TURNPIKE PROPERTY IN A PARTICULAR CLASS OF STOCHASTIC LOTKA-VOLTERRA SYSTEMS

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Abstract

We study a controlled stochastic Lotka-Volterra model for a one-predator-two-prey model with white noise. We consider an optimal control problem for this model and the stability of its solutions. We discuss a type of stability for state trajectories, optimal control and adjoint trajectories, called the Turnpike property. We assume linear growth and Lipschitz conditions in the drift and diffusion terms, prove the Turnpike property, applying the Hamiltonian formalism and the maximum principle to this stochastic control problem, and express the optimal control in terms of state and adjoint variables. Finally, we illustrate our results with an example in which we numerically solve the stochastic differential equation systems of the model.

Key words

Lotka-Volterra model, Turnpike property, stochastic optimal control, maximum stochastic principle.

1 Introduction

In mathematical ecology, the Lotka-Volterra systems represent one of the most important models to analyze population dynamics, because they describe very well many aspects of interactions between species in competition, such as persistence, extinction, and stability of its solutions, [Lotka, 1956], [Volterra, 1931], [Chapman, 1967]. These models are more realistic if we consider natural random environmental variations, introducing Wiener processes. Recently, many papers have been written about stochastic Lotka-Volterra models with white noise, studying persistence, extinction, boundedness, local, stability and more properties. In [Romero et al., 2021] we analyze the turnpike property of a stochastic controlled Lotka–Volterra system with white noise for two species. Also, in [Bao, 2012] was studied the asymptotic convergence of a general stochastic population dynamics of the type Lotka-Volterra and driven by Lévy noise, given some important asymptotic pathwise estimation assuming different conditions over the Poisson's process coefficient, but they don't consider any control functions in the processes. The Lotka-Volterra equations have also been applied to laser physics (optical and photonic devices), to describe population inversion and the number of emitted photons, as in [Aboites et al., 2022]. In this case, the authors study their regions of stability and the transformation of a fixed point into a limit cycle. The model to consider here is given by the following non-linear stochastic ordinary differential equations system with initial and final conditions:

$$\begin{aligned} dx_1 &= \eta x_1(t) - \beta x_1(t)x_2(t) - \delta x_1(t)x_3(t) \\ &\quad - A_1 x_1(t)u_1(t)dW_1(t) \\ dx_2 &= \omega x_2(t) - \beta x_2(t)x_1(t) - \epsilon x_2(t)x_3(t) \\ &\quad - A_2 x_2(t)u_2(t)dW_2(t) \\ dx_3 &= -\kappa x_3(t) + \delta x_3(t)x_1(t) + \epsilon x_3(t)x_2(t) \\ &\quad - A_3 x_3(t)u_3(t)dW_3(t), \end{aligned} \tag{1}$$

$$\begin{aligned} x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_{30}, \\ x_1(T) = x_{11}, \quad x_2(T) = x_{21}, \quad x_3(T) = x_{31}, \end{aligned} \tag{2}$$

where η, ω, κ are positive constants in $(0, 1]$, being the intrinsic growth rates of two preys and predator population, respectively, β, δ, η and ϵ in $(0, 1]$, are positive constants, meaning the contact rates per unit of time between prey-prey, predator-first prey, and predator-second prey, respectively. $u_1(t), u_2(t), u_3(t)$ are the controls, representing, by example by the hunting in each population, for which we have modulated their effect with constants $A_1, A_2, A_3 \in (0, 1]$. To take into account environmental fluctuations on the prey and predator populations, we introduce standard independent Wiener processes $W_1(t), W_2(t), W_3(t)$ with parameters $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$, respectively, in three independent random variations for each population, defined over a probability space $(\Omega, \mathcal{F}, P)$. We establish the following Stochastic Optimal Control Problem:

Stochastic Optimal Control Problem (SOCP): To find the controls $u_1(t)$, $u_2(t)$, $u_3(t)$ in system (1) with conditions (2), which minimize the following expected cost functional in the Lagrange form:

$$J(u_1, u_2, u_3) = E\Big\{\frac{1}{2}\int_0^T \sum_{i=1}^{3}\Big(x_i^2(t)+u_i^2(t)\Big)\,dt\Big\},\ a.s. \tag{3}$$

Definition 1. The control $u^*(t) = (u_1^*(t), u_2^*(t), u_3^*(t))$ associated to system (1) is said to be an optimal control if it satisfies $J(u^*(\cdot)) = \min_{u(\cdot)} J(u(\cdot))$. The corresponding state $x^*(t)$ is called the optimal state, and $(x^*(t), u^*(t))$ is called the pair optimal. Besides, consider the complete steady-state solution $\{\bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t)\}$ of system (1) with the cost functional (3), adjoint system (7) and variable $q(t)$.

