}(0) = 0, \quad u(T) = X \tag{6}$$

in the given time T and minimizes a objective function $J[u]$ in the class U of admissible controls:

$$J[u] \to \min_{u \in U}, \quad J = W(T). \tag{7}$$

The above formulated optimization problem (1)–(7) means active damping the elastic vibrations in the initially perturbed mechanical system with distributed parameters via point control input.

To solve the initial-boundary value problem (1)–(4)), we apply the MIDR, described in [**?**], [**?**]–[**?**], [**?**], [**?**], [**?**], in which some strict local equalities are replaced by an integral relation. In the case under consideration it is possible to reduce problem (1)–(4) to a variational problem. If a solution p^*, m^*, and w^* exists then the following functional Φ_1 reaches its absolute minimum on this solution over all admissible functions p, m, and w under local constraints (1), (3), (4)

$$\begin{aligned} \Phi_1(p^*, m^*, w^*) &= \min_{p,\, m,\, w} \Phi_1(p, m, w) = 0, \\ &\qquad \Phi_1 = \int_0^T \int_0^L \varphi_1(p, m, w) dy dt, \\ \varphi_1 &= \frac{(p - \rho\dot{w})^2}{2\rho} + \frac{(m - EIw'')^2}{2EI}. \end{aligned} \tag{8}$$

Note that the integrand φ_1 in Eq. (8) has the dimension of the energy linear density and is nonnegative. Hence, the corresponding integral is nonnegative for any arbitrary functions p, m, and w ($\Phi_1 \geq 0$).

Denote the actual and arbitrary admissible linear momentum densities, bending moments, and displacements by p^*, m^*, w^* and p, m, w, respectively, and specify that $p = p^* + \delta p$, $m = m^* + \delta m$, $w = w^* + \delta w$. Then $\Phi_1(p, m, w) = \Phi_1(p^*, m^*, w^*) + \delta_p \Phi_1 + \delta_m \Phi_1 +$

$\delta_w\Phi_1 + \delta^2\Phi_1$, where $\delta_p\Phi_1$, $\delta_m\Phi_1$, $\delta_w\Phi_1$ are the first variations of the functional Φ_1 with respect to p, m, w and $\delta^2\Phi_1$ is its second variation.

Integrating the first variations by parts and taking into account Eqs. (1), (3), and (4) result in the following relations

$$\begin{aligned}
&\delta_w\Phi_1 = \\
&\int_0^T\int_0^L \left[(\dot{p} - \rho\ddot{w}) - (m - EIw'')''\right]\delta w dy\, dt - \\
&\int_0^L \left[p - \rho\dot{w}\right]_{t=T}\delta w dy + \\
&\int_0^T \left[EIw''\delta w' - EIw'''\delta w\right]_{y=L} dt, \\
&\delta_p\Phi_1 = \int_0^T\int_0^L \left[\frac{p}{\rho} - \dot{w}\right]\delta p dy\, dt, \\
&\delta_m\Phi_1 = \int_0^T\int_0^L \left[\frac{m}{EI} - w''\right]\delta m dy\, dt.
\end{aligned} \tag{9}$$

It is seen from Eq. (9) that the fist variation of the functional Φ_1 is equal to zero for any admissible variations δp, δm, δw if the following equations are valid:

$$p = \rho\dot{w}, \quad m = EIw''.$$

Hence, the stationary conditions (9) of the functional Φ_1 are equivalent to relations (2) and together with constraints (1), (3), and (4) constitute the full system of dynamical equations for the beam.

The second variation

$$\delta^2\Phi_1 = \int_0^T\int_0^L \varphi_1(\delta p, \delta m, \delta w)dydt \geq 0,$$

which is quadratic with respect to the variations δp, δm, δw, is nonnegative because the integrand $\varphi_1(\delta p, \delta m, \delta w) \geq 0$.

