

ON ALMOST SURE ASYMPTOTIC STABILITY OF LINEAR STOCHASTIC SYSTEMS WITH MULTIPLICATIVE DISTURBANCES AND RANDOM COEFFICIENTS

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Abstract

This paper considers the behavior of solutions to systems of linear stochastic differential equations with multiplicative disturbances and random coefficients. We present conditions for almost sure exponential stability in terms of integrated-time averages of logarithmic matrix norms. The study also analyzes instability conditions. To complete the analysis, we investigate the pathwise asymptotic boundedness and unboundedness of solutions. We then illustrate this approach by applying it to a two-dimensional cooperative-competitive system with sine-Wiener noise perturbations.

Key words

linear stochastic system, multiplicative disturbances, random coefficients, almost sure stability, sine-Wiener process

1 Introduction

Consider a linear stochastic system with multiplicative disturbances and random time-varying coefficients. Assume that on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, we are given an n -dimensional stochastic process X_t , $t \geq 0$, with the dynamics

$$dX_t = A_t X_t dt + \sum_{i=1}^r C_t^{(i)} X_t dw_{it}, \quad X_0 = x, \quad (1)$$

where x is a non-random initial state, and w_t , $t \geq 0$, is an r -dimensional standard Wiener process with components w_{it} , $t \geq 0$, $i = 1, \dots, r$. The terms A_t and $C_t^{(i)}$ denote random time-varying matrices with $\mathcal{F}_t$ -adapted and integrable entries, satisfying $\int_0^t (\|A_s\| +$

$\sum_{i=1}^r \|C_s^{(i)}\|^2) ds < \infty$ for all $t \geq 0$ almost surely (a.s.), where $\|\cdot\|$ represents the Frobenius matrix norm, which is equivalent to the Euclidean vector norm of the vectorized matrix. Under these conditions, equation (1) admits a unique strong solution (see [Tang, 2003]).

If $C_t^{(i)} \equiv 0$ for all $i = 1, \dots, r$, then (1) reduces to a random ordinary differential equation (RODE) (see [Arnold, 2006; Caraballo and Han, 2017; Ladde and Lakshmikantham, 1980a; Neckel and Rupp, 2013]) used to describe random interactions and coupling [Aghajani and Touri, 2022; Feng et al., 2006; Cure and Neri, 2023; Ladde, 1977a]. Otherwise, it specifies a stochastic differential equation (SDE) for an Itô process. When A_t and C_t are diagonal matrices, the system represents geometric Brownian motion (GBM) type returns and volatility in economics and finance [Föllmer, 1994; Shen and Wei, 2016; Rieder and Wopperer, 2012]. In the multivariate case, this setup extends to various applications in engineering and physics, including complex chemical reactions [Mondal and Bag, 2015], stationary SDOF models [Huntington and Lyrintzis, 1996], time-space structures [Guo et al., 2014; Gao et al., 2011], and electrical circuits [Farnoosh et al., 2011]. Furthermore, setting $A_t = \rho_t A$ and $C_t = \sqrt{\rho_t} C$ with $\rho_t \geq 0$ introduces a stochastic time-scale (random clock) (see [Kobayashi, 2011]). Linear system frameworks can also facilitate the analysis of nonlinear systems, such as those featuring isolated subsystems [Ladde and Lakshmikantham, 1980b]. Specifically, (1) covers certain classes of stochastic bilinear systems, as demonstrated below.

We investigate the long-run behavior of solutions, focusing on convergence, boundedness, or growth. In particular, we aim to establish conditions for an exponential convergence rate. Equation (1) belongs to a class of

stochastic processes driven by semimartingales, which can be seen by writing $dX_t = \mathcal{B}(X)dZ_t$ with the semimartingale $Z_t = \text{vec} \left(\int_0^t A_s ds + \sum_{i=1}^r C_s^{(i)} dw_{is} \right)$, where $\mathcal{B}(X)$ is a block matrix and $\text{vec}(\cdot)$ denotes the vectorization operation. When studying such processes, it is standard to assume various ergodicity properties [Mohammed and Scheutzw, 1996], uniform boundedness of coefficients [Li and Yu, 2025], or constraints ensuring that integrated time averages remain bounded away from zero [Mao, 1991, Section 3].

Due to the random nature of the coefficients, an approach based on time-varying Lyapunov functions would require analyzing an infinite-horizon backward stochastic differential equation (BSDE) and evaluating its solution with an unknown adjoint process. For arbitrary processes A_t and $C_t^{(i)}$, this framework does not yield explicit stability conditions. However, employing the standard quadratic Lyapunov function $V(X) = X'X$ (where $'$ denotes the transpose) alongside the logarithmic matrix norm directly accounts for the specific features of the coefficients and provides verifiable conditions.

The remainder of the paper is organized as follows. Section 2 introduces the necessary definitions and presents the main theorems regarding the stability and instability conditions of the solution to (1). Section 3 provides an illustrative example analyzing a bivariate cooperative-competitive system under sine-Wiener noise perturbations. Section 4 applies the developed theoretical framework to the stability analysis of a variable stiffness robotic joint, demonstrating the practical engineering implications of the derived sufficient condition. Finally, Section 5 formulates the conclusions.

2 Main Results

In this section, we present our main results concerning the asymptotic behavior of solutions to (1).

2.1 Almost Sure Exponential Stability

The following definition is adapted from Khasmiskii [Khasmiskii, 2011, Section 5.8]. Let $X_t^x, t \geq 0$, be a solution to (1) with the initial state $X_0 = x$.

Definition 1. *The solution X_t is said to be almost surely (a.s.) exponentially stable if there exists a constant $\kappa > 0$ such that*

$$\|X_t^x\| \leq \kappa_0 \exp(-\kappa t) \quad \text{a.s. for all } t \geq 0,$$

where $\kappa_0 = \kappa(x)$ is an a.s. finite random variable. Conversely, if there exists a constant $\kappa > 0$ such that

$$\|X_t^x\| \geq \kappa_0 \exp(\kappa t) \quad \text{a.s. for all } t \geq 0,$$

where $\kappa_0 = \kappa(x) > 0$ a.s. and $x \neq 0$, then X_t is called a.s. exponentially unstable.