Definition 2. Given a real polynomial

$$f(y) = y^n + a_1 y^{n-1} + ... + a_n. \tag{4}$$

whose zeros all lie in the left half-plane is called Hurwitz polynomial. Given the polynomial (4), let us define the corresponding matrix called Hurwitz matrix of f by:

$$H(f) = \begin{pmatrix} a_1 & a_3 & a_5 & \dots & 0 \\ 1 & a_2 & a_4 & \dots & 0 \\ 0 & . & . & \dots & 0 \\ . & . & . & \dots & 0 \\ . & . & . & \dots & 0 \\ 0 & . & . & \dots & a_n \end{pmatrix}.$$

and the determinants $\Delta_1 = a_1$, and

$$\Delta_k = \begin{pmatrix} a_1 & a_3 & a_5 & \dots & a_{2k-1} \\ 1 & a_2 & a_4 & \dots & a_{2k-2} \\ 0 & a_1 & a_3 & \dots & a_{2k-3} \\ 0 & a_0 & a_2 & \dots & a_{2k-4} \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a_k \end{pmatrix}.$$

for $k = 2, 3, \cdots n$, with $a_j = 0$ for $j > n$.

Definition 3. We will say that a Stochastic Optimal Control Problem 3 satisfies the Turnpike property iqf there exist constants C_1, and C_2 such that:

$$\begin{aligned} &E||x_T(t) - \bar{x}(t)||^2 + E||u_T(t) - \bar{u}(t)||^2 \\ &+E||p_T(t) - \bar{p}(t)||^2 \\ &\le C_2 e^{-2C_1(t-t_0)}. \end{aligned} \tag{5}$$

Some nonlinear control systems have the following property called turnpike property: the optimal trajectory, the optimal control, and the corresponding adjoint vector remain exponentially close to a steady state. The Turnpike property of a solution of an optimal control problem means that an optimal trajectory for most of the time could stay in a neighborhood of a balanced equilibrium path, corresponding to the optimal steady-state solution. This property is a characteristic of the Turnpike theory which was introduced in 1958 in mathematical economics and recently has been applied in Control Theory in [Trélat et al., 2015], [Zaslavski, 2006] and [Sun et al., 2022]. We will analyze the stability of the optimal-trajectory Turnpike property of the solutions of the stochastic controlled Lotk-Volterra model.

2 Preliminaries

Considering the stochastic differential system (1), we assume the following hypothesis related to the Lipschitz and linear growth conditions in the variable x:

(H1) There exist constants $\kappa_1 < \infty$, $\kappa_2 < \infty$, $\kappa_3 < \infty$ and $\kappa_4 < \infty$ such that $f(x, t, u)$ and $g(x, t, u)$ satisfy:

a) At most linear growth condition:

$$\begin{aligned} ||f(x, t, u)||^2 \le \kappa_1(1 + ||x||^2), \\ ||g(x, t, u)||^2 \le \kappa_2(1 + ||x||^2), \end{aligned}$$

b) Lipschitz continuity:

$$\begin{aligned} ||f(x, t, u) - f(y, t, u)||^2 \le \kappa_3||x - y||^2, \\ ||g(x, t, u) - g(y, t, u)||^2 \le \kappa_4||x - y||^2, \end{aligned}$$

(H2) Controls are bounded: there exists $\kappa_5 < \infty$, such that $\forall t \in R$: $||u(t)|| \le \kappa_5$

