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OPTIMAL FEEDBACK CONTROL OF TRAVELING WAVE IN A PIECEWISE LINEAR FITZHUGH–NAGUMO MODEL

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Abstract

In a previous study [Konishi, Takeuchi, Shimizu, Chaos 2011], the authors proposed a simple systematic design procedure that employed periodic impulsive and time-continuous feedback forces to eliminate traveling waves in a piecewise linear FitzHugh–Nagumo (FHN) model. As the previous study used only the integral control method in *classical* control theory, it was not easy to specify the system performance. Therefore, the present paper introduces an optimal control method in *modern* control theory that can specify the system performance. Furthermore, we show that the designed force is valid not only for the piecewise linear function but also for a class of smooth nonlinear FHN models.

Key words

FitzHugh–Nagumo model, traveling wave, optimal feedback control, piecewise linear model.

1 Introduction

Excitable media such as cardiac tissue and the Belousov–Zhabotinsky (BZ) reaction have received considerable attention in the field of nonlinear science (Mikhailov and Showalter, 2006). It is known that spatial waves and spatiotemporal chaos in cardiac tissue induce major health problems, because irregular activation, such as ventricular tachycardia and ventricular fibrillation, decreases the ability of the heart to pump blood. A current treatment for irregular activation is applying a *high*-voltage electric shock to the patient's chest. However, this shock often causes physical and mental strain to the patient. Therefore, a practical use of *low*-voltage electric shock is anticipated (Sinha and Sridhar, 2008; Takagi et al., 2004). In addition, the BZ reaction have attracted increasing interest (Mikhailov and Showalter, 2006). It was reported that light-intensity feedback control can stabilize and track unstable propagating waves in a photosensitive BZ reaction (Mihaliuk et al., 2002; Sakurai et al., 2002).

A number of researchers have proposed various feedback control methods to eliminate spatiotemporal behavior in excitable media (Sinha and Sridhar, 2008). Yuan, Chen, and Yang showed that an external force injected into resting regions can counter and eliminate propagation waves (Yuan et al., 2007). The global feedback control proposed by Yoneshima, Konishi, and Kokame uniformly applies a force to the medium by sensing the medium activity (Yoneshima et al., 2008). Guo *et al.* proposed local feedback control that identifies the spiral tip areas and makes them unexcitable. This method can experimentally eliminate spiral turbulence (Guo et al., 2010). Sakaguchi and Nakamura eliminated breathing spiral waves in the Aliev–Panfilov model by using delayed feedback control (Sakaguchi and Nakamura, 2010).

From the practical viewpoint, it would be useful to understand the systematic design of the external forces involved in the elimination of spatiotemporal behavior. This is because a systematic design would not require trial-and-error testing. However, most studies on the elimination of spatiotemporal behavior employed only numerical simulations (Sinha and Sridhar, 2008). A systematic design procedure for a single impulsive nonfeedback force was provided by Osipov and Collins (Osipov and Collins, 1999). Although periodic impulsive and time-continuous feedback forces have the potential to achieve low-amplitude elimination, their procedure cannot be used for such forces. Our previous study proposed a simple systematic design procedure for such forces (Konishi et al., 2011). This study focused on a one-dimensional FitzHugh–Nagumo (FHN) model with a piecewise linear function (Ohta and Kiyose, 1996; Ohta et al., 1997; Koga, 1993; Rinzel and Keller, 1973; Tonnelier, 2003b; Tonnelier, 2003a) and obtained simple analytical results. The proposed procedure is useful for designing nonfeedback and feedback control systems. However, in our previous study, the following problems, which are important subjects from a practical viewpoint, remained un-

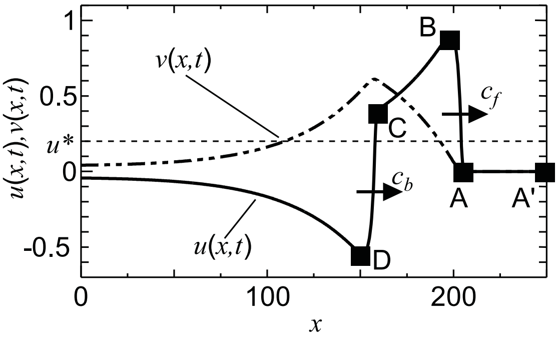

Figure 1. Spatial distribution of the traveling wave.

solved: (i) it is impossible to specify the system performance including the transient time for elimination and the amplitude of the external force for designing the feedback controller; (ii) it is unclear whether the procedure can be used for an FHN model with smooth nonlinear functions.