Definition 2. [Ladde, 1977b] *Let $\mathcal{A}_t$ be a square matrix of size n with $\mathcal{F}_t$ -adapted and integrable entries on $[0, t]$*

for $t \geq 0$, and let $\|\cdot\|_{\text{ind}}$ denote an induced (subordinated) matrix norm. Then, the logarithmic norm $\mu(\mathcal{A}_t)$ is defined as

$$\mu(\mathcal{A}_t) = \lim_{h \rightarrow 0^+} \frac{\|I + h\mathcal{A}_t\|_{\text{ind}} - 1}{h} \quad \text{a.s.}, \quad (2)$$

where I is the identity matrix satisfying $\|I\|_{\text{ind}} = 1$.

The correspondence between various matrix norms and their associated logarithmic matrix norms is given below (see [Ladde, 1977b; Soderlind, 2024]):

$$\begin{aligned} \|\mathcal{A}_t\|_2 &= \lambda_{\max} \left(\sqrt{\mathcal{A}_t' \mathcal{A}_t} \right), & \mu_2(\mathcal{A}_t) &= \lambda_{\max} \left(\frac{\mathcal{A}_t + \mathcal{A}_t'}{2} \right); \\ \|\mathcal{A}_t\|_1 &= \max_j \sum_{i=1}^n |a_{ij,t}|, & \mu_1(\mathcal{A}_t) &= \max_j \left(a_{jj,t} + \sum_{i \neq j} |a_{ij,t}| \right); \\ \|\mathcal{A}_t\|_\infty &= \max_i \sum_{j=1}^n |a_{ij,t}|, & \mu_\infty(\mathcal{A}_t) &= \max_i \left(a_{ii,t} + \sum_{j \neq i} |a_{ij,t}| \right), \end{aligned}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix, and $|\cdot|$ represents the absolute value.

As noted above, in the absence of multiplicative noise, one can employ conditions on $\lambda_{\max}(\mathcal{A}_t)$ or $\mu(\mathcal{A}_t)$ to study the asymptotic behavior of solutions to (1) viewed as a RODE (see [Arnold, 2006; Ladde and Lakshmikantham, 1980a; Siegert, 2008]). In the SDE case, we apply a transformation via the quadratic Lyapunov function $V(X) = X'X$ to obtain a scalar linear SDE, which allows us to formulate stability conditions for a symmetric time-varying matrix $\mathcal{A}_t$.

Set

$$\mathcal{A}_t = A_t + A_t' + \sum_{i=1}^r (C_t^{(i)})' C_t^{(i)} \quad (3)$$

Theorem 1. *If*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mu(\mathcal{A}_s) ds < 0 \quad \text{a.s.}, \quad (4)$$

where $\mathcal{A}_t$ is given by (3), then the solution X_t to (1) is almost surely exponentially stable.

Proof. Let $y_t = X_t' X_t$ for $t \geq 0$ with $X_0 = x$. Applying Itô's formula yields

$$\begin{aligned} d(X_t' X_t) &= X_t' \left(A_t + A_t' + \sum_{i=1}^r (C_t^{(i)})' C_t^{(i)} \right) X_t dt \\ &\quad + 2 \sum_{i=1}^r X_t' C_t^{(i)} X_t dw_{it}, \end{aligned}$$

which can be rewritten as the following scalar linear stochastic differential equation:

$$dy_t = a_t y_t dt + 2y_t \left(\sum_{i=1}^r \sigma_{it} dw_{it} \right),$$

where the time-varying coefficients a_t and σ_{it} are defined as

$$a_t = \frac{X'_t \left(A_t + A'_t + \sum_{i=1}^r (C_t^{(i)})' C_t^{(i)} \right) X_t}{X'_t X_t},$$

$$\sigma_{it} = \frac{X'_t C_t^{(i)} X_t}{X'_t X_t}, \quad i = 1, \dots, r.$$

Then, the solution to (5) can be explicitly written as

$$y_t = y_0 \exp \left\{ \int_0^t a_v dv + 2 \sum_{i=1}^r \int_0^t \sigma_{iv} dw_{iv} - 2 \sum_{i=1}^r \int_0^t \sigma_{iv}^2 dv \right\}. \quad (5)$$

Since $a_t = \frac{X'_t \mathcal{A}_t X_t}{X'_t X_t}$ and the matrix $\mathcal{A}_t$ is symmetric, it follows that $a_t \leq \lambda_{\max}(\mathcal{A}_t)$. Moreover, the inequality $\lambda_{\max}(\mathcal{A}_t) \leq \mu(\mathcal{A}_t)$ holds (see [Ladde, 1977b]), which implies

$$\int_0^t a_v dv \leq \int_0^t \mu(\mathcal{A}_v) dv. \quad (6)$$

For each $i = 1, \dots, r$, let $M_{it} = 2 \int_0^t \sigma_{iv} dw_{iv}$ denote a local martingale, with its quadratic variation given by $\langle M_i \rangle_t = 4 \int_0^t \sigma_{iv}^2 dv$. The stochastic integral in (5) can then be expressed as $M_{it} - \frac{1}{2} \langle M_i \rangle_t$. Applying the convergence theorem for local martingales (see [Liptser and Shiriyayev, 2012, Section 2.7]), we obtain

$$\limsup_{t \rightarrow \infty} \left(M_{it}(\omega) - \frac{1}{2} \langle M_i \rangle_t(\omega) \right) \leq \chi(\omega) \quad \text{a.s.}, \quad (7)$$

where $\chi(\omega) = 0$ if $\langle M_i \rangle_\infty(\omega) = \infty$, and $\chi(\omega)$ is an a.s. finite random variable if $\langle M_i \rangle_\infty(\omega) < \infty$ for $\omega \in \Omega$. Consequently, the sum in the exponent of (5) is bounded from above by an a.s. finite random variable, meaning that the asymptotic behavior of y_t is governed by the state-independent bound (6). It follows that the condition presented in Theorem 1 is sufficient to ensure the convergence $y_t \rightarrow 0$ a.s. as $t \rightarrow \infty$ at an exponential rate. Returning to the original state variables via $\|X_t\| = \sqrt{y_t}$, we conclude that the solution X_t is almost surely exponentially stable.