We are interested in exponential stability of solutions of SOCP and in a kind of stability of the controls and optimal trajectories, called turnpike property. The turnpike property of a solution in an optimal control problem means that an optimal trajectory, for most of the time could remain exponentially close to a balanced equilibrium path, corresponding to the optimal steady-state solution. To solve this SOCP, we use the stochastic maximum principle [Oksendal, et al., 2007], [Gu et al., 2016], which is the extension of the Pontryagin maximum principle corresponding to Ito diffusions (Kushner, Bismut) and jump diffusions (Oksendal). We define a generalized Hamiltonian function $H(x(t), p(t), q(t), u(t))$ associated to SOCP:

$$\begin{aligned} H = &\eta x_1(t)p_1(t) - \beta p_1(t)x_1(t)x_2(t) \\ &-\delta p_1(t)x_1(t)x_3(t) - A_1 p_1(t)x_1(t)u_1(t) \\ &+\omega p_2(t)x_2(t) - \beta p_2(t)x_2(t)x_1(t) \\ &-\epsilon p_2(t)x_2(t)x_3(t) - A_2 x_2(t)p_2(t)u_2(t) \\ &-\kappa p_3(t)x_3(t) + \delta p_3(t)x_3(t)x_1(t) \\ &+\epsilon p_3(t)x_3(t)x_2(t) - A_3 p_3(t)x_3(t)u_3(t) \\ &-\frac{1}{2}\sum_{i=1}^{3}\Big(x_i^2(t) + u_i^2(t) - 2\alpha_i q_{ii}\Big). \end{aligned} \tag{6}$$

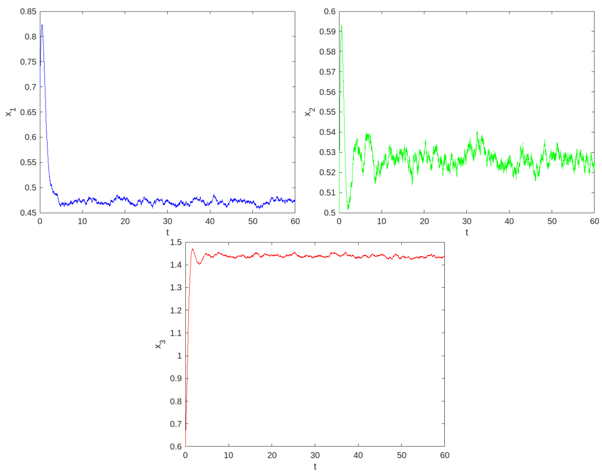

Figure 1. The limit trajectory of the optimal states $x_1(t)$, $x_2(t)$, $x_3(t)$, using the Runge-Kutta scheme.

So, the adjoint equations in the unknown process $p(t)$, corresponding to process $x(t)$, are the following backward differential stochastic equations:

$$\begin{aligned} dp_1(t) &= \Big(\eta x_1(t) - p_1(t) + \beta p_1(t)x_2(t) \\ &\quad +\delta p_1(t)x_3(t) + A_1p_1(t)u_1(t) \\ &\quad +\beta p_2(t)x_2(t) - \delta p_3(t)x_3(t)\Big)dt \\ &\quad +\textstyle\sum_{i=1}^{3} q_{1i}\, dW_1(t) \\ dp_2(t) &= \Big(\omega x_2(t) - p_2(t) + \beta p_2(t)x_1(t) \\ &\quad +\epsilon p_2(t)x_3(t) + A_2p_2(t)u_2(t) \\ &\quad +\beta p_1(t)x_1(t) - \epsilon p_3(t)x_3(t)\Big)dt \\ &\quad +\textstyle\sum_{i=1}^{3} q_{2i}\, dW_2(t) \\ dp_3(t) &= \Big(\kappa x_3(t) + p_3(t) - \delta p_3(t)x_1(t) \\ &\quad -\epsilon p_3(t)x_2(t) + A_3p_3(t)u_3(t) \\ &\quad +\delta p_1(t)x_1(t) + \epsilon p_2(t)x_2(t)\Big)dt \\ &\quad +\textstyle\sum_{i=1}^{3} q_{3i} dW_3(t), \end{aligned} \tag{7}$$

We denote $\dfrac{\partial H}{\partial u}$ by H_u, $\dfrac{\partial^2 H}{\partial u^2}$ by H_{uu}, $\cdots$ and so on for H_x, H_u, H_p, H_q, H_{ux}, H_{up}, , H_{uq}, H_{xu}, H_{xp}, H_{xq}, H_{pu}, H_{px}, H_{pq}, H_{qu}, H_{qx}, H_{qp}, H_{xx}, H_{pp}, H_{qq}.