The present paper shows that problems (i) and (ii) can be solved using *modern* control theory and a smooth nonlinear function, respectively. As our previous study (Konishi et al., 2011) used only the integral control method in *classical* control theory, problem (i) could not be systematically solved. On the other hand, this paper introduces an optimal feedback control method in *modern* control theory to systematically solve problem (i). Furthermore, for problem (ii), the force designed by our procedure for the piecewise linear FHN model is applied to a smooth nonlinear FHN model. We find that the designed force is valid not only for the piecewise linear model but also for a class of smooth nonlinear models.

2 Piecewise linear FHN model

Now consider the one-dimensional piecewise linear FHN model (Ohta and Kiyose, 1996; Ohta et al., 1997; Koga, 1993; Rinzel and Keller, 1973):

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = f[u(x,t)] - v(x,t) + D\dfrac{\partial^2 u(x,t)}{\partial x^2} \\ \dfrac{\partial v(x,t)}{\partial t} = \varepsilon\{u(x,t) - \gamma v(x,t)\} + e(t) \end{cases}, \tag{1}$$

$$f[u] = H[u - u^*] - u, \tag{2}$$

where $u(x,t)$ and $v(x,t)$ are fast and slow variables, respectively. x denotes position and t is continuous time. The diffusion coefficient for the fast variable is denoted by $D > 0$. Here $0 < \varepsilon \ll 1$ and $\gamma \in (0, u^*/(1-u^*))$ are parameters. $u^* \in (0, 1/2)$ is the threshold of $f[u]$ and H represents the step function. The weak external force $e(t)$ is applied with spatial uniformity to the slow dynamics. Let us assume that a traveling wave propagates through a one-dimensional space $x \in (-\infty, +\infty)$. Figure 1 charts the spatial distribution of the traveling wave.

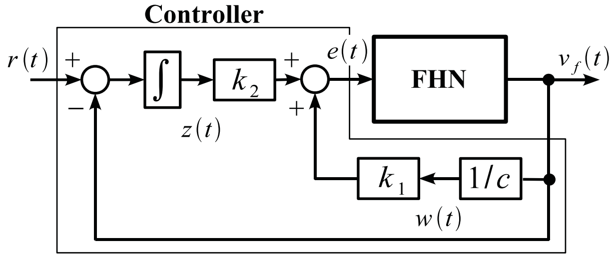

Figure 2. Block diagram of state feedback with integral control.

3 Feedback control

In this section, we derive a linear time invariant system (Konishi et al., 2011) and design an optimal feedback controller on the basis of a linear quadratic regulator.

3.1 Linear time invariant system

This subsection reviews our previous work (Konishi et al., 2011). The velocity of the traveling wave at the wavefront (curve AB in Fig. 1), c_f, is given by

$$c_f = \{1 - 2(u^* + v_f)\}\sqrt{\frac{D}{(u^* + v_f)(1 - u^* - v_f)}}. \tag{3}$$

The wavefront velocity c_f depends on $v(x,t) = v_f$ in front of the wave (A–A' region in Fig. 1). $u(x,t)$ and $v(x,t)$ have spatial uniformity in the A–A' region. As the external force $e(t)$ is applied in a spatially uniform manner to the entire medium, $u(x,t)$ and $v(x,t)$ maintain their spatial uniformity in this region. They can be considered as variables $u_f(t)$ and $v_f(t)$ in this region. In addition, the wavefront velocity satisfying Eq. (3) can also be a time variable $c_f(t)$. Because $u(x,t)$ and $v(x,t)$ in this region have spatial uniformity, we can neglect the diffusion term in Eq. (1). As $u_f(t) < u^*$, the nonlinear function is simplified to $f[u] = -u$. The parameter ε is assumed to be a sufficiently small positive value; thus, we ignore the fast mode and focus only on the slow dynamics. In consequence, the force $e(t)$ and variable $v_f(t)$ are approximately given by the linear time invariant (LTI) system

$$\begin{cases} \dot{w}(t) = aw(t) + be(t) \\ v_f(t) = cw(t) \end{cases}, \tag{4}$$

where $w(t)$ describes the slow dynamics and the system parameters (a, b, c) are written as

$$a := \eta^{(+)},\ b := -\frac{1}{\eta^{(+)} - \eta^{(-)}},\ c = -1 - \eta^{(+)},$$

$$\eta^{(\pm)} := \{-(1 + \varepsilon\gamma) \pm \sqrt{(1 - \varepsilon\gamma)^2 - 4\varepsilon}\}/2.$$