3 Relation of variational principles

Let us formulate two conventional variational principles for dynamical beam problems. If the functions of elastic and kinetic energy density exist, the displacements w as a function of coordinates y are given in the initial ($t = 0$) and final ($t = T$) instants, and boundary conditions (3) do not change under displacement variations, Hamilton's principle follows the principal of virtual work (see [**?**]):

$$\begin{aligned}
\delta\mathrm{H} &= \delta\int_0^T [\mathrm{T} - \Pi]\, dt = 0, \\
\mathrm{T} &= \frac{1}{2}\int_0^L \rho\dot{w}^2 dy, \\
\Pi &= \frac{1}{2}\int_0^L EI(w'')^2 dy; \\
w(t,0) &= u, \quad w'(t,0) = 0; \\
w(0,x) &= w^0, \quad w(T,x) = w^f.
\end{aligned} \tag{10}$$

Note that the displacement fields w must rigorously satisfy the displacement boundary conditions at $y = 0$ (see Eq. (3)). The equilibrium relation in Eq. (1) and moment boundary conditions at $y = L$ in Eq. (3) are implicitly contained in the functional H in Eq. (10) as its stationary conditions.

If there exist functions of elastic and kinetic energy density expressed in terms of the momentum density and bending moment, the momentum density p as functions of the coordinates y are given in the initial ($t = 0$) and final instants ($t = T$), and boundary conditions (3) do not change under momentum and moment variations, the stationary principle for complementary energy follows the principal of virtual complementary work:

$$\begin{aligned}
\delta\mathrm{H_c} &= \delta\int_0^T [\mathrm{T} - \Pi]\, dt = 0, \\
\mathrm{T_c} &= \frac{1}{2\rho}\int_0^L p^2 dy, \\
\Pi_c &= \frac{1}{2EI}\int_0^L m^2 dy + u\, m'|_{y=0}; \\
w(t,0) &= u, \quad w'(t,0) = 0; \\
w(0,x) &= w^0, \quad w(T,x) = w^f.
\end{aligned} \tag{11}$$

According to this principle the fields of momentum density p and moment m must rigorously satisfy the equilibrium equation in Eq. (1) and boundary conditions at $y = L$ in Eq. (3). The displacement boundary conditions at $y = 0$ are Euler's equations for the functional H_c in Eq. (11).

It was shown in [**?**] for static linear elasticity problems that the method of integrodifferential relations gives one the possibility to formulate various variational principles. Analogously to the static case, one can introduce into consideration parametric families of quadratic functionals, for example,

$$\begin{aligned}
\Phi &= \int_0^T\int_0^L \varphi(p, m, w)dydt, \\
\varphi &= \frac{(p - \rho\dot{w})^2}{2\rho} + a\frac{(m - EIw'')^2}{2EI}.
\end{aligned} \tag{12}$$

which stationary conditions are equivalent to relations (2). Here $a \neq 0$ is a real constant. At $a = 1$ integral (12) is equivalent to the nonnegative functional Φ_1 defined in Eq. (8). Let us consider in more detail a functional $\Phi_2 = \Phi|_{a=-1}$ in this family at $a = -1$ and formulate the following variational problem: to find unknown functions p^*, m^*, w^* providing the stationary value for this functional and satisfying constraints (2)–(4). It is possible to transform the integral Φ_2, which has the dimension of action, to the form

$$
\begin{aligned}
\Phi_2 &= \Theta_{pm} + \Theta_w - 2\Theta, \\
\Theta &= \frac{1}{2}\int_0^T \int_0^L \left[p\dot{w} - m w''\right] dydt, \\
\Theta_w &= \frac{1}{2}\int_0^T \int_0^L \left[\rho \dot{w}^2 - EI(w'')^2\right] dydt, \\
\Theta_{pm} &= \int_0^T \int_0^L \left[\frac{p^2}{2\rho} - \frac{m^2}{2EI}\right] dydt. \qquad (13)
\end{aligned}
$$

Note that the integral Θ_w depends only on the displacements w, whereas Θ_{pm} is independent of this function. The bilinear functional Θ is independent explicitly of elastic and inertial material properties, and after integrating by parts and using relations (1) it can be presented as

$$
2\Theta = \int_0^L pw|_{t=0}^{t=T}\, dy - \int_0^T u\; m'|_{y=0}\, dt. \qquad (14)
$$

After substituting Eq. (14) into Eq. (13), the functional Φ_2 will have the form

$$
\begin{aligned}
\Phi_2 = \Theta_w + \Theta_{pm} - \\
\int_0^T u\; m'|_{y=0} + \int_0^L pw|_{t=0}^{t=T}\, dy. \qquad (15)
\end{aligned}
$$