We will now define some matrices that will play an important role in the next section.

$$\begin{aligned} A &= H_{px} - H_{pu}H_{uu}^{-1}H_{ux}, \\ R &= -H_{xx} + H_{xu}H_{uu}^{-1}H_{ux}, \\ B &= H_{pu}, \end{aligned}$$

the 6×6 matrix $Q = \begin{pmatrix} -H_{pu}H_{uu}^{-1}H_{pu}^{*} \\ -H_{px} + H_{xu}H_{uu}^{-1}H_{uq} \end{pmatrix}$, and $\hat{M} = \begin{pmatrix} A & -BH_{uu}^{-1}B^{*} \\ R & -A^{*} \end{pmatrix}$, where A^{*}, B^{*}, H_{pu}^{*} denotes transposition of A, B, H_{pu}, respectively.

3 Turnpike Property

We will start by establishing some criteria on the asymptotic stability of the equilibria of stochastic first order differential system. First, we have the Hurwitz criterion:

Theorem 1. [Routh-Hurwitz criterion [Routh, 1905]].

If all the determinants Δ_k defined above are positive, the polynomial $f(z)$ has only zeros with negative real parts, i.e. $f(z)$ is a Hurwitz polynomial. Considering the n th-order, linear equation with constant coefficients [Arnold, 1974],

$$y^n + a_1y^{n-1} + \dots + a_ny = 0, \tag{8}$$

the corresponding equation to (8) with noisy coefficients

$$Y_t^n + (a_1+\xi_1(t))Y_t^{n-1} + \dots + (a_n+\xi_n(t))Y_t = 0, \tag{9}$$

where $\xi_1(t), \cdots \xi_n(t)$ are in general correlated Gaussian white noise processes with covariance

$$E(\xi_i(t)\xi_j(s)) = S_{ij}\delta(t-s), \tag{10}$$

is rewritten as a stochastic first-order differential equation

$$dX_t^n = -\sum_{i=1}^{n} b_iX_t^{n+1-i}\, dt - \sum_{i=1}^{n}\sum_{j=1}^{n} G_{ij}X_t^{n+1-i}\, dW_t^j, \tag{11}$$

with G an $n \times n$ such that $GG' = S$.
The asymptotic stability of (11) is determined by the following theorem.

Theorem 2. [Khasminskiy criterion [Khas'minskiy, 1969]] The equilibrium position of (11) is asymptotically stable in mean square if and only if $\Delta_k > 0$, $i = 1, \cdots, n$ and $\Delta_n > \Delta/2$

$$\Delta = \begin{pmatrix} \bar{q}_{nn}^{(0)} & \bar{q}_{nn}^{(1)} & \bar{q}_{nn}^{(2)} & \dots & \bar{q}_{nn}^{n-1} \\ 1 & a_2 & a_4 & \dots & 0 \\ 0 & . & . & \dots & 0 \\ . & . & . & \dots & 0 \\ . & . & . & \dots & 0 \\ 0 & . & . & \dots & a_n \end{pmatrix}.$$

where

$$\begin{aligned} \bar{q}_{nn}^{(n-k-1)} &= \sum_{i+j=2(n-k)} \bar{S}_{ij}(-1)^{j+1} \\ \bar{S} &= SS^T. \end{aligned} \tag{12}$$

We will now present the main result of this work concerning the stability of optimal solutions of the stochastic Lotka-Volterra model, the turnpike property. The turnpike property means that the most important fact about the behavior of solutions is the optimality criterion considered and it is irrelevant the choice of time interval or the data used, for times far from the

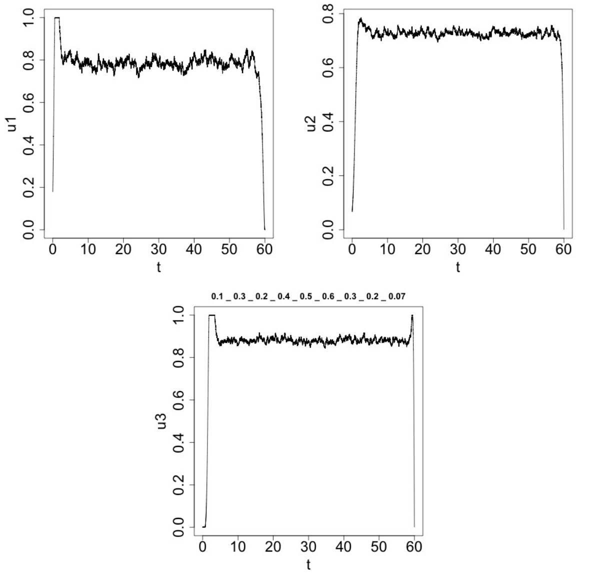

Figure 2. The limit trajectory of the optimal controls $u_1(t)$, $u_2(t)$, $u_3(t)$, using the Runge-Kutta scheme.

endpoints of the time interval.