To obtain the a.s. boundedness of X_t as $t \rightarrow \infty$, we require a weaker condition than that in Theorem 1. From (5)–(7) it immediately follows that:

Corollary 1. *If $\mu^* = \limsup_{t \rightarrow \infty} \int_0^t \mu(\mathcal{A}_s) ds \leq 0$ a.s., then $\limsup_{t \rightarrow \infty} \|X_t\|^2 \leq \bar{c}$ a.s., where $\bar{c} \geq 0$ is an a.s. finite random variable. Moreover, for $\mu_* = -\infty$, we have $\limsup_{t \rightarrow \infty} \|X_t\|^2 = 0$ a.s.*

2.2 Almost Sure Exponential Instability

The study of conditions for a.s. exponential instability is associated with the analysis of the adjoint process for (5).

Let $\epsilon > 0$ be arbitrarily small. Introduce

$$\begin{aligned} \tilde{\mathcal{A}}_t = & -A'_t - A_t \\ & + \left(\frac{1}{2} + \epsilon \right) \sum_{i=1}^r \left((C_t^{(i)})'^2 + (C_t^{(i)})^2 \right) \\ & + \frac{1}{2} \sum_{i=1}^r \left((C_t^{(i)} C_t^{(i)})' - (C_t^{(i)})' C_t^{(i)} \right) \\ & + \epsilon \sum_{i=1}^r \left((C_t^{(i)})' C_t^{(i)} + (C_t^{(i)}) C_t^{(i)'} \right). \end{aligned} \quad (8)$$

Theorem 2. *If*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mu(\tilde{\mathcal{A}}_s) ds < 0 \quad \text{a.s.}, \quad (9)$$

where $\tilde{\mathcal{A}}_t$ is given by (8), then X_t is exponentially unstable a.s. Moreover, if

$$c^* = \limsup_{t \rightarrow \infty} \frac{\int_0^t \sum_{i=1}^r \|C_s^{(i)}\|^2 ds}{t} < \infty \quad \text{a.s.}, \quad (10)$$

then the assertion holds for $\epsilon = 0$ in (8).

Proof. Following [Dragan et al., 2006, Section 1], the process $\tilde{y}_t$ is defined as an adjoint process for (5) if $y_t \tilde{y}_t = 1$. This relation implies that the almost sure (a.s.) exponential stability of $\tilde{y}_t$ guarantees the exponential instability of y_t . Using (5), we obtain

$$\begin{aligned} \tilde{y}_t = \tilde{y}_0 \exp \left\{ - \int_0^t a_v dv - 2 \sum_{i=1}^r \int_0^t \sigma_{iv} dw_{iv} \right. \\ \left. + 2 \sum_{i=1}^r \int_0^t \sigma_{iv}^2 dv \right\}, \end{aligned} \quad (11)$$

where $\tilde{y}_0 = 1/y_0$.

For a chosen small $\epsilon_0 > 0$, we can decompose the exponent in (11) as follows:

$$\begin{aligned} & - \int_0^t a_v dv - 2 \sum_{i=1}^r \int_0^t \sigma_{iv} dw_{iv} + 2 \sum_{i=1}^r \int_0^t \sigma_{iv}^2 dv \\ & = - \int_0^t a_v dv + (2 + \epsilon_0) \sum_{i=1}^r \int_0^t \sigma_{iv}^2 dv \\ & \quad - 2 \sum_{i=1}^r \int_0^t \sigma_{iv} dw_{iv} - \epsilon_0 \sum_{i=1}^r \int_0^t \sigma_{iv}^2 dv \\ & = - \int_0^t a_v dv + (2 + \epsilon_0) \sum_{i=1}^r \int_0^t \sigma_{iv}^2 dv \\ & \quad + \sum_{i=1}^r (2M_{it} - \epsilon_0 \langle M_i \rangle_t), \end{aligned}$$

where $M_{it} = -\int_0^t \sigma_{iv} dw_{iv}$ are local martingales and $\langle M_i \rangle_t = \int_0^t \sigma_{iv}^2 dv$ are their quadratic variations. By the limit theorem for local martingales, the term $\sum_{i=1}^r (M_{it} - (\epsilon_0/2)\langle M_i \rangle_t)$ is bounded from above a.s. on $[0, \infty)$ for any $\epsilon_0 > 0$. Thus, to ensure the a.s. exponential stability of $\tilde{y}_t$, it is sufficient to satisfy

$$\limsup_{t \rightarrow \infty} \frac{-\int_0^t a_v dv + (2 + \epsilon_0) \sum_{i=1}^r \int_0^t \sigma_{iv}^2 dv}{t} < 0 \quad \text{a.s.}$$

Recall that the terms σ_{it}^2 can be written via the symmetric part of the matrices:

$$\sigma_{it}^2 = \left(\frac{X_t' C_t^{(i)} X_t}{X_t' X_t} \right)^2 = \frac{1}{4} \left(\frac{X_t' ((C_t^{(i)})' + C_t^{(i)}) X_t}{X_t' X_t} \right)^2.$$

Applying the Cauchy-Schwarz inequality for quadratic forms yields the upper bound:

$$\sigma_{it}^2 \leq \frac{1}{4} \frac{X_t' ((C_t^{(i)})' + C_t^{(i)})^2 X_t}{X_t' X_t}.$$

Setting $\epsilon = \epsilon_0/4$, we combine $-a_t$ and the bounded sum of σ_{it}^2 :

$$\begin{aligned} -a_t + (2 + \epsilon_0) \sum_{i=1}^r \sigma_{it}^2 &\leq \frac{1}{X_t' X_t} X_t' \left\{ -A_t' - A_t - \sum_{i=1}^r (C_t^{(i)})' C_t^{(i)} \right. \\ &\quad \left. + \left(\frac{1}{2} + \epsilon \right) \sum_{i=1}^r ((C_t^{(i)})' + C_t^{(i)})^2 \right\} X_t. \end{aligned}$$

