SPEED GRADIENT CONTROL OVER QUBIT STATES

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Abstract

We discuss the model of a quantum bit driven by an external classical field without decay in the rotating wave approximation. In such a model, the whole evolution of the qubit states takes place on the Bloch sphere. We reformulate the model as a unitless set of real ordinary differential equations and use the normalized external field as a feedback control parameter.

The closed-loop algorithm is designed in the form of the speed gradient, driving the dynamical system towards minimizing a given nonnegative goal function expressed via the qubit variables. We investigate the achievability of the control goal, and focus on the most important features of the speed gradient algorithm applied to a quantum system in comparison with classical systems.

Our approach is valid for the control over the ground and excited population levels, and over the qubit phase variables.

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Key words

Semi-classical model, rotating-wave approximation, Bloch sphere, speed gradient feedback, Fradkov-Pogromsky's theorem.

1 Introduction

The fast development of technologies based on quantum bit (qubit) applications [Chae et al., 2024] created a great demand for theoretical formulation and experimental realization of control approaches to engineer and drive efficiently the qubit states [Werninghaus, 2024]. The modern experimental setup is already advanced enough to perform virtually different existing control algorithms [Niknam et al., 2022; Kuzmanović et al., 2024; Réglade et al., 2024].

The energy and closely related properties of qubits can be efficiently controlled by optimal [Rebentrost et al., 2009; Wang and Belavkin, 2012] and gradient suboptimal [Pechen and Borisenok, 2015; Pechen et al., 2022] feedback algorithms. Nevertheless, the application of control feedback methods developed mostly for classical systems has its own specific features while we study quantum systems.

Here we discuss the model of qubit driven by an external classical field without decay in the rotating wave approximation. In such an approach, the whole evolution of the qubit states takes place on the surface of the Bloch sphere.

In Section 2 we reformulate the model as a unitless set of real ordinary differential equations, and use the normalized external field as a feedback control parameter. The closed-loop algorithm is designed in the form of speed gradient feedback [Fradkov, 2007], driving the dynamical system towards minimizing a given nonnegative goal function expressed via the qubit variables.

Then in Section 3 we investigate the achievability of the control goal, and focus on the most important features of the speed gradient algorithm applied to a quantum system in comparison with classical systems.

2 Control Model for Qubit

To formulate the control model, we use the rotating-wave approximation originated in quantum optics and magnetic resonance [Scully and Zubairy, 1997]. It neglects the rapidly oscillating terms in a Hamiltonian, and it is focused on the near-resonant transitions and the low intensities of external driving fields.

Then we develop the speed gradient (SG) feedback approach [Fradkov, 2007] as an efficient suboptimal algorithm to engineer qubit states [Pechen and Borisenok, 2015; Pechen et al., 2022].

2.1 The Rotating-Wave Approximation for Qubit

In the absence of decay in the quantum dynamical system, the qubit preserves its evolution on the surface of the Bloch sphere:

$$x^2 + y^2 + z^2 = 1\,, \tag{1}$$

with the set of equations in the unitless form [Pechen and Borisenok, 2015]:

$$\begin{aligned} \dot{x}(t) &= u(t)z(t)\,; \\ \dot{y}(t) &= z(t)\,; \\ \dot{z}(t) &= -y(t) - u(t)x(t)\,. \end{aligned} \tag{2}$$

where x, y, z are real variables expressed via the density matrix complex elements as:

$$\begin{aligned} x &= \rho_{22} - \rho_{11}\,; \\ y &= \rho_{12}e^{i\omega t} + \rho_{21}e^{-i\omega t}\,; \\ z &= i[\rho_{12}e^{i\omega t} - \rho_{21}e^{-i\omega t}]\,, \end{aligned} \tag{3}$$

and the energy difference between the two levels is:

$$\omega = \frac{E_2 - E_1}{\hbar}\,. \tag{4}$$

The function $u(t)$ stands for the unitless external classical field driving the qubit. The time variable in (2) is also normalized by ω to a unitless form.

