# CHAOS DEATH AND COMPLETE SYNCHRONIZATION IN CONJUGATE COUPLED CHAOTIC OSCILLATORS

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**Abstract**
When nonlinear oscillators are mutually coupled via dissimilar (or conjugate) variables, they show different regimes of synchronous behavior. In identical conjugate coupled chaotic oscillators complete synchronization occurs only by chaos suppression, when the coupled subsystems drive each other into a regime of periodic dynamics. In contrast to complete synchronization via diffusive coupling in similar variables, the coupling terms do not vanish but rather act as an internal drive. We study the phenomenon of chaos death and complete synchronization in a mutually conjugate coupled funnel Rössler system.

**Key words**
Conjugate coupling, Chaos death, Complete synchronization

## 1 Introduction

The study of coupled nonlinear dynamical systems is important from both theoretical and practical perspectives. Natural systems are rarely isolated and are coupled to one another on a range of spatial and temporal scales. Interactions between two or more systems typically give rise to new phenomena—synchronization being the most common. Although it has long been known that weak coupling of oscillators leads to synchronization [Pikovsky, Rosenblum and Kurths, 2001] recent studies have focused on coupling nonlinear systems where chaos synchronization, hysteresis, phase locking, phase shifting, phase-flip and amplitude death can occur [Pikovsky, Rosenblum and Kurths, 2001; Kaneko, 1993; Pecora and Carroll, 1990; A. Prasad, Iasemmidis, Sabesan and Tsakalis, 2005; Ott, 1993].

In a recent study [Karnatak, Ramaswamy and Prasad, 2007] we examined the effect of coupling systems via dissimilar (or *conjugate*) variables. The motivation came from the fact that although the coupled systems might be identical, it is at times not possible to couple them via the same variables due to various practical reasons, for instance in a coupled semiconductor laser experiment [Kim, 2005; Kim, Roy, Aron, Carr and Schwartz, 2005], a signal proportional to the photon intensity fluctuation from one laser is used to modulate the injection current of the other and vice versa. Other examples can be drawn from the epidemiology literature [Goleniewski, 1996] or from studies of electrical circuits, where the dynamical variables correspond to currents or voltages which are cross coupled [Ueta, Kawakami, 2003].

An important question to ask is whether conjugate coupled systems can show complete synchronization (CS) or not ? Note that in directly diffusively coupled oscillators, for complete synchronization the coupling terms vanish, implying that the synchronous state is thus an orbit of the uncoupled system. When the coupling involves conjugate variables, the coupling term typically *cannot* vanish. Nevertheless, we find that conjugate coupled systems can show complete synchronization when they drive each other into a regime of periodic dynamics, although the synchronous state is a new orbit that is characteristic of the coupled system. We term this change in dynamics from being instrinsically chaotic to periodic as a consequence of the interaction between oscillators as *chaos death*.

## 2 Conjugate coupling

Consider two coupled dynamical systems,

$$\left.\begin{aligned}\dot{\mathbf{X}}_{\mathbf{1}} &= \mathbf{f}(\mathbf{X}_{\mathbf{1}}) + \epsilon\mathbf{g}_{\mathbf{1}}(\mathbf{X}'_{\mathbf{2}}, \mathbf{X}_{\mathbf{1}}) \\ \dot{\mathbf{X}}_{\mathbf{2}} &= \mathbf{f}(\mathbf{X}_{\mathbf{2}}) + \epsilon\mathbf{g}_{\mathbf{2}}(\mathbf{X}'_{\mathbf{1}}, \mathbf{X}_{\mathbf{2}}).\end{aligned}\right\} \tag{1}$$

In the absence of coupling (when $\epsilon = 0$) these are identical, with $\mathbf{f}$ a general nonlinear function, and the variables $\mathbf{X_i}$'s are taken to be dimensionless. The superscript $'$ denotes the fact of conjugate coupling, namely that *dissimilar* variables appear in the arguments of the coupling functions $\mathbf{g_1}$ and $\mathbf{g_2}$, and here we only consider linear $\mathbf{g}$.

In our previous work we have shown that in specific instances this form of the coupling can lead the dynamics into a regime of amplitude death, namely a situation where both the oscillators are driven to stable fixed points that may be created by the coupling. Underlying this phenomenon is the fact that conjugate variables, especially in oscillatory dynamics, are effectively similar to considering time–delay, and it is known [Reddy, Sen and Johnston, 1998] that time delay coupling can cause amplitude death (AD).