As established previously, expanding the square $((C_t^{(i)})' + C_t^{(i)})^2$ and subtracting $\sum_{i=1}^r (C_t^{(i)})' C_t^{(i)}$ leads directly to the definition of $\tilde{A}_t$ in (8). Therefore, we obtain:

$$-a_t + (2 + \epsilon_0) \sum_{i=1}^r \sigma_{it}^2 \leq \frac{X_t' \tilde{A}_t X_t}{X_t' X_t} \leq \mu(\tilde{A}_t). \quad (12)$$

Integrating this inequality and applying condition (9) directly implies the a.s. exponential stability of $\tilde{y}_t$. Furthermore, if condition (10) holds, we analyze the limit as $\epsilon \rightarrow 0$. Set $\epsilon = 0$ in (8). Let $-\kappa^* < 0$ denote the corresponding limit in (9). From the uniform boundedness guaranteed by $c^* < \infty$, we obtain:

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{-\int_0^t a_v dv + 2 \sum_{i=1}^r \int_0^t \sigma_{iv}^2 dv}{t} &\leq \limsup_{t \rightarrow \infty} \frac{\int_0^t \{\mu(\tilde{A}_s)|_{\epsilon=0} + 2\epsilon \sum_{i=1}^r \|C_s^{(i)}\|^2\} ds}{t} \\ &\leq -\kappa^* + 2\epsilon c^*. \end{aligned}$$

Since $\epsilon > 0$ can be chosen arbitrarily small such that $-\kappa^* + 2\epsilon c^* < 0$, the stability property holds for $\epsilon = 0$ in (8). Given the inverse relationship between the stability of $\tilde{y}_t$ and the stability of y_t , the claim is proven.

Applying the fundamental inequality for logarithmic matrix norms, $\mu(\tilde{A}_t) + \mu(-\tilde{A}_t) \geq 0$, inequality (9) implies to the following:

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mu \left[(A_t' + A_t) \right. \\ - \left(\frac{1}{2} + \epsilon \right) \sum_{i=1}^r ((C_t^{(i)})' + C_t^{(i)})^2 \\ - \frac{1}{2} ((C_t^{(i)})' C_t^{(i)} + (C_t^{(i)} C_t^{(i)})') \\ \left. - \epsilon \sum_{i=1}^r ((C_t^{(i)})' C_t^{(i)} + (C_t^{(i)} C_t^{(i)})') \right] ds > 0 \end{aligned} \quad (13)$$

almost surely. The condition (13) shows that, if instability arises, then the diffusion matrix C_t cannot counteract the impact of the potentially unstable state matrix A_t , thus avoiding the so-called “stabilization by noise” possibility. Combining the definition of the adjoint process (11), estimate (12), Corollary 1, and the reciprocal relation $\limsup_{t \rightarrow \infty} (1/\|X\|_t^2) = (\liminf_{t \rightarrow \infty} \|X\|_t^2)^{-1}$, we obtain the following:

Corollary 2. *If $\tilde{\mu}^* = \limsup_{t \rightarrow \infty} \int_0^t \mu(\tilde{A}_s) ds \leq 0$ a.s., then $\liminf_{t \rightarrow \infty} \|X_t\|^2 \geq \tilde{c}$ a.s., where $\tilde{c} > 0$ is an a.s. finite random variable. Moreover, for $\mu_* = -\infty$, we have $\liminf_{t \rightarrow \infty} \|X_t\|^2 = +\infty$ a.s.*

3 Linear bivariate stochastic system under sine-Wiener noise perturbations

This section analyzes a linear bivariate system subject to sine-Wiener noise perturbations across all coefficients. As a prominent class of bounded noise, sine-Wiener processes are widely utilized to model phenomena across the natural sciences and physical applications [D’Onofrio, 2013]. Notably, this framework is frequently employed to capture rate uncertainty [Guo et al., 2019; Wang et al., 2019; Lacitignola and Martiradonna, 2025]. While various properties of sine-Wiener noise were studied in [Wang et al., 2021], its incorporation into multivariate equations was subsequently investigated in [Ping, 2023]. Our primary focus is to derive stability and instability conditions using Theorems 1 and 2 from Section 1.

Let $n = 2$ and $r = 1$ in (1), and assume that

$$\begin{aligned} A_t &= \begin{pmatrix} a_{11} & a_{12} \sin(W_t) \\ a_{21} \sin(W_t) & a_{22} \end{pmatrix}, \\ C_t &= \begin{pmatrix} 0 & \sigma_1 \sin(W_t) \\ \sigma_2 \sin(W_t) & 0 \end{pmatrix}, \end{aligned}$$

where $C_t^{(1)} = C_t$ and $w_{1t} = w_t$. Here, $W_t, t \geq 0$, is an $\mathcal{F}_t$ -Wiener process independent of $\{w_t, t \geq 0\}$, and a_{ij}, σ_i are non-random constants for $i, j = 1, 2$ such that $\sigma_1 \sigma_2 \neq 0$.

In component-wise form, the system is written as

$$\begin{aligned} dX_{1t} &= a_{11}X_{1t}dt + a_{12}\sin(W_t)X_{2t}dt \\ &\quad + \sigma_1\sin(W_t)X_{2t}dw_t, \\ dX_{2t} &= a_{21}\sin(W_t)X_{1t}dt + a_{22}X_{2t}dt \\ &\quad + \sigma_2\sin(W_t)X_{1t}dw_t, \end{aligned} \quad (14)$$

with initial states $X_{10} = x_1$ and $X_{20} = x_2$. Equations (14) place a special emphasis on modeling interaction uncertainty. Specifically, consider the terms

$$(a_{12} + \sigma_1\xi_t)\sin(W_t) \quad \text{and} \quad (a_{21} + \sigma_2\xi_t)\sin(W_t),$$

where ξ_t represents Gaussian white noise.