Theorem 3. Given the steady-state solution $(\bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t))$ of system (1) with the corresponding cost functional, and the solution $\{x(t), u(t), p(t), q(t)\}$ of system (1) with the cost functional (3), if there exist the H_{uu}^{-1} matrix and if the norm of the functional matrix Q is bounded, then the solutions to SOCP satisfy the Turnpike property: there exist constants C_1, and C_2 such that:

$$
\begin{aligned}
&E||x_T(t) - \bar{x}(t)||^2 + E||u_T(t) - \bar{u}(t)||^2 \\
&+E||p_T(t) - \bar{p}(t)||^2 \\
&\leq C_2 e^{-2C_1(t-t_0)}. \qquad (13)
\end{aligned}
$$

Proof. We consider a perturbation of variables $x(t), u(t), p(t), q(t)$ around the steady-state solution $\bar{x}(t), \bar{u}(t), p(t), q(t), \bar{p}(t), \bar{q}(t)$, as follow:

$$
\begin{aligned}
x_T(t) &= \bar{x}(t) + \delta x(t), \\
u_T(t) &= \bar{u}(t) + \delta u(t), \\
p_T(t) &= \bar{p}(t) + \delta p(t), \\
q_T(t) &= \bar{q}(t) + \xi(t)\delta q(t), \qquad (14)
\end{aligned}
$$

where $\xi(t)$ is the Gaussian variable that models the white noise: $dW(t) = \xi(t)dt$ .Using the Hamiltonian perturbed and the stochastic Maximum Principle, we obtain:

$$
\begin{aligned}
\delta u(t) &= -H_{uu}^{-1}\Big(H_{ux}\delta x(t) + H_{up}\delta p(t) + H_{uq}\delta q(t)\xi(t) \\
\delta \dot{x}(t) &= \delta H_p + \delta H_q = H_{px}\delta x(t) + H_{pu}\delta u(t) \\
&\quad + H_{pp}\delta p(t) + H_{pq}\delta q(t). \qquad (15)
\end{aligned}
$$

Observing that $H_{pp} = H_{pq} = 0$ and $\delta H_q = 0$, we obtain:

$$
\begin{aligned}
\delta \dot{x}(t) = H_{px}\delta x(t) - H_{pu}H_{uu}^{-1}\Big(&H_{ux}\delta x(t) + H_{up}\delta p(t) \\
&+ H_{uq}\delta q(t)\xi(t)\Big), \qquad (16)
\end{aligned}
$$

or equivalently:

$$
\begin{aligned}
\delta \dot{x}(t) = &\big(H_{px} - H_{pu}H_{uu}^{-1}H_{ux}\big)\delta x(t) \\
&- H_{pu}H_{uu}^{-1}H_{up}\delta p(t) \\
&- H_{pu}H_{uu}^{-1}H_{uq}\delta q(t)\xi(t).
\end{aligned}
$$

Analogous for $p(t)$ we get:

$$
\begin{aligned}
\delta \dot{p}(t) &= -H_{xx}\delta x(t) - H_{xu}\delta u(t) \\
&\quad - H_{xp}\delta p(t) - H_{xq}\delta q(t) \\
&= (-H_{xx} + H_{xu}H_{uu}^{-1}H_{ux})\delta x(t) \\
&\quad + (-H_{xp} + H_{xu}H_{uu}^{-1}H_{up})\delta p(t) \\
&\quad - \Big(H_{xq} - H_{xu}H_{uu}^{-1}H_{uq}\xi(t)\Big)\,\delta q(t). \qquad (17)
\end{aligned}
$$

Now, following [Trélat et al., 2015] for our stochastic extended system, we define

$$
\begin{aligned}
Z(t) &= (\delta x(t), \delta p(t))^\top, \\
d\hat{W}(t) &= (dW(t), dW(t))^\top,
\end{aligned}
$$

where $dW(t) = (dW_1(t), dW_2(t), dW_3(t))$, $dW_i = \xi_i(t)\delta q_i(t)$ and

$$
Z_0 = \begin{pmatrix} x_T(0) - \bar{x} \\ p_T(0) - \bar{p} \end{pmatrix}.
$$

So, we may write the systems (1) and (7) with conditions (2) as follows:

$$
\begin{aligned}
dZ(t) &= \hat{M}Z(t)dt + Qd\hat{W}(t) \\
Z(0) &= Z_0. \qquad (18)
\end{aligned}
$$