If at both initial time $t = 0$ and finale time $t = T$ either the displacement or momentum density function is given then the functional Φ_2 decomposes into two independent parts

$$
\begin{aligned}
\Phi_2 &= \mathrm{H}_w + \mathrm{H}_{pm}, \\
\mathrm{H}_w &= \Theta_w + \Xi_w, \\
\mathrm{H}_{pm} &= \Theta_{pm} - \int_0^T u\; m'|_{y=0} + \Xi_p. \qquad (16)
\end{aligned}
$$

Here the functional H_w depends only on the displacement fields w, H_{pm} is a function of the moment m and momentum density p, Ξ_w and Ξ_p are space integrals which are depended on w and p, respectively, and defined by initial and terminal conditions. In Eqs. (17)–(21) the forms of the integrals Ξ_w and Ξ_p are presented for several kinds of such conditions.

$$
\begin{aligned}
& w(0,y) = w^0(y), \quad w(T,y) = w^f(y): \qquad (17) \\
& \Xi_u = 0, \\
& \Xi_p = \int_0^L \left[w^f\; p|_{t=T} - w^0\; p|_{t=0}\right] dy; \\
& p(0,y) = p^0(y), \quad p(T,y) = p^f(y): \qquad (18) \\
& \Xi_w = \int_0^L \left[p^f\; w|_{t=T} - p^0\; w|_{t=0}\right] dy, \\
& \Xi_p = 0; \\
& w(0,y) = w^0(y), \quad p(T,y) = p^f(y): \qquad (19) \\
& \Xi_w = \int_0^L \left[p^f\; w|_{t=T}\right] dy, \\
& \Xi_p = -\int_0^L \left[u^0\; p|_{t=0}\right] dy; \\
& p(0,y) = p^0(y), \quad w(T,y) = w^f(y): \qquad (20) \\
& \Xi_w = -\int_0^L \left[p^0\; w|_{t=0}\right] dy, \\
& \Xi_p = \int_0^L \left[u^f\; p|_{t=T}\right] dy; \\
& w(0,y) = w(T,y), \quad p(0,y) = p(T,y): \qquad (21) \\
& \Xi_w = 0, \quad \Xi_p = 0.
\end{aligned}
$$

The first four cases (17)–(20) correspond to different space-time boundary value problems, and the last problem (21) describes periodic body motions.

The stationary conditions for the functional Φ_2 under the equilibrium relation and space-time boundary conditions discussed above can be written as

$$
\delta\Phi_2 = \delta_w \mathrm{H}_w + \delta_m \mathrm{H}_{pm} + \delta_p \mathrm{H}_{pm} = 0. \qquad (22)
$$

Consequently, this constrained variational problem is equivalent to two independent problems:

1. To find a displacement vector w^* providing the stationary value for the functional H_w

   $$
   \delta_w \mathrm{H}_w = 0 \qquad (23)
   $$

   and strictly satisfying displacement boundary conditions at $y = 0$ in Eq. (3) as well as initial and/or terminal displacement conditions shown in Eqs. (17)–(21).
2. To find functions of bending moment m^* and momentum density p^* guarantying the stationarity for the functional H_{pm}

   $$
   \delta_m \mathrm{H}_{pm} + \delta_p \mathrm{H}_{pm} = 0 \qquad (24)
   $$

   and obeying equilibrium relation in Eq. (1), moment constraints at $y = L$ in Eq. (3), and initial and/or terminal momentum conditions from Eqs. (17)–(21).

The displacement fields w^* obtained from Problem 1 let one find corresponding moment $EI(w^*)''$ and momentum density $\rho\dot{w}^*$. On the other hand the moment and momentum density fields m^*, p^* obtained from Problem 2 can be related to the following bending and velocity fields: $m^*/(EI)$, p^*/ρ. It is difficult enough to identify corresponding displacements w since one has to solve the overdetermined system of differential equations $w'' = m^*/(EI)$ and $\dot{w} = p^*/\rho$.

It is important to emphasize that if displacements are given at the beginning and end the motion (Eq. (17)) then Problem 1 is equivalent to Hamilton's principle (10) ($\mathrm{H}_w = \mathrm{H}$). If initial and terminal momentum fields are given (Eq. (18)) then Problem 2 coincides with complementary principle (11) ($\mathrm{H}_{pm} = \mathrm{H}_c$). Both conventional principles are valid in the case of periodic time conditions (Eq. (21)).

In contrast to variational principles (10) and (11) which are formulated for space-time boundary value problems, the functional Φ_2 can be applied as well to the initial boundary value problem (1)–(4). The initial conditions (4) do not allow to separate the functional Φ_2, because the last term in relation (15) depends explicitly on both displacements w and momentum density p. In this case, as for the functional Φ_1, the variational problem has to be solved simultaneously with respect to unknown displacement, moment, and momentum density functions.