The first source of noise reflects parametric uncertainty in the interaction rates, incorporating $a_{12} + \sigma_1\xi_t$ and $a_{21} + \sigma_2\xi_t$, which leads to multiplicative disturbances in (14). The second source, the bounded sine-Wiener noise, acts as an environmental perturbation that strengthens or weakens the coupling effects. Both coupling coefficients are random and time-varying:

$$\hat{a}_{12,t} = a_{12}\sin(W_t), \quad \hat{a}_{21,t} = a_{21}\sin(W_t).$$

For a given sample path $\omega \in \Omega$ and time t , if $\hat{a}_{ij,t}(\omega) > 0$ ($i \neq j$), cooperative behavior occurs; if $\hat{a}_{ij,t}(\omega) < 0$, it models competition; and if $\hat{a}_{12,t}(\omega) \cdot \hat{a}_{21,t}(\omega) < 0$, it characterizes a predator-prey relationship.

Also note that (14) corresponds to a reduction of a 4-dimensional bilinear stochastic system. Let Y_t be the state, where $Y_{1t} = X_{1t}$, $Y_{2t} = X_{2t}$, $Y_{3t} = \sin(W_t)$, $Y_{4t} = \cos(W_t)$. Then,

$$\begin{aligned} dY_{1t} &= a_{11}Y_{1t}dt + a_{12}Y_{2t}Y_{3t}dt + \sigma_1Y_{2t}Y_{3t}dw_t, \\ dY_{2t} &= a_{21}Y_{1t}Y_{3t}dt + a_{22}Y_{2t}dt + \sigma_2Y_{1t}Y_{3t}dw_t, \\ dY_{3t} &= -\frac{1}{2}Y_{3t}dt + Y_{4t}dW_t, \\ dY_{4t} &= -\frac{1}{2}Y_{4t}dt - Y_{3t}dW_t, \end{aligned}$$

with $Y_{10} = x_1$, $Y_{20} = x_2$, $Y_{30} = 0$, and $Y_{40} = 1$.

We first apply Theorem 1 to study the conditions on a_{ij} and σ_i that guarantee exponential stability a.s. For the system (14), the matrix $\mathcal{A}_t = A_t + A'_t + C'_tC_t$ takes the form

$$\mathcal{A}_t = \begin{pmatrix} 2a_{11} + \sigma_2^2 \sin^2(W_t) & (a_{12} + a_{21}) \sin(W_t) \\ (a_{12} + a_{21}) \sin(W_t) & 2a_{22} + \sigma_1^2 \sin^2(W_t) \end{pmatrix}.$$

To evaluate the stability condition (4), we find the logarithmic norm $\mu(\mathcal{A}_t)$, which can be estimated as

$$\begin{aligned} \mu(\mathcal{A}_t) &\leq \max(2a_{11} + \sigma_2^2 \sin^2(W_t), \\ &\quad 2a_{22} + \sigma_1^2 \sin^2(W_t)) + |a_{12} + a_{21}| |\sin(W_t)| \\ &\leq \max(2a_{11} + \sigma_2^2, 2a_{22} + \sigma_1^2) \\ &\quad + |a_{12} + a_{21}| |\sin(W_t)|. \end{aligned} \quad (15)$$

Next, we integrate inequality (15) from 0 to t and divide by t :

$$\begin{aligned} \frac{1}{t} \int_0^t \mu(\mathcal{A}_s) ds &\leq \max(2a_{11} + \sigma_2^2, 2a_{22} + \sigma_1^2) \\ &\quad + |a_{12} + a_{21}| \frac{1}{t} \int_0^t |\sin(W_s)| ds. \end{aligned} \quad (16)$$

Utilizing the ergodic properties of the sine-Wiener process, the asymptotic behavior of the integral in (16) can be explicitly evaluated using its invariant measure, i.e., $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\sin(W_s)| ds = \frac{2}{\pi}$ a.s. Indeed, since $W_t, t \geq 0$ is a standard Wiener process, its modulo 2π wrapped distribution converges to the uniform distribution on $[-\pi, \pi]$ as $t \rightarrow \infty$ (see [Mardia and Jupp, 2000, Section 3.5.3]). Consequently, the limit follows from the ergodic theorem for diffusion processes [Revuz and Yor, 1999, Section 10.3].

Taking the limit superior as $t \rightarrow \infty$ on both sides of (16), we arrive at:

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mu(\mathcal{A}_s) ds &\leq \max(2a_{11} + \sigma_2^2, 2a_{22} + \sigma_1^2) \\ &\quad + \frac{2}{\pi} |a_{12} + a_{21}| \quad \text{a.s.} \end{aligned}$$

According to Theorem 1, the system (14) is exponentially stable a.s. if the right-hand side of the integrated limit is strictly negative. Therefore, a sufficient condition for the almost sure exponential stability is given by:

$$\max(2a_{11} + \sigma_2^2, 2a_{22} + \sigma_1^2) + \frac{2}{\pi} |a_{12} + a_{21}| < 0. \quad (17)$$

The robustness of the system to noise intensities σ_1, σ_2 can be investigated using Theorem 2. In this case, setting $\epsilon = 0$, the matrix $\tilde{\mathcal{A}}_t$ takes the form:

$$\tilde{\mathcal{A}}_t = \begin{pmatrix} -2a_{11} + (\sigma_1\sigma_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)) \sin^2(W_t) & -(a_{12} + a_{21}) \sin(W_t) \\ -(a_{12} + a_{21}) \sin(W_t) & -2a_{22} + (\sigma_1\sigma_2 + \frac{1}{2}(\sigma_2^2 - \sigma_1^2)) \sin^2(W_t) \end{pmatrix}.$$

Note that we are justified in setting $\epsilon = 0$ in (8) since the condition (10) holds. Specifically, using the Frobenius norm of C_t , we find:

$$\begin{aligned} c^* &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|C_s\|^2 ds \\ &= (\sigma_1^2 + \sigma_2^2) \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sin^2(W_s) ds \\ &= \frac{\sigma_1^2 + \sigma_2^2}{2} < \infty \quad \text{a.s.} \end{aligned}$$

Furthermore, we can rewrite the diagonal trigonometric terms using the identity:

$$\sin^2 W_t = \frac{1}{2}(1 - \cos(2W_t)).$$

Notice that $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \cos(2W_s) ds = 0$ a.s. due to the ergodicity of the functional $\cos(2W_t)$ defined on the wrapped Brownian motion.