2.2 Bloch Sphere Formulation

The time evolution of x, y, z is constrained by (1), which allows us to reduce the number of variables.

Let's reformulate (2) in the polar coordinates (r, θ, ϕ):

$$\begin{aligned} x &= r\cos\theta\,; \\ y &= r\sin\theta\cos\phi\,; \\ z &= r\sin\theta\sin\phi\,, \end{aligned} \tag{5}$$

such that:

$$\begin{aligned} \dot{\theta}(t) &= -u(t)\sin\phi(t)\,; \\ \dot{\phi}(t) &= -1 - \frac{u(t)\cos(\phi(t))}{\tan(\theta(t))}. \end{aligned} \tag{6}$$

The third variable $r = 1$. That reflects the fact that the whole dynamical evolution of the qubit system takes place on the surface of the Bloch sphere.

3 Stabilization of Qubit States

It has been demonstrated that the speed gradient algorithm can be efficiently applied for different types of control to classical [Andrievsky and Guzenko, 2014] and quantum [Borisenok, 2021] systems, both for purposes of stabilization and tracking. Here we use it to stabilize the state of qubit for the parameters ϕ and θ.

3.1 Speed Gradient Feedback

The goal of control is to *stabilize* the qubit state at some desired level θ_* and ϕ_*. To design the SG algorithm, let's choose the non-negative goal function:

$$G(t) = \frac{1}{2}\left[\phi(t) + t - \phi_*\right]^2 + \frac{1}{2}\left[\theta(t) - \theta_*\right]^2\,. \tag{7}$$

The explicit time term t in RHS(7) stands for the compensation of eigen-revolution of the system for the coordinate ϕ, see the term -1 in the second equation (6).

To minimize the goal function (7), the control signal field u in SG is defined as follows:

$$u = -\Gamma\frac{\partial}{\partial u}\left[\frac{dG(t)}{dt}\right]\,, \tag{8}$$

with the positive constant: $\Gamma > 0$. The gradient for the one dimensional control signal is represented as the partial derivative in RHS(8).

By (6) and (7), we get:

$$\begin{aligned} \frac{dG}{dt} = -u\cdot\Bigg[\frac{(\phi + t - \phi_*)\cos\phi}{\tan\theta} &+ \\ +(\theta - \theta_*)\sin\phi\Bigg]\,, \end{aligned} \tag{9}$$

and the SG control signal (8) takes the form:

$$u = \Gamma\left[\frac{(\phi + t - \phi_*)\cos\phi}{\tan(\theta)} + (\theta - \theta_*)\sin\phi\right]\,. \tag{10}$$

Now we can use (10) as the external field to design the target qubit states.

3.2 Numerical Simulations

Now we check numerically whether the algorithm (10) allows us to achieve the control goal (7) for an arbitrary initial state of the dynamical system.

First, let's demonstrate in Fig.1 the successful stabilization of the qubit state for the target variables $\phi_* = 0.38$, $\theta_* = 0.55$, and $\Gamma = 2.4$.

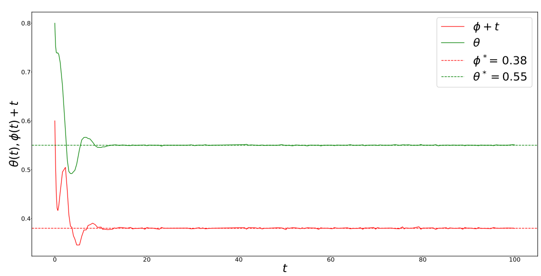

Figure 1. SG stabilization of the qubit state.

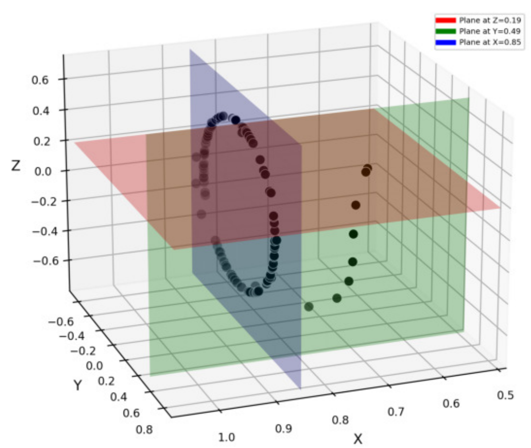

Figure 2. Dynamics of the controlled qubit in (x, y, z).