The value of the nonnegative functional Φ_1, which has the dimension of action can serve to estimate the quality of approximate solutions for all variational problems proposed in the paper.

4 An approximation algorithm

To find an approximate solution of the optimal control problem for the motion of a beam which is undeformed at the beginning (a special case of problem Eqs. (1), (3)–(8)) a polynomial representation of the unknown functions has been used in [?], [?], [?]. In this work the functions p, m, and w are approximated by bivariate piece-wise polynomial splines defined on rectangular meshes.

As shown in Fig. 2 let us divide the time-space domain $\Omega = (0, T) \times (0, L)$ on $N \times M$ rectangles Ω_{kl} which vertices have coordinates $Q_{k-1,\,l-1}$, $Q_{k-1,\,l}$, $Q_{k,\,l-1}$, Q_{kl}, where $Q_{kl} = (t_k, y_l)$; $t_k > t_{k-1}$, $k = 1, \ldots, N$; $y_l > y_{l-1}$, $l = 1, \ldots, M$; $t_0 = 0$, $t_N = T$, $y_0 = 0$, $y_M = L$. Let also the boundary edges of these time-space rectangles be named $L_{kl} = (Q_{k,l-1}, Q_{kl})$, $k = 0, \ldots, N$, $l = 1, \ldots, M$, and $T_{kl} = (Q_{k-1,l}, Q_{kl})$, $k = 1, \ldots, N$, $l = 0, \ldots, M$. For all rectangles Ω_{kl}

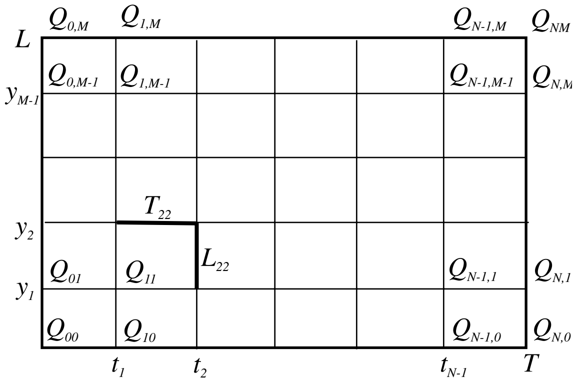

Figure 2. Rectangular mesh in the time-space domain Ω.

the polynomial approximating functions are given

$$\tilde{p}_{kl} = \sum_{i+j=0}^{N^p_{kl}} p^{(ij)}_{kl} t^i y^j,$$
$$\tilde{m}_{kl} = \sum_{i+j=0}^{N^m_{kl}} m^{(ij)}_{kl} t^i y^j,$$
$$\tilde{w}_{kl} = \sum_{i+j=0}^{N^w_{kl}} w^{(ij)}_{kl} t^i y^j. \qquad (25)$$

The control function u is restricted to a set of time splines

$$U_N = \left\{ \begin{array}{l} u: \quad u = \sum_{i=0}^{N^u_k} u^{(i)}_k t^i; \\ t \in (t_{k-1}, t_k), \quad k = 1, \ldots, N; \end{array} \right\}. \qquad (26)$$

Here $p^{(ij)}_{kl}$, $m^{(ij)}_{kl}$, $w^{(ij)}_{kl}$, and $u^{(i)}_k$ are unknown real coefficients. The basis functions are chosen so that the approximations can exactly satisfy boundary and piece-wise polynomial initial conditions (3), (4), and the equilibrium equation (1) on the rectangles Ω_{kl} by suitably selected integers N^p_{kl}, N^m_{kl}, N^w_{kl}, and N^u_k. In addition, to apply the variational formulation given above the following conformed interelement relations must be satisfied

$$\begin{aligned} &\tilde{w}_{kl}(t_k, y) = \tilde{w}_{k+1,\,l}(t_k, y), \\ &\tilde{p}_{kl}(t_k, y) = \tilde{p}_{k+1,\,l}(t_k, y), \\ &\quad (t_k, y) \in L_{kl}, \\ &\qquad k = 1, \ldots, N-1, \quad l = 1, \ldots, M; \\ &\tilde{w}_{kl}(t, y_l) = \tilde{w}_{k,\,l+1}(t, y_l), \\ &\tilde{w}'_{kl}(t, y_l) = \tilde{w}'_{k,\,l+1}(t, y_l), \\ &\tilde{m}_{kl}(t, y_l) = \tilde{m}_{k,\,l+1}(t, y_l), \\ &\tilde{m}'_{kl}(t, y_l) = \tilde{m}'_{k,\,l+1}(t, y_l), \\ &\quad (t, y_l) \in T_{kl}, \\ &\qquad k = 1, \ldots, N, \quad l = 1, \ldots, M-1. \end{aligned} \qquad (27)$$