Arguing similarly to the previous stability analysis, it can be shown that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mu(\tilde{A}_s) ds \\ \leq -2 \min(a_{11}, a_{22}) + \frac{\sigma_1 \sigma_2}{2} + \frac{1}{4} |\sigma_1^2 - \sigma_2^2| \\ + \frac{2}{\pi} |a_{12} + a_{21}| \end{aligned}$$

almost surely.

Therefore, a sufficient condition for a.s. exponential instability is given by

$$\begin{aligned} -2 \min(a_{11}, a_{22}) + \frac{\sigma_1 \sigma_2}{2} \\ + \frac{1}{4} |\sigma_1^2 - \sigma_2^2| + \frac{2}{\pi} |a_{12} + a_{21}| < 0. \end{aligned} \quad (18)$$

In particular, this implies that if $\sigma_1 \sigma_2 < 0$ and the magnitudes of σ_1 and σ_2 are close to each other, then for sufficiently large $|\sigma_1|$ and $|\sigma_2|$, the multiplicative disturbances destabilize the system. Conversely, for $\sigma_1 \sigma_2 > 0$ and large noise intensities σ_1 and σ_2 , it is clear that neither the stability nor the instability condition holds. However, the system may still remain stable, as shown below by considering particular cases of (14) where $A_t \equiv 0$.

Example 1: Consider the case where the diffusion coefficients have opposite signs. Equations (14) can be rewritten as

$$dX_t = \sin(W_t) \tilde{C}^{(a)} X_t dw_t, \quad (19)$$

where

$$\tilde{C}^{(a)} = \begin{pmatrix} 0 & -\tilde{\sigma}_1 \\ \tilde{\sigma}_2 & 0 \end{pmatrix}.$$

The condition (18) suggests exponential instability if $|\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2| < 2\tilde{\sigma}_1 \tilde{\sigma}_2$. Moreover, this instability arises for any $\tilde{\sigma}_1, \tilde{\sigma}_2 > 0$. Indeed, the solution of (19) admits a closed-form estimate. By setting the scaling factor $k = \frac{\tilde{\sigma}_1}{\tilde{\sigma}_2} > 0$, we obtain

$$d(X_{1t}^2 + kX_{2t}^2) = \sin^2(W_t)(X_{1t}^2 + kX_{2t}^2)dt,$$

which yields

$$\begin{aligned} X_{1t}^2 + kX_{2t}^2 &= \exp\left\{\int_0^t \sin^2(W_v) dv\right\} (X_{10}^2 + kX_{20}^2) \\ &= \exp\left\{\frac{1}{2}t - \frac{1}{2}\int_0^t \cos(2W_v) dv\right\} \\ &\quad \times (X_{10}^2 + kX_{20}^2). \end{aligned}$$

This confirms the statement of Theorem 2.

Example 2: Next, consider the case:

$$dX_t = \sin(W_t) \tilde{C}^{(b)} X_t dw_t, \quad (20)$$

where

$$\tilde{C}^{(b)} = \begin{pmatrix} 0 & \tilde{\sigma}_1 \\ \tilde{\sigma}_2 & 0 \end{pmatrix}.$$

In this example, neither requirement (17) nor (18) is met, as $\max(\tilde{\sigma}_1^2, \tilde{\sigma}_2^2) > 0$ and $\tilde{\sigma}_1 \tilde{\sigma}_2 > 0$.

Let $k = \sqrt{\frac{\tilde{\sigma}_1}{\tilde{\sigma}_2}} > 0$. Then we have

$$\begin{aligned} d(kX_{1t}) &= k\tilde{\sigma}_1 X_{2t} \sin(W_t) dw_t, \\ dX_{2t} &= \frac{\tilde{\sigma}_2}{k} (kX_{1t}) \sin(W_t) dw_t, \end{aligned}$$

where $k\tilde{\sigma}_1 = \frac{\tilde{\sigma}_2}{k} = \sqrt{\tilde{\sigma}_1 \tilde{\sigma}_2} = \sigma$. Thus, system (20) can be rewritten as

$$d \begin{pmatrix} kX_{1t} \\ X_{2t} \end{pmatrix} = \sigma \sin(W_t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} kX_{1t} \\ X_{2t} \end{pmatrix} dw_t. \quad (21)$$

Equation (21) involves a symmetric matrix, which is orthogonally diagonalizable:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P, \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Setting $\tilde{X}_t = (kX_{1t}, X_{2t})'$ and $\tilde{Y}_t = P' \tilde{X}_t$, we obtain

$$d\tilde{Y}_t = \sigma \sin(W_t) D \tilde{Y}_t dw_t,$$

which yields the closed-form solution:

$$\tilde{Y}_t = \mathcal{M}_t \exp\left(-\sigma^2/2 \int_0^t \sin^2(W_v) dv\right) Y_0,$$

where

$$\mathcal{M}_t = \begin{pmatrix} m_t & 0 \\ 0 & m_t^{-1} \end{pmatrix}, \quad m_t = \exp\left(\sigma \int_0^t \sin(W_v) dw_v\right).$$

Consequently, the solution in the original coordinates is given by the formula $\tilde{X}_t = P \tilde{Y}_t \exp\{-(\sigma^2/2) \int_0^t \sin^2(W_v) dv\} \mathcal{M}_t P' \tilde{X}_0$.

From this, it follows that

$$\begin{aligned} \|X_t\| &\leq c(k) \exp\left(-\frac{\sigma^2}{2} \int_0^t \sin^2(W_v) dv\right) \\ &\quad \times \|P\|^2 \|\mathcal{M}_t\| \|X_0\| \end{aligned}$$

which simplifies to

$$\|X_t\| \leq c(k) \exp\left(-\frac{\sigma^2}{4}t + \frac{\sigma^2}{4} \int_0^t \cos(2W_v) dv\right) \times \|P\|^2 \|\mathcal{M}_t\| \|x\|.$$

Taking into account that $\mathcal{M}_t$ is a diagonal matrix with martingale entries m_t and $1/m_t$ that have linearly bounded quadratic variations, we arrive at the estimate $\|X_t\| \leq \kappa_0(x) \exp(-\kappa t)$ as $t \rightarrow \infty$ a.s., where $0 < \kappa < \sigma^2/4$. Thus, the process X_t from Example 2 is exponentially stable a.s. for any $\bar{\sigma}_1, \bar{\sigma}_2 > 0$.