Existence-and-uniqueness of solution of equation (18) is guaranteed by assumption H1 and [Oksendal, et al., 2007]. Besides, since $\hat{M}$ is a matrix time-independent [Trélat et al., 2015], the solution of system (18), is given by:

$$
Z(t) = e^{\hat{M}(t-t_0)}Z_0 + \int_{t_0}^{t} e^{\hat{M}(s-s_0)}Q\,d\hat{W}(s). \qquad (19)
$$

Now, we focus the matrix $\hat{M}$ in the deterministic part of equation (19) to apply the methods of the Riccati theory used in [Trélat et al., 2015] to found the constant C_1: considering the algebraic Riccati equation:

$$
XA + A^*X - XBH_{uu}^{-1}B^*X - R = 0, \qquad (20)
$$

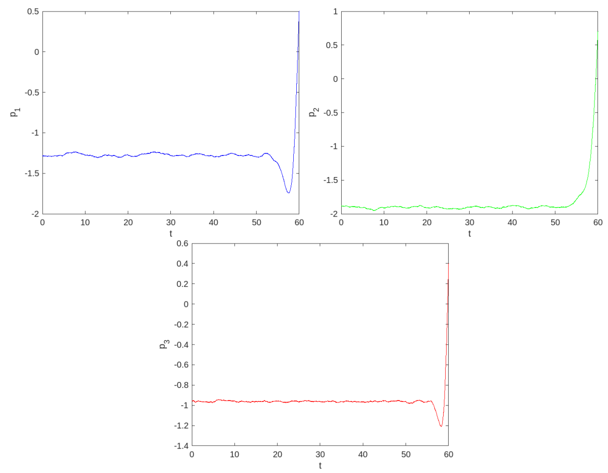

Figure 3. The limit trajectory of the optimal co-states $p_1(t)$, $p_2(t)$, $p_3(t)$, using the Runge-Kutta scheme.

and its minimal symmetric negative definite matrix solution, E_-, whose existence and uniqueness is guaranteed in Lemma 6 and Theorem 6 of [Molinari, 1977], allowing to obtain a diagonal matrix equivalent to $\hat{M}$ that satisfies equation (19), whose upper diagonal is $A + BH_{uu}^{-1}B^*E_-$ and its eigenvalues have negative real parts, . So, defining

$$C_1 = -\max\{\mu | \mu \in Spec(A + BH_{uu}^{-1}B^*E_-)\}, \quad (21)$$

$C_1 > 0$ and we have, [Sun et al., 2022]:

$$||e^{\hat{M}(t-t_0)}|| \le e^{-C_1(t-t_0)}. \quad (22)$$

By [Oksendal, et al., 2007], from (19), we apply the Schwarz inequality, Ito's isometry and we use the inequality

$$\Big|\sum_{i=1}^{n} x_i\Big|^2 \le n\sum_{i=1}^{n}|x_i|^2, \quad n \in N, \quad (23)$$

for $n = 3$, to obtain:

$$E||Z(t)||^2 \le 3(E||e^{-C_1(t-t_0)}Z_0||^2 + E||\int_{t_0}^{t} e^{-C_1(s-s_0)}Qd\hat{W}(s)||^2. \quad (24)$$

The previous inequality yields:

$$E||Z(t)||^2 \le 3\Big(e^{-2C_1(t-t_0)}||Z_0||^2 + e^{-2C_1(t-t_0)}\int_{t_0}^{t} E||Q||^2 ds\Big),$$

and, by assumptions $(H1a)$, $(H2)$ and the boundedness or matrix Q, we deduce:

$$E||Z(t)||^2 \le 3e^{-2C_1(t-t_0)}\Big(||Z_0||^2 + \int_{t_0}^{t} E||Q||^2 ds\Big) \le 3e^{-2C_1(t-t_0)}\Big(||Z_0||^2 + ||Q||^2\Big).$$

Finally, considering $||Z_0|| < \infty$ and $||Q|| < \infty$, from Gronwall inequality and setting $C_2 = 3(||Z_0||^2 + ||Q||^2)$, the inequality (25) can be rewritten in the form:

$$E||Z(t)||^2 \le C_2 e^{-2C_1(t-t_0)}, \quad (25)$$

from which the proof is complete.

