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ASYMPTOTIC METHOD APPLIED TO THE LOCALIZATION OF VIBRATIONS IN A WEAKENED COLUMN

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Abstract. *This paper is devoted to the dynamic analysis of two-connected beam-columns with a variation of the bending connection and minor perturbations of the length of each span. The point of reduced bending stiffness represented by a rotational spring may result from a crack. This rotational spring can also be associated to semi-rigid connection in the field of steel or composite structures for instance. Dynamics of this axially loaded two-span weakened column appears to exhibit strong localization for small values of flexibility of the rotational spring. The vibration mode shapes indicate a strong confinement of the vibration level to a fraction of the column. A quantitative criterion of localization is established and is correlated to well known phenomena such as curve veering effect or close eigenvalues. Such a result is quite encouraging as localization is strongly associated to the flexibility values of the rotational spring. When considering the open crack analogy, localization only appears for severely damaged* column*. It can then be understood as an indicator of the damage level of the global structure.*

Keywords: Asymptotic method, Localization, Buckling, Vibration, Crack, Semi-rigid connection.

1. INTRODUCTION

Localization can be understood as the concentration of a representative parameter (generally the strain or the displacement) in a part of the structure. Such a phenomenon has to be well understood in order to avoid (or control) structural collapse. Mode localization, whereby a particular vibration mode (or buckling mode) may be confined to a limited region of the structure has been noticed in elastic structures (see the bibliographical study of El Naschie, 2000, or the one of Bendiksen, 2000). Indeed, localization phenomenon has been found in many elastic stability (and dynamics) problems of continuous structures, such as local buckling and overall buckling of thin-walled members or buckling (and dynamics) of repetitive structures in presence of irregularities (Pierre et al, 1987; Pierre and Plaut, 1989; Zingales and Elishakoff, 2000). Localization in a multi-span elastic column may indeed appear by slightly perturbing the length of each column in presence of additional external springs. Localization means that one span vibrates with a larger amplitude than the other spans.

In this paper, localization in a multi-span axially loaded elastic column is considered again, with the introduction of an internal bending connection located at an intermediate support. The fundamental analysis is restricted to a two-span weakened column although similar results are expected for the more general multi-span column. The length of each span is slightly perturbed, in order to break the symmetry of the initial structural model. The buckling problem was already treated by Challamel et al (2006). The point of reduced bending stiffness represented by a rotational spring can be associated to semi-rigid connection (Gurfinkel and Robinson, 1965; Plaut and Yang, 1995 or Wang et al, 2004 for instance), with particular applications in the field of steel or composite structures for instance. Such a junction may also describe a stiffened human limb joint or a robotic arm joint (Wang et al, 2004). The rotational spring may also result from a crack occurring at the intermediate support. For both analogies (semi-rigid connection or crack model), it is hoped that asymptotic results will reveal the nature of the singular coupling between material property of the connection, and geometrical property of the two-connected weakened column (with the breaking of symmetry of the structural model).

When considering the open crack analogy, it will be shown that localization appears for severely damaged columns. Buckling or dynamics of cracked elastic structures has been the subject of numerous investigations. Dimarogonas (1996) gave a state of the art review of methods developed to analyse cracked structures. The crack may be modelled by a simplified elastic rotational spring, whose flexibility can be easily identified for the case of one-sided crack (see for instance, Dimarogonas, 1996). The equivalent flexibility depends on the depth of the crack and on the height of the cross section of the beam. A lot of papers have been devoted to the vibrations of cracked structural components. Recently, Binici (2005) investigates the effect of axial force on the vibration of beams with multiple open cracks. Drewko and Hien (2005) study the eigenvalue sensitivity associated to a cracked beam. The link between localization and vibration of weakened columns was not studied in these previous works.

2. STRUCTURAL MODEL

A two-span column is considered in Figure 1 with length l and constant bending stiffness EI. It is assumed that the rotational spring, whose flexibility is denoted by k, is located at the intermediate support. It is recalled that this rotational spring may model a one-sided crack, or more generally semi-rigid connection in civil engineering. The column is subjected to axial compressive load P. The uniform mass per unit of length is denoted by $\overline{m}$.