4 Application: Stability Analysis of a Variable Stiffness Robotic Joint

Consider a single-degree-of-freedom (SDOF) mechanical system describing the rotational dynamics of a single-joint robotic manipulator under stochastic perturbations:

$$m\ddot{\theta}(t) + \left(c_0 + \sigma_c \xi_t^{(1)}\right) \dot{\theta}(t) + (k_0 + \xi_t^{(2)})^2 \theta(t) = 0$$

where $\theta(t)$ represents the joint angular position, $\dot{\theta}(t)$ is the angular velocity, and m denotes the mass moment of inertia of the robotic link.

This formulation models a modern Variable Stiffness Actuator (VSA) subject to two distinct types of physical disturbances:

1. The term $\xi_t^{(1)}$ represents a high-frequency white noise process in the sensory feedback loop. When the digital controller applies active software damping to stabilize the arm, the measurement errors from the encoders induce a multiplicative noise proportional to the velocity $\dot{\theta}(t)$.
2. The term $\xi_t^{(2)}$ represents an internal colored noise process, modeled as a zero-mean Ornstein-Uhlenbeck (OU) process with stationary variance $\sigma_{st}^2 = \sigma_{ou}^2/2\theta_{ou}$: $d\xi_t^{(2)} = -\theta_{ou}\xi_t^{(2)}dt + \sigma_{ou}dW_t$, $\xi_0^{(2)} = 0$. It captures the physical inertia and thermal/pressure fluctuations within the VSA actuation mechanism. Due to the geometric and structural properties of mechanical VSAs, the physical restoring torque scales quadratically with this internal activation parameter, leading to the squared expression $(k_0 + \xi_t^{(2)})^2$.

Expanding the quadratic stiffness term as $-(k_0 + \xi_t^{(2)})^2 = -k_0^2 - 2k_0\xi_t^{(2)} - (\xi_t^{(2)})^2$ and introducing the state vector $Y_t = (\theta, \dot{\theta})'$, the scalar SDOF equation can be transformed into a two-dimensional linear stochastic differential equation in the Itô sense:

$$dY_t = \left(A + \Delta A_t\right)Y_t dt + CY_t dw_t$$

where w_t is a standard scalar Wiener process, and the structural matrices are defined as:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k_0^2}{m} & -\frac{c_0}{m} \end{pmatrix}, \quad \Delta A_t = \begin{pmatrix} 0 & 0 \\ -\kappa_t & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_c}{m} \end{pmatrix}$$

Here, $\kappa_t = \frac{2k_0\xi_t^{(2)} + (\xi_t^{(2)})^2}{m}$ encapsulates the time-varying parametric stiffness disturbances.

To satisfy the requirements of Theorem 1 and eliminate the positive eigenvalues inherent in the standard Euclidean matrix drift of SDOF mechanical models, it is necessary to perform a coordinate transformation to a stable Lyapunov-weighted space via $X_t = P^{1/2}Y_t$, where the symmetric positive-definite matrix $P > 0$ solves the deterministic core equation:

$$A'P + PA + C'PC + I = 0$$

Solving this system of linear algebraic equations yields the exact components of the matrix P :

$$\begin{aligned} p_{12} &= \frac{m}{2k_0^2}, \\ p_{22} &= \frac{m^2(m + k_0^2)}{k_0^2(2c_0m - \sigma_c^2)}, \\ p_{11} &= \frac{c_0}{2k_0^2} + \frac{m(m + k_0^2)}{2c_0m - \sigma_c^2} \end{aligned}$$

By Sylvester's criterion, the existence of a valid positive-definite solution $P > 0$ strictly requires $2c_0m > \sigma_c^2$, which matches the standard physical condition for the mean-square stabilizability of the joint under multiplicative sensory noise.

In the transformed coordinate system X_t , the full auxiliary system matrix defined in the condition of Theorem 1 takes the form $\bar{\mathcal{A}}_t = \bar{A} + \bar{A}' + \Delta\bar{A}_t + \bar{\Delta}\bar{A}_t' + \bar{C}'\bar{C}$, where

$$\begin{aligned} \bar{A} &= P^{1/2}AP^{-1/2}, \\ \Delta\bar{A}_t &= P^{1/2}\Delta A_tP^{-1/2}, \quad \bar{C} = P^{1/2}CP^{-1/2}. \end{aligned}$$

Evaluating the spectral logarithmic norm $\mu_2(\bar{\mathcal{A}}_t)$ in the original state space via the generalized Rayleigh quotient yields the following sharp unapproximated inequality:

$$\mu_2(\bar{\mathcal{A}}_t) \leq -\lambda_{\min}(P^{-1}) + \frac{p_{22}}{\sqrt{\det(P)}}|\kappa_t|$$

Applying the triangle inequality to isolate the distinct components of the stiffness noise κ_t , we establish the upper bound:

$$\begin{aligned} \mu_2(\bar{\mathcal{A}}_t) &\leq -\lambda_{\min}(P^{-1}) + \frac{2k_0p_{22}}{m\sqrt{\det(P)}}|\xi_t^{(2)}| \\ &\quad + \frac{p_{22}}{m\sqrt{\det(P)}}(\xi_t^{(2)})^2 \end{aligned}$$

To employ Theorem 1, we evaluate the integrated-time averages of the logarithmic norm on an infinite horizon.