4 Numerical simulations

In this section we carry out numerical simulation of the solutions of the example here presented for our model driven by white noise. Being the parameters in the equations (1) and (7) the intrinsic growth rates of two preys and predator population and the contact rates per unit of time between species and being constants in $(0, 1]$, they have been chosen for the convenience of the simulations the following values: $\eta = 1$, $\delta = 0.30$, $\beta = 1$, $\omega = 1$, $\epsilon = 0.30$, $\kappa = 1$, $A_1 = 0.4$, $A_2 = 0.2$, $A_3 = 0.5$, although the values $\eta = 0.50$, $\delta = 0.30$, $\beta = 0.20$, $\omega = 0.40$, $\epsilon = 0.30$, $\kappa = 0.70$, $A_1 = 0.4$, $A_2 = 0.40$, $A_3 = 0.19$ were also tested. Solving the steady-state system we obtain $\bar{x} = (0.47, 0.52, 1.45)$, $\bar{p} = (-1.3, -1.9, -0.9)$ and $\bar{u} = (0.25, 0.2, 0.7)$. We obtain so the Hessian matrix H_{px}, H_{pu}, H_{uu}^{-1}, H_{xx} and H_{ux}, also the matrix R, B, A and E_-. So, H_{px} is the matrix which columns are: $\begin{pmatrix} 1 - x_2 - x_3 - 0.4u_1 \\ -x_2 \\ -x_3 \end{pmatrix}$, $\begin{pmatrix} -x_1 \\ 1 - x_1 - x_3 - 0.2u_2 \\ -x_3 \end{pmatrix}$ and $\begin{pmatrix} -x_1 \\ -x_2 \\ 1 - x_1 - x_2 - 0.5u_3 \end{pmatrix}$. The other matrix are:

$$H_{pu} = \begin{pmatrix} -0.4 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.5 \end{pmatrix}, \ H_{uu}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$H_{ux} = \begin{pmatrix} -0.4p_1 & -p_1 - p_2 & -p_1 + p_2 \\ -p_1 - p_2 & -1 & -p_2 + p_3 \\ -p_1 + p_3 & -p_2 + p_3 & -1 \end{pmatrix},$$

$$H_{xx} = \begin{pmatrix} -1 & -p_1 - p_2 & -p_1 + p_2 \\ -p_1 - p_2 & -1 & -p_2 + p_3 \\ -p_1 + p_3 & -p_2 + p_3 & -1 \end{pmatrix},$$

$$R = \begin{pmatrix} 1.12 & 3.2 & 0.4 \\ 3.2 & 1.38 & -1 \\ -0.4 & -1 & 1.45 \end{pmatrix}, \ B = \begin{pmatrix} -0.4 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.5 \end{pmatrix},$$

$$A = \begin{pmatrix} -1.278 & -0.47 & -0.47 \\ -0.52 & -1.036 & -0.52 \\ -1.45 & -1.45 & -0.565 \end{pmatrix}.$$

Beside, we find the solution of Riccati equation

$$XA + A^T X - XBH_{uu}^{-1}B^T X - R = 0,$$

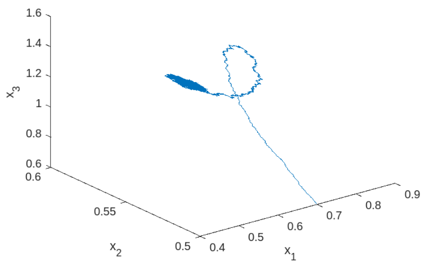

Figure 4. (x_1, x_2, x_3)-limit trajectory in the phase space, using the Runge-Kutta scheme.

using MatLab to obtain:

$$E_- = \begin{pmatrix} -0.9510 & -0.0635 & 1.4880 \\ -0.0635 & -1.9559 & 2.4749 \\ 1.4880 & 2.4749 & -3.4496 \end{pmatrix}.$$