3.2 Controller

This paper proposes state feedback with integral control (Kuo and Golnaraghi, 2003), as shown in Fig. 2, to eliminate traveling waves. The external force $e(t)$ is given by

$$\begin{cases} e(t) = k_1 w(t) + k_2 z(t) \\ \dot{z}(t) = r(t) - v_f(t) \end{cases}, \tag{5}$$

where $z(t)$ is the additional variable and $k_{1,2}$ are the feedback gains to be designed. Our main goal is to stop the traveling wavefront (i.e., $\lim_{t\to+\infty} c_f(t) = 0$); thus, according to Eq. (3), the controller must track $v_f(t)$ to $1/2 - u^*$. To this end, the reference signal $r(t)$ should be a step input with amplitude $1/2 - u^*$. We use the additional variable $z(t)$ because the integral unit is required to perform tracking without a steady-state error.

Let us design the gains $k_{1,2}$ by using a linear quadratic regulator (Zak, 2002). Combining LTI system (4) and controller (5), we have

$$\begin{cases} \dot{\mathbf{x}}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{b}e(t) + \begin{bmatrix} 0 \ 1 \end{bmatrix}^T r(t) \\ v_f(t) = \boldsymbol{c}\mathbf{x}(t) \end{cases}, \tag{6}$$

$$e(t) = \boldsymbol{k}\mathbf{x}(t), \tag{7}$$

where $\mathbf{x}(t) := \begin{bmatrix} w(t) \ z(t) \end{bmatrix}^T$ is the system variable. The system matrices ($\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$) and the feedback gain vector $\boldsymbol{k}$ are written as

$$\boldsymbol{A} := \begin{bmatrix} a & 0 \\ -c & 0 \end{bmatrix}, \ \boldsymbol{b} := \begin{bmatrix} b \\ 0 \end{bmatrix}, \ \boldsymbol{c} := \begin{bmatrix} c \\ 0 \end{bmatrix}^T, \ \boldsymbol{k} := \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}^T. \tag{8}$$

Because it is obvious that $(\boldsymbol{A}, \boldsymbol{b})$ is controllable (i.e., $\det \begin{bmatrix} \boldsymbol{b} \ \boldsymbol{Ab} \end{bmatrix} = -b^2 c \neq 0$), the feedback gain $\boldsymbol{k}$ can be designed by a simple procedure. To obtain optimal system performance, this study employs a linear quadratic regulator: controller (7) is designed as follows such that the performance index is minimized:

$$J = \int_0^\infty \left\{ \bar{\mathbf{x}}(t)^T \boldsymbol{Q} \bar{\mathbf{x}}(t) + \mu \bar{e}(t)^2 \right\} \mathrm{d}t, \tag{9}$$

$$\bar{\mathbf{x}}(t) := \mathbf{x}(t) - \mathbf{x}_\infty, \ \bar{e}(t) := e(t) - e_\infty,$$

where $\mathbf{x}_\infty := \lim_{t\to\infty} \mathbf{x}(t)$ and $e_\infty := \lim_{t\to\infty} e(t)$. Here $\boldsymbol{Q} > 0$ and $\mu > 0$ are the weights that can be arbitrarily chosen. The feedback gain is given by

$$\boldsymbol{k} = -\frac{1}{\mu} \boldsymbol{b}^T \boldsymbol{P}, \tag{10}$$

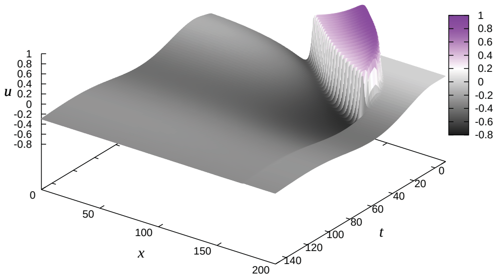

Figure 3. Spatial distribution of the traveling wave with integral control ($k_1 = 0, k_2 = 0.003$).