Utilizing the ergodicity of the stationary OU process, the time integrals of $|\xi_t^{(2)}|$ and $(\xi_t^{(2)})^2$ converge almost surely to their spatial mathematical expectations under the invariant Gaussian measure $\mathcal{N}(0, \sigma_{st}^2)$:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\xi_s^{(2)}| ds = \sqrt{\frac{2}{\pi}} \sigma_{st} \quad \text{a.s.},$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (\xi_s^{(2)})^2 ds = \sigma_{st}^2 \quad \text{a.s.}$$

Substituting these limits into the stability condition (4), the final verifiable parametric condition for the almost sure exponential stability of the robotic joint is formulated as:

$$\frac{p_{22}}{m \sqrt{\det(P)}} \left(2k_0 \sqrt{\frac{2}{\pi}} \sigma_{st} + \sigma_{st}^2 \right) < \lambda_{\min}(P^{-1}) \quad (22)$$

The derived parametric inequality (22) establishes a fundamental balance between the internal actuation disturbances and the noise-modified dissipative capacity of the robotic joint, revealing a non-trivial algebraic mechanism under bounded noise constraints:

1. Bounded Noise Limits and Singularity Avoidance:

A crucial feature of the derived criterion is the behavior of the geometric transmission multiplier $\frac{p_{22}}{\sqrt{\det(P)}}$. Since the sensory white noise intensity is strictly bounded by its physical upper limit ($\sigma_c^2 \leq \sigma_{\max}^2 < 2c_0 m$), the system is guaranteed to operate safely away from the mean-square stabilizability singularity. Within this well-defined bounded domain, the elements of the Lyapunov matrix P are strictly contained, ensuring that the geometric transmission factor remains non-singular and strictly bounded. This prevents arbitrary amplification of the parametric stiffness variations through the noise transmission channel.

2. Preservation of the Effective Dissipation Rate:

Because the sensory noise intensity is locked below the critical threshold, the spectral expansion of P is strictly constrained. The right-hand side stability margin $\lambda_{\min}(P^{-1}) = 1/\lambda_{\max}(P)$, which represents the effective "dissipation rate" of the closed-loop system, maintains a strictly positive lower bound. This ensures that the deterministic active damping gain c_0 provided by the controller is always sufficient to maintain a robust energy dissipation channel.

3. Spectral Balance of Bounded Multi-Noise Dynamics:

The final criterion explicitly underscores a nested multi-noise interaction: to guarantee almost sure exponential stability, the active software damping c_0 must provide a sufficient safety envelope to suppress the combined linear and quadratic components of the colored OU process energy ($2k_0 \sqrt{\frac{2}{\pi}} \sigma_{st} + \sigma_{st}^2$), scaled by the non-singular geometric factor of the robotic joint.

From a practical design perspective, the stationary variance of the internal actuator noise, $\sigma_{st}^2 = \sigma_{ou}^2 / 2\theta_{ou}$, serves as a key optimization target for the robotic control loops. Engineers can actively suppress this destabilizing factor through two independent technological channels:

1. **Hardware Noise Reduction (σ_{ou}):** This approach minimizes the primary source intensity by utilizing high-precision zero-backlash harmonic drives to eliminate cogging torques, introducing active low-pass power filters to stabilize motor currents, or expanding pneumatic receiver capacities to dampen internal pressure fluctuations within the VSA.
2. **Algorithmic Bandwidth Enhancement (θ_{ou}):** Alternatively, the correlation time of the colored noise can be significantly reduced by increasing the loop execution frequency of the digital controller (e.g., from 100 Hz to 2 kHz). Implementing a high-gain localized inner feedback loop to command the actuator's internal parameters forces a much faster reversion to the mean. By driving $\theta_{ou} \rightarrow \infty$, the state-dependent deviations are rapidly suppressed, asymptotically attenuating the stationary variance ($\sigma_{st}^2 \rightarrow 0$). This effectively limits the cumulative energy injected by the parametric disturbances and forces the joint dynamics to remain within the safe bounds defined by the final stability inequality.

The primary constructive value of the derived sufficient stability condition lies in its direct application to the analytical design and optimization of diverse dynamical processes, ranging from electrical circuits and chemical kinetics to physical oscillators, quadcopters, and active robotic controllers, aligning with recent advances in non-stationary and delayed mechanical systems [Imangazieva, 2024]. In many practical frameworks, the baseline parameters – such as damping coefficients, loop gains, or chemical reaction rates – function as adjustable or software-defined inputs under severe environmental uncertainty. For higher-dimensional systems with simpler noise profiles, such as quadcopters, this framework provides a scalable analytical baseline to handle increased state variables. Consequently, the explicit algebraic inequality serves as a rigorous, hardware-verifiable synthesis tool that naturally complements modern robust control strategies developed for systems with high stochastic sensitivity and potential multi-stability phenomena [Kolinichenko, 2024].

By utilizing this sharp analytical boundary, engineers and physicists can directly compute safe operating envelopes and parameter configurations without relying on empirical trial-and-error adjustments or computationally expensive real-time Monte Carlo rollouts. Furthermore, this scalar inequality provides an ideal terminal constraint for advanced predictive optimization algorithms, such as Stochastic Model Predictive Control (SMPC), matching recent robust frameworks for investigating complex physical media and pressure variations

subject to parametric fluctuations [Ankilov et al., 2024]. Embedding this algebraic envelope directly into finite-horizon decision-making routines guarantees infinite-horizon almost sure asymptotic convergence, successfully bridging the gap between localized stochastic modeling and global closed-loop stability across various complex physical and engineering systems.

5 Conclusions

We investigated the asymptotic behavior of solutions to a linear SDE with multiplicative noise and random coefficients. From this analysis, we derived sufficient conditions for exponential stability and instability, expressed via the asymptotic integrated averages of logarithmic norms for auxiliary matrices (Section 1). These auxiliary matrices, which incorporate the entries of the state and diffusion matrices, arise directly from constructing a Lyapunov function in the form of the squared norm of the state vector. Furthermore, examining a two-dimensional system with sine-Wiener perturbed coefficients (Section 3) highlighted the impact of the signs of noise volatilities.

To bridge the gap between abstract theory and engineering applications, Section 4 established the methodological framework for applying these results to practical systems, such as variable stiffness robotic joints and higher-dimensional quadcopter models under multi-noise environments. This analysis conceptually demonstrated how the derived sufficient stability condition serves as an analytical synthesis tool to define safe operating boundaries for industrial PD loops and Stochastic Model Predictive Control (SMPC) without requiring online numerical computations. The dynamics demonstrated in the examples reflect the fact that the conditions in Theorems 1 and 2 are strictly sufficient. Applying these stability tests is especially justified in engineering and physical systems where closed-form analytical solutions are unavailable.

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