Since

$$H_{uq} = 0, \, -H_{px} = \begin{pmatrix} -1.07 & -0.47 & -0.47 \\ -0.52 & -0.96 & -0.52 \\ -1.45 & -1.45 & -0.34 \end{pmatrix},$$

so we have

$$Q = \begin{pmatrix} -0.40 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.20 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.50 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.070 & -0.470 & -0.470 \\ 0 & 0 & 0 & -0.520 & -0.960 & -0.520 \\ 0 & 0 & 0 & -1.450 & -1.450 & -0.340 \end{pmatrix},$$

and $||Q|| = 2.5391$. Now we find the matrix $A - BH_{uu}^{-1}B^T E_-$:

$$A - BH_{uu}^{-1}B^T E_- = \begin{pmatrix} -0.9510 & -0.0635 & 1.4880 \\ -0.0635 & -1.9559 & 2.4749 \\ 1.4880 & 2.4749 & -3.4496 \end{pmatrix},$$

and its characteristic polinomial $p(\lambda)$:

$$p(\lambda) = \lambda^3 + 3.9718\lambda^2 + 3.5433\lambda + 0.7848,$$

which eigenvalues are: $\lambda_1 = -2.8104, \lambda_2 = -0.8215, \lambda_3 = -0.3399$. Since all eigenvalues are negative, the hypotheses of the Routh-Hurwitz theorem [Routh, 1905] are satisfied, so we can choose $C_2 = 2.8104$.

The Hurwitz matix associated to matrix $A - BH_{uu}^{-1}B^T E_-$ is

$$H(p) = \begin{pmatrix} 3.9718 & 0.7848 & 0 \\ 1 & 3.5433 & 0 \\ 0 & 3.9718 & 0.7848 \end{pmatrix}.$$

So, we calculate Δ_1, Δ_2, Δ_3 and Δ:

$$\Delta_1 = 3.9718,$$

$$\Delta_2 = det \begin{pmatrix} 3.9718 & 0.7848 \\ 1 & 3.5433 \end{pmatrix} = 13.28847,$$

$$\Delta_3 = det \begin{pmatrix} 3.9718 & 3.5433 & 0 \\ 1 & 3.5433 & 0 \\ 0 & 3.9718 & 0.7848 \end{pmatrix} = 10.4287983,$$

$$\Delta = det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3.5433 & 0 \\ 0 & 3.9718 & 0.7848 \end{pmatrix} = 5.7848184,$$

to verify that $\Delta_3 > \frac{\Delta}{2}$ and we can choose $C_1 = (2\Delta_n)^2$. Finally, we have verified the conclusion of the theorem.

In Fig. 1 we present the states limit trajectory $x_1(t)$, $x_2(t)$ and $x_3(t)$, using the Runge-Kutta. Also, in Fig. 2 we present the limit trajectory of the optimal controls $u_1(t)$, $u_2(t)$ and $u_3(t)$ and, in Fig. 3, the adjoint states limit trajectory $p_1(t)$, $p_2(t)$ and $p_3(t)$ are presented. It can be noted that in all cases, the optimal solution graph is divided into three parts: the start of the process in which the solution adjusts to reach the stationary equilibrium solution, once reached, it remains in a vicinity of it for a long time, until just before reaching the target, where it adjusts again to reach it. Finally, in Fig. 4, we have obtained (x_1, x_2, x_3)-limit trajectory in the phase space. To better appreciate the different white noise and jump disturbances in the phase space, we have presented the figures with each disturbance separately. We can observe that all these trajectories have the Turnpike property.

5 Conclusions

In this paper we have studied an optimal control problem applied to a controlled stochastic Lotka-Volterra model. The results of this work suggest that, in this biological system of species in competition, to reach the optimal state of prevalence, especially in long time horizons, the system tends to remain into a near-optimal equilibrium for a considerable time, going away only shortly at the beginning and end of the process spending most of its time near a optimal equilibrium. This behavior, called Turnpike property, also appears in the field of competition between other species, such as microbial or algal species. [Djema, 2021]. We have shown that its solutions exhibit the Turnpike property, by means of the Stochastic Maximum Principle. We have found expressions for the optimal controls that allowed us to solve numerically the coupled stochastic differential equations in the state variable and the adjoint variable. Using the Stochastic Maximum Principle approach, we have been able to verify that the studied model presents the Turnpike property in the controls, the states and the co-states.

We have showed by means of an example that the Turnpike property is indeed satisfied and we have performed a simulation of the results. Finally, comparing the results of this controlled stochastic Lotka-Volterra model in randomly disturbed environments, with the deterministic model also studied previously, we can conclude that this model is more realistic and that still the Turnpike property, despite containing disturbances due to white noise. Since during the simulations we noticed a high sensitivity of the results to the values of the parameters associated with white noise, it would be desirable to carry out in future research the stochastic sensitivity function technique, as in [Kolinichenko, 2024], [Bashkirtseva, 2023]. It is important to provide methods that allow measurement of parameters noise effects and prediction of system behavior, involving stochastic sensitivity functions.

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