Note that complete synchronization is not likely to occur unless there are special symmetries in the functions $\mathbf{f}, \mathbf{g_1}$ and $\mathbf{g_2}$. Indeed, in the direct case, when similar variables appear in the coupling terms, the synchronization manifold is one where the effective coupling vanishes, and this is not explicitly possible here.

## 3 System

In order to present our results, we focus on the chaotic Rössler system, where the nonlinear function $\mathbf{f}$ is specified by (we use the notation $\mathbf{X_i} \equiv x_i, y_i, z_i$),

$$\left.\begin{array}{l} \dot{x}_1 = -\omega_1 y_1 - z_1 \\ \dot{y}_1 = \omega_1 x_1 + a y_1 \\ \dot{z}_1 = b + z_1(x_1 - c). \end{array}\right\} \tag{2}$$

As has been extensively demonstrated in numerous studies, over a range of the parameters $a, b,$ and $c$ the dynamics is oscillatory, and can be chaotic [Rössler, 1976]. The second system is identical, except that the variables have subscript 2. The parameters in the two systems are taken to be identical (or nearly identical). Furthermore, we mainly consider linear (diffusive) coupling functions $\mathbf{g_i}$. We have, however studied a number of different forms of the coupling and find similar behaviour (results not presented here).

## 4 Complete Synchronization and chaos death

Consider mutually coupled identical chaotic oscillators,

$$\left.\begin{array}{l} \dot{x}_{1,2} = -\omega_{1,2} y_1 - z_1 + \epsilon(y_{2,1} - x_{1,2}) \\ \dot{y}_{1,2} = \omega_{1,2} x_{1,2} + a y_{1,2} \\ \dot{z}_{1,2} = b + z_{1,2}(x_{1,2} - c). \end{array}\right\} \tag{3}$$

As explicitly shown in the inset of Fig. 1(a) the coupling term does not vanish as it does in the case of diffusive coupling with similar variables, although owing to the symmetric form, both the coupling terms $g_{1,2} = (y_{2,1} - x_{1,2})$ are indeed identical. It is therefore natural to consider the order parameter

$$\sigma = \langle|\frac{g_1}{g_2}\rangle| - 1 \tag{4}$$

(here $\langle g_i \rangle$ corresponds to time averaged value of the coupling term.) so that for CS, $\sigma = 0$ while it is nonzero otherwise. The variation of the order parameter $\sigma$ with $\epsilon$ is shown in in Fig. 1(b). Near $\epsilon \sim 0.3$ there is a reverse Hopf bifurcation that causes a loss of CS. On further increasing $\epsilon$, the system undergoes amplitude death to identical fixed points, again leading to a vanishing of $\sigma$.

A consequence of the fact that the coupling does not vanish on the synchronization manifold is that the synchronized dynamics is typically not present in the uncoupled system. Here, for instance, the two chaotic oscillators drive each other into a regime of periodic motion (as can be seen from the dynamics of the $g_i$'s in inset of Fig. 1(a), for example). In a sense this is a case of "chaos death"—the killing of chaotic variations in amplitude—similar to the suppression of all oscillations when amplitude death occurs.

Regardless of whether the coupling involves similar or conjugate variables, the synchronized solution lies on the manifold $\mathbf{M}(x_1 = x_2, y_1 = y_2, z_1 = z_2)$. With similar variables, the coupling vanishes, which means that the subsystems effectively decouple. Then if the subsystems have chaotic motion, the resulting synchronized motion will also be chaotic with the perturbations transverse to the manifold $\mathbf{M}$ being stable while the ones on $\mathbf{M}$ grow exponentially.

With conjugate variables in the coupling term, the subsystems are still interacting in the CS regime: this however requires symmetries, and as already seen, the coupling terms become identical. We can therefore consider this as an instance of a common signal (since $g_1(t) \equiv g_2(t)$) driving identical subsystems: this is effectively an instance of generalized synchronization [Abarbanel, Rulkov and Sushchik, 1996].