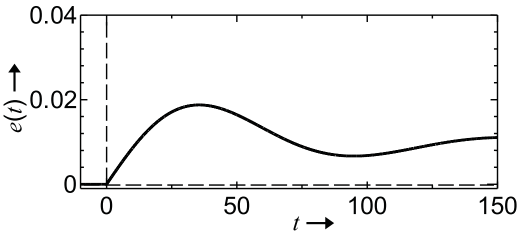

Figure 4. External force with integral control ($k_1 = 0, k_2 = 0.003$).

where $\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} > 0$ satisfies the algebraic Riccati equation

$$\boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{PA} + \boldsymbol{Q} - \frac{1}{\mu} \boldsymbol{P} \boldsymbol{b} \boldsymbol{b}^T \boldsymbol{P} = \boldsymbol{0}. \tag{11}$$

Now let us design the feedback gain $\boldsymbol{k}$ according to the above procedure. For simplicity, the weight $\boldsymbol{Q}$ is fixed at $\boldsymbol{Q} = \mathrm{diag}\{q_1, q_2\}$. Substituting Eq. (8) into Eq. (11), we have

$$\begin{aligned} p_{12} &= \sqrt{\mu q_2 / b^2}, \\ p_{11} &= \left\{ a\mu + \sqrt{a^2\mu^2 - b^2\mu(2cp_{12} - q_1)} \right\} / b^2, \\ p_{22} &= p_{12}(a - b^2 p_{11}/\mu)/c. \end{aligned}$$

Here we obtain optimal gain (10),

$$k_1 = -\frac{1}{\mu} b p_{11}, \ k_2 = -\frac{1}{\mu} b p_{12}. \tag{12}$$

Therefore, we have an optimal feedback controller (5) with gain (12).

3.3 Numerical examples

Throughout this paper, we assume that the parameters of FHN model (1) are known and fixed at

$$u^* = 0.2, \ D = 1.0, \ \varepsilon = 0.03, \ \gamma = 0.1. \tag{13}$$

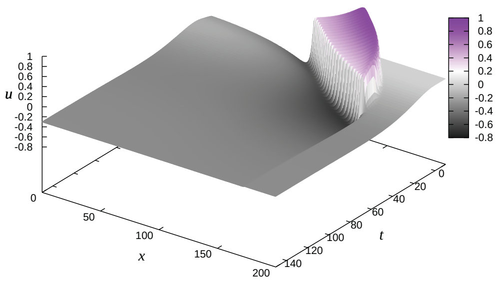

Figure 5. Spatial distribution of the traveling wave with optimal control (case (I): $k_1 = 0.0598, k_2 = 0.0041$).

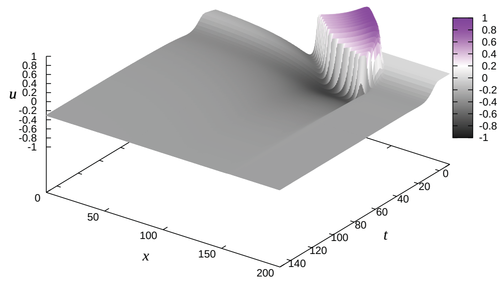

Figure 7. Spatial distribution of the traveling wave with optimal control (case (II): $k_1 = 0.3135, k_2 = 0.0632$).

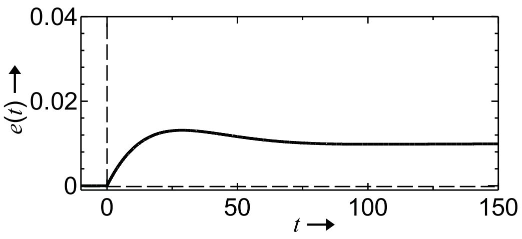

Figure 6. External force with optimal control (case (I): $k_1 = 0.0598, k_2 = 0.0041$).

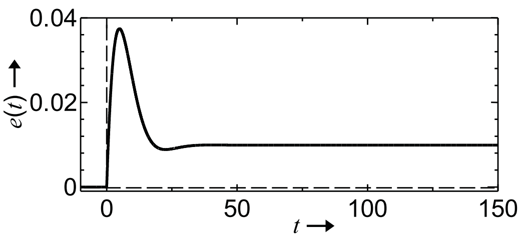

Figure 8. External force with optimal control (case (II): $k_1 = 0.3135, k_2 = 0.0632$).