These approximations give one the possibility to obtain numerical solutions and analyze its quality and convergence rate for resulting finite dimensional systems by using the distribution of the discretization error φ and the value of the functional Φ.

After satisfying constraints (1), (3)–(6), and (27) the resulted finite-dimensional unconstrained minimization problem (8) yields an approximate solution $\tilde{p}^*(t,y,u)$, $\tilde{m}^*(t,y,u)$, $\tilde{w}^*(t,y,u)$ for an arbitrary control $u \in U_N$, where U_N is the set of piece-wise polynomial functions with a given degree N_k^u on the intervals Ω_{kl}. The optimal control $u^*(t,y)$ is found by minimizing the objective function J in Eq. (7). Let us consider a functional $J[u]$ quadratic with respect to the control parameters $u_k^{(i)}$. In this case if $U \subset U_{NM}$ in Eq. (7) then the corresponding optimization problem is reduced to a system of linear equations with respect to unknown control parameters $u_k^{(i)}$.

The value of the functional

$$\tilde{\Phi} = \Phi(\tilde{p}^*(t,y,u^*), \tilde{m}^*(t,y,u^*), \tilde{w}^*(t,y,u^*))$$

is an absolute integral criterion of the optimal solution quality. To define relative integral errors the following nonnegative functionals with the dimension of action can be used

$$\begin{aligned}
\Phi &= \Psi_{pm} + \Psi_w - 2\Psi, \\
\Psi_{pm} &= \int_0^T \int_0^L \psi_{pm}(p,m,w)\,dydt, \\
&\qquad \psi_{pm} = \frac{p^2}{2\rho} + \frac{m^2}{2EI}; \\
\Psi_w &= \int_0^T \int_0^L \psi_w(p,m,w)\,dydt, \\
&\qquad \psi_w = \frac{\rho \dot{w}^2}{2} + \frac{EI\,(w'')^2}{2}; \\
\Psi &= \int_0^T \int_0^L \psi(p,m,w)\,dydt, \\
&\qquad \psi = \frac{p\dot{w}}{2} + \frac{mw''}{2}.
\end{aligned} \tag{28}$$

After substituting optimal functions $\tilde{p}^*(t,y,u^*(t,y))$, $\tilde{m}^*(t,y,u^*(t,y))$, $\tilde{w}^*(t,y,u^*(t,y))$ in the functionals Ψ, Ψ_{pm}, Ψ_w various dimensionless values can be generated, for example,

$$\begin{aligned}
\Delta_1 &= \frac{\Psi_{pm} + \Psi_w}{2\Psi} - 1, \\
\Delta_2 &= 1 - \frac{2\Psi}{\Psi_{pm} + \Psi_w}, \\
\Delta_3 &= \frac{\Phi}{2\Psi_{pm}}, \quad \Delta_4 = \frac{\Phi}{2\Psi_w},
\end{aligned} \tag{29}$$

and so on. The integrand φ in Eq. (8) can serve as a local criterion of solution quality, whereas the

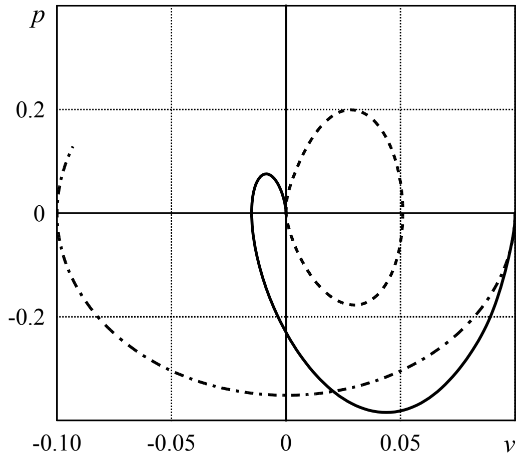

Figure 3. Phase trajectories of the beam end points.

functions $\tilde{\psi}(t,y) = \psi(\tilde{p}^*, \tilde{m}^*, \tilde{w}^*)$, $\tilde{\psi}_{pm}(t,y) = \psi_{pm}(\tilde{p}^*, \tilde{m}^*, \tilde{w}^*)$, $\tilde{\psi}_w(t,y) = \psi_w(\tilde{p}^*, \tilde{m}^*, \tilde{w}^*)$ at $u = u^*(t,y)$ estimate the linear density of the total mechanical energy in the beam.