Figure 1 demonstrates that SG can stabilize both variables, and the typical time of stabilization is defined by $1/\Gamma$.

In the Cartesian coordinates (x, y, z), the stabilization takes as the superposition with the rotational component on the (x, z)-plane due to the term t in the goal function definition, see Fig.2.

The numerical parameters for Fig.2 are the same as for Fig.1.

3.3 Some Features of the Controlled Qubit Dynamics

Nevertheless, for some initial conditions and target values, it can be difficult to reach the goal for one of the coordinates, as shown in Fig.3. The graph displays the following parameter values: $\phi_* = 0.38$, $\theta_* = 0$, $\Gamma = 2.4$, and the initial values: $\phi(0) = 0.6$ and $\theta(0) = 0.8$.

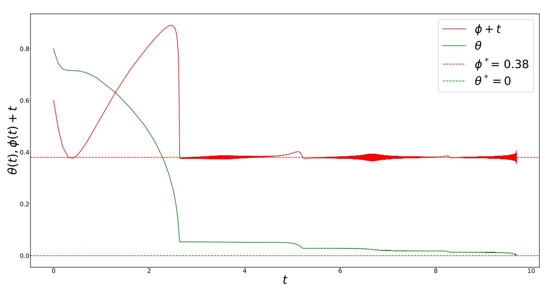

Figure 3. SG fails to stabilize the qubit state.

Figure 3 demonstrates that in some situations the control algorithm cannot drive the system towards the target values. Instead of it, the controlled variable becomes 'locked' at the plane performing the rotational motion without stabilization.

3.4 Fradkov-Pogromsky's Theorem

The reason that for some initial conditions and goal functions SG cannot achieve the control goal is the fact that quantum systems may not satisfy the set of conditions of Fradkov-Pogromsky's theorem [Fradkov and Pogromsky, 1998]:

1. *The regularity condition*: RHS(6) and the partial derivative $\partial(dG/dt)\partial u$ are bounded on any bounded set of the variables ϕ, θ, and u.
2. *The convexity condition*: dG/dt is convex in u.
3. *The achievability condition*: There exist a real u_* and scalar uniformly continuous in each bounded region non-negative function $\rho(\phi, \theta)$; $\rho(0) \equiv 0$ such that:

$$\rho(\phi, \theta) + \frac{dG(\phi, \theta, u_*)}{dt} \leq 0$$

for all ϕ, θ.

4. *The boundedness condition*: If the function G is bounded, then RHS(6) are bounded as well. If the conditions 1.-4. are valid, SG algorithm (10) can achieve the control goal (7) for an arbitrary set of the initial conditions.

In the general case, the quantum system (6) does not satisfy the achievability condition of Fradkov-Pogromsky's theorem, as one can easily see from (9) for the regularity and convexity conditions.

Thus, the 'classical' formulation of the SG algorithm must be sufficiently modified.

4 Discussions and Conclusion

In principle, the speed gradient approach developed here is valid for the control over the ground and excited population levels, and over the qubit phase variables.

Speed gradient control over qubit state variables gives a way for the effective feedback algorithm to engineer different characteristics of quantum systems. It can be used in quantum computations to design quantum logic gates, master the initial qubit states, and manipulate their energy parameters. In perspective, our approach can be extended to multi-qubit cases.

Yet, the numerical analysis of the qubit under the speed gradient control demonstrates that for some initial conditions and target values, it can be difficult to reach the goal for one of the coordinates. Such features are not typical for the application of the gradient control algorithms to classical systems.

The solution could be a significant reformulation of the algorithm and its application to the elements of the density matrix. At the same time, the simplicity of use, the independence of achieving the goal from the initial conditions, as well as the low computational cost will remain the main advantages of the speed gradient method.

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