Although the dynamics could be either regular or chaotic, internal consistency requires that the motion be nonchaotic. The argument is as follows. For generalized synchrony, all subsystem Lyapunov exponents should be negative, but if the coupling term is chaotic, then requires that the subsystem Lyapunov exponent should be positive. This contradiction is resolved if both the coupling and the individual motions are nonchaotic; see Fig. 1(a)).

To summarize, for mutually conjugate coupled oscillators, complete synchrony becomes possible *only* with chaos suppression, leading to synchronized periodic dynamics.

## 5 Conclusion

The coupling between identical dynamical systems may naturally occur through conjugate variables and this can give rise to the regimes of synchrony. We have

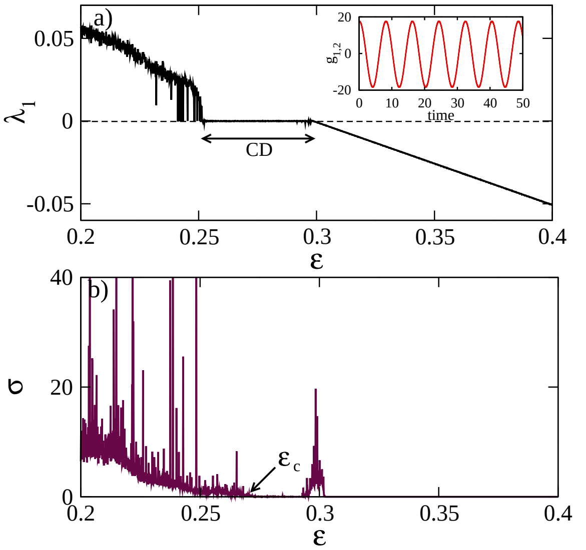


Figure 1. The largest Lyapunov exponent as a function of the coupling parameter $\epsilon$ for the coupld Rössler system is shown in (a). Order parameter, $\sigma$, as function of the coupling parameter is shown in (b). Transition in dynamics from being chaotic to periodic is clearly visible in (a) with the regime of chaos death (periodic behavior) highlighted as $CD$. The threshold for complete synchronization (where $\sigma$ vanishes) for the coupled system is marked as $\epsilon_c$ in (b). The inset figure shows the oscillations of the coupling term $g_{1,2}$ at $\epsilon = 0.28$. The oscillator parameters are $a = 0.3$, $b = 0.1$, $c = 8.5$ and $\omega_{1,2} = \omega_0 = 0.98$.

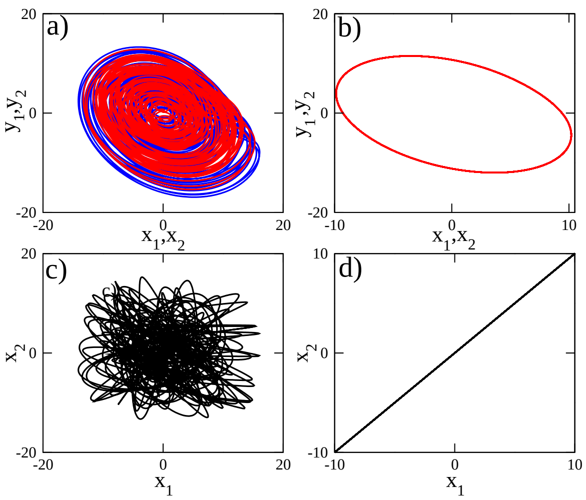


Figure 2. Chaotic and periodic dynamical regimes before and after $\epsilon_c$ are shown in (a) and (b). (c) and (d) show corresponding uncorrelated and completly synchronized dynamics. (a) and (c) are plotted for $\epsilon = 0.22$ while (b) and (d) are for $\epsilon = 0.28$

explored complete synchronization and the mechanism of chaos death in this paper.

Complete synchronization occurs here in a manner that is distinct from the situation when the coupling is in similar variables: the coupling does not vanish on the synchronization manifold, and instead each of the systems is driven to a dynamical state that cannot occur in the absence of the interaction. The only completly synchronized solution possible in conjugate coupling is necessarily periodic. When the systems are not identical phase synchronization is possible, while generalized synchrony occurs when the coupling is unidirectional.

We believe that the above results are generally applicable, and that such regimes of complete synchronization and chaos death will occur in other conjugate coupled chaotic systems. Furthermore, they hold for a range of parameter mismatch, so that the phenomena observed here are robust and should be observable in experiments.

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