5 Numerical example

For numerical modeling, the following dimensionless parameters are used: the beam length $L = 1$, its linear mass density $\rho = 1$, bending rigidity $EI = 1$, control time $T = 1$, final truck displacement $X = 0$.

Let us remind that the optimal control problem (1)–(7) of reduction of beam vibrations is considered. In a fixed time the system should be moved from a given initial state to the terminal state where the total mechanical energy of the system reaches its minimal value. The quadratic objective function J in the proposed variational formulation can be expressed in terms of the energy density function ψ_{pw} defined in Eq. (5). For the illustration the simple control law u is chosen as a time polynomial of the fixed degree N_u

$$u = \sum_{i=1}^{N_u} u_i (T-t) t^{i+2}, \quad t \in (0,T). \tag{30}$$

The unknown function u satisfy terminal conditions $u = 0$ at the time T and must minimize the functional J in Eq. (5). In the case under investigation the control u in (**??**) contains N_u unknown optimizing parameters.

To demonstrate the effectiveness of the numerical algorithm proposed in Section 3 the following mesh and approximation parameters are assigned: $N = 3$, $M = 1$, $N_{kl}^p = N_{kl}^m = N_{kl}^w = 10$. To improve the integral quality of numerical solution a time nonuniform mesh with the parameters $t_1 = 1/6$ and $t_2 = 1/2$ was found for this optimal problem.

As a initial state of the beam in Eq.(4) the zero distribution of the momentum density $g(y) = 0$ and the first

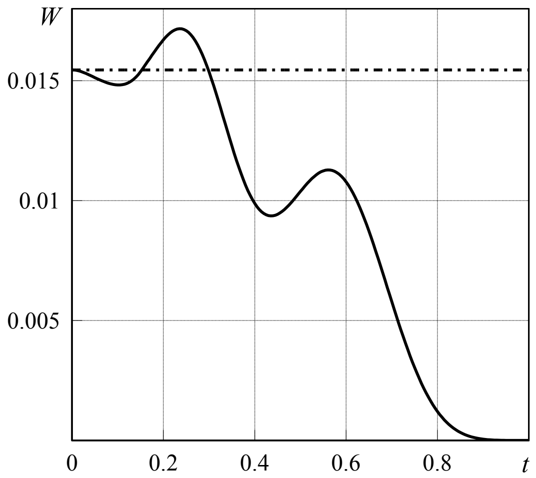

Figure 4. Total mechanical energy W vs time t.

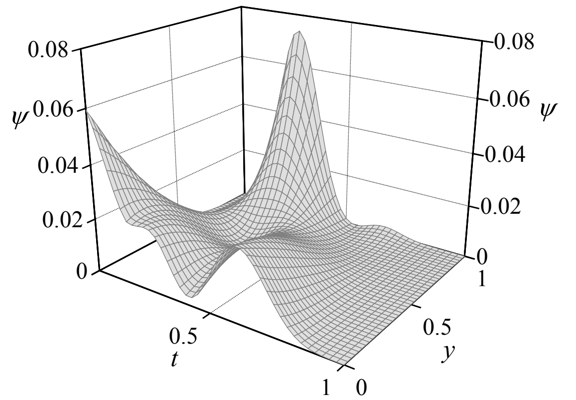

Figure 5. Distribution of mechanical energy density $\psi_{pm}(t, y)$.