The explicit Euler method is used with time step $\delta t = 0.01$ and space step $\delta x = 0.3$. Let us review the numerical result of the integral control, which corresponds to a particular case of controller (5) (i.e., $k_1 = 0$ and $k_2 > 0$), discussed in our previous work (Konishi et al., 2011). Figures 3 and 4 show the spatial distribution of the traveling wave and the external force just before and after the control start time ($t = 0$). For $t > 0$, the wavefront AB slows down and the wave back CD maintains its velocity, and then, the wave back catches up with the wavefront. Eventually, the traveling wave disappears at $t \approx 34.5$.

Now we design the optimal controller proposed in the preceding section. Consider the two specifications of system performance: (I) preference for a low peak force and (II) preference for the rapid elimination of traveling waves.

For case (I), the weights in index (9) are set to $q_1 = q_2 = 1.0$ and $\mu = 6 \times 10^4$. From these weights and parameters (13), optimal gain (12) can be obtained: $k_1 = 0.0598$ and $k_2 = 0.0041$. The spatial distribution of the traveling wave and the external force are shown in Figs. 5 and 6. It can be observed that compared with the results of integral control shown in Figs. 3 and 4, the peak force is small.

For case (II), the weights are set to $q_1 = q_2 = 1.0$ and $\mu = 250$. Optimal gain (12) can be obtained: $k_1 = 0.3135$ and $k_2 = 0.0632$. The spatial distribution of the traveling wave and the external force are shown in Figs. 7 and 8. We observe that the traveling wave rapidly disappears. From the two cases, it can be numerically confirmed that the optimal controller works well.

4 FHN model with smooth nonlinear functions

This section investigates whether the force designed for the piecewise linear FHN model is valid for smooth nonlinear FHN models. Let us introduce the smooth nonlinear function shown in Fig. 9:

$$f[u] = -u + 0.5 + 0.5 \tanh(\alpha(u - u^*)). \qquad (14)$$

Note that this function converges on piecewise linear function (2) as $\alpha \to +\infty$. We have observed that the traveling wave can propagate in the FHN model with function (14) for $\alpha \geq 10$; however, it cannot propagate for $\alpha < 10$. We have numerically confirmed that the integral controller ($k_1 = 0, k_2 = 0.003$) and the optimal controller ($k_1 = 0.3135$, $k_2 = 0.0632$) designed for the piecewise linear FHN model are valid for the smooth nonlinear FHN model for $\alpha \geq 10$. Figure 10 shows the optimal control of the traveling wave in the FHN model with function (14) ($\alpha = 10$). It can be observed that the designed controller works well even for the FHN model with smooth function (14).

5 Conclusion

This paper provided a systematic procedure for designing an optimal feedback controller to eliminate the traveling waves in the piecewise linear FHN model. The designed controller achieves our goal: rapid disappearance of traveling waves and a low peak force. Furthermore, it was shown that the controller designed for the piecewise linear FHN model is valid for a class of smooth nonlinear FHN models.

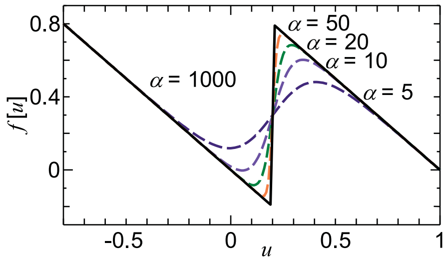

Figure 9. Smooth nonlinear function $f[u]$ ($u^* = 0.2$).

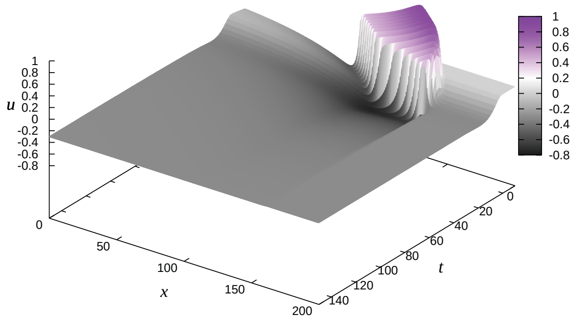

Figure 10. Spatial distribution of the traveling wave in optimal controlled FHN model with function (14) ($\alpha = 10$, $k_1 = 0.3135$, $k_2 = 0.0632$).

Acknowledgments

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