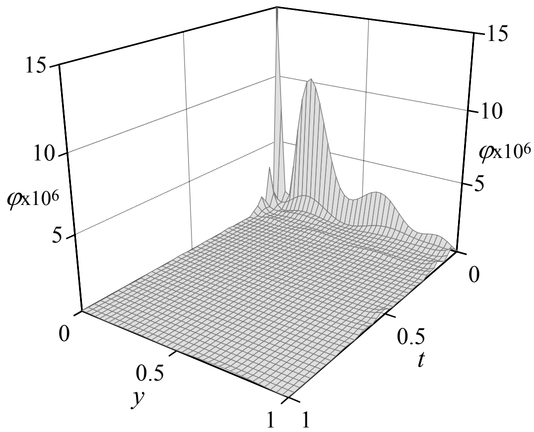

Figure 6. Distribution of local solution error.

eigenform for the displacement function

$$\begin{aligned} f(y) = C_1 \left[\cos(\lambda_1 y) - \cosh(\lambda_1 y)\right] \\ +C_2 \left[\sin(\lambda_1 y) - \sinh(\lambda_1 y)\right], \qquad (31) \\ \lambda_1 \simeq 1.8751, \\ C_1 \simeq -1.3622 C_2, \quad C_2 \simeq -0.0367 \end{aligned}$$

are chosen. Using the least quare method this function is approximated by the polynomial $\tilde{w}_{11}(0, y)$ of order $N_{11}^{w} = 10$ defined in Eq. (1)

$$\Delta f = \int_0^l \left[f(y) - \tilde{w}_{11}(0, y)\right]^2 dy \to \min_{w_{11}^{(ij)}}$$

under the relative integral error in the initial displacement condition $\sqrt{\Delta f / ||f||} = 10^{-10}$. The initial displacement of the beam free end is $w(0, l) = 0.1$.

To estimate the polynomial optimization resources for beam dynamic problems an admissible truck position $u(t) = 0$ is specified as a sample motion (free vibration. As the number of free parameters of the polynomial control in the optimization problem (1), (3)–(8) increases, the total mechanical energy of the beam at the terminal time $t = T$ reduces considerably. The optimal control motion of the beam was obtained by the MIDR for $N_u = 10$. In Fig. 3 the optimal state of the beam end clamped on the truck is shown in the displacement-momentum phase plane (w, p) by the solid curve. The sample uncontrolled vibration of the beam free tip are presented in this figure by the dash-dot curve to compare with the optimal motion of this point (dash line).

In Fig. 4 the pattern of the total beam energy during the optimal control process is shown by the solid curve. At the beginning of this motion the value of the energy W defined in Eq. (5) is approximately equal to $W(0) \simeq 0.01545$ and remains constant if the truck is at rest (dash line). In the case of the polynomial control law with 10 control parameters, the value of W can be reduced to $J = W(T) \simeq 3.4 \times 10^{-8}$ by more than 2×10^{-6} times.

The distribution of energy density characterizes the dynamic process taking place in the beam during the optimal control. In Fig. 5 the function $\psi_{pm}(t, y)$ defined in Eq. (**??**) is presented. As it is seen from the picture the mechanical energy is reduced noticeably at the end of the optimal process.

The value of the functional Φ_1 can be considered as an integral performance criterion for the optimal solution whereas the integrand φ_1 in (8) is a local quality characteristic. Figure 6 shows the distribution of the function $\varphi_1(t, y)$ for $N_u = 5$. It can be seen that its value is small almost everywhere, except for the vicinity of $t = 0$ with its maximum at the point $y = 0$. For the defined parameters the value of the functional is equal to $\Phi \simeq 6.158 \times 10^{-8}$ whereas its relative value, for example, $\Delta_3 \simeq 3.3 \times 10^{-6}$.

6 Conclusions

The presented method of integrodifferential relations to solve mechanical initial-boundary value problems can be considered as an alternative to conventional approaches. Based on the MIDR a variational principle which stationary conditions are equivalent to the constitutive beam relations was formulated. For this principle the nonnegative quadratic functional under minimization can serve as integral criteria of the solution quality, whereas its integrand characterizes the local error distribution. The principle proposed is also applicable to the beam problems with the mixed boundary conditions (e.g., elastic support).

The finite element algorithm developed enables one to construct effective bilateral estimates for various integral characteristics (elastic energy, displacements, etc.). The polynomial spline technique allows one to take into account nonhomogeneous inertial and elastic properties. This FEM realization gives one the possibility to work out various strategies of p-h adaptive mesh refinement by using the local error estimates. The computational cost of the algorithm proposed is stipulated by the efficiency of particular FEM realization.

Acknowledgements

This research was supported by the Russian Foundation for Basic Research (grant No. 08-01-00234) and the Program "State Support for Leading Scientific Schools" (NSh-169.2008.1, NSh-4315.2008.1).

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