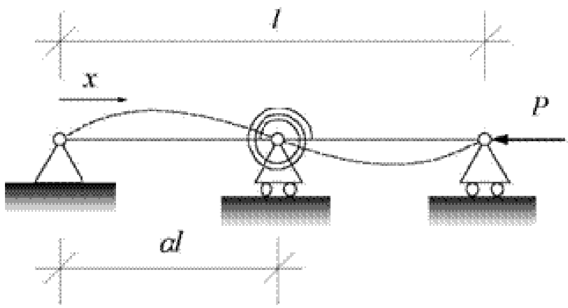

Figure 1 – The physical system

Equations of the free bending vibrations of this Euler-Bernoulli beam are given by:

$$\left| \begin{array}{l} x \in [0;\alpha l] : EI\dfrac{\partial^4 y_1}{\partial x^4} + P\dfrac{\partial^2 y_1}{\partial x^2} + \overline{m}\dfrac{\partial^2 y_1}{\partial t^2} = 0 \\ x \in [\alpha l; l] : EI\dfrac{\partial^4 y_2}{\partial x^4} + P\dfrac{\partial^2 y_2}{\partial x^2} + \overline{m}\dfrac{\partial^2 y_2}{\partial t^2} = 0 \end{array} \right. \quad (1)$$

where $y_i \,(i \in \{1,2\})$ is the transverse deflection of each span, function of the time t and the spatial co-ordinate x. α is a dimensionless parameter which characterises the position of the intermediate support $(\alpha \in [0;1])$. The solution of Eq. (1) is sought of the form:

$$y_i(x,t) = w_i(x)\sin(\omega t) \quad (2)$$

Substituting Eq. (2) into Eq. (1) yields the differential equation for modal displacements:

$$\left| \begin{array}{l} x \in [0;\alpha l] : EI\dfrac{d^4 w_1}{dx^4} + P\dfrac{d^2 w_1}{dx^2} - \overline{m}\omega^2 w_1 = 0 \\ x \in [\alpha l; l] : EI\dfrac{d^4 w_2}{dx^4} + P\dfrac{d^2 w_2}{dx^2} - \overline{m}\omega^2 w_2 = 0 \end{array} \right. \quad (3)$$

The boundary conditions associated to each outer support are:

$$\left| \begin{array}{l} w_1(0) = 0 \\ \dfrac{d^2 w_1}{dx^2}(0) = 0 \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} w_2(l) = 0 \\ \dfrac{d^2 w_2}{dx^2}(l) = 0 \end{array} \right. \quad (4)$$

The deflection is vanishing at the intermediate support:

$$\left| \begin{array}{l} w_1(\alpha l) = 0 \\ w_2(\alpha l) = 0 \end{array} \right. \quad (5)$$

Finally, the last conditions express the continuity of the bending moment at the intermediate support, and the crack (or connection) constitutive behavior:

$$\left| \begin{array}{l} EI\dfrac{d^2 w_1}{dx^2}(\alpha l) = EI\dfrac{d^2 w_2}{dx^2}(\alpha l) \\ EI\dfrac{d^2 w_1}{dx^2}(\alpha l) = k\left[\dfrac{dw_2}{dx}(\alpha l) - \dfrac{dw_1}{dx}(\alpha l)\right] \end{array} \right. \quad (6)$$

Let us denote the fundamental differences between the present model, and the model studied by Pierre (1988), or Pierre and Plaut (1989), who introduced an external spring, whereas the model developed in the paper considers an internal spring. The following dimensionless parameters are introduced:

$$\xi = \frac{x}{l} \quad ; \; \lambda^2 = \omega l^2 \sqrt{\frac{m}{EI}} \; ; \; \chi^2 = \frac{Pl^2}{EI} \; \text{ and } \; \gamma = \frac{kl}{EI} \tag{7}$$

$()'$ denotes the derivative with respect to the dimensionless spatial coordinate ξ. The new governing equations are:

$$\left| \begin{array}{ll} w_1^{(4)} + \chi^2 w_1'' - \lambda^4 w_1 = 0 & for \quad \xi \in [0;\alpha] \\ w_2^{(4)} + \chi^2 w_2'' - \lambda^4 w_2 = 0 & for \quad \xi \in [\alpha;1] \end{array} \right. \tag{8}$$

The general integrals of each fourth order linear differential equation are:

$$w_i(\xi) = A_i \cosh(\phi\xi) + B_i \sinh(\phi\xi) + C_i \cos(\psi\xi) + D_i \sin(\psi\xi)$$

$$\text{with} \quad \left| \begin{array}{l} \phi = \sqrt{\dfrac{-\chi^2 + \sqrt{\chi^4 + 4\lambda^4}}{2}} \\ \psi = \sqrt{\dfrac{\chi^2 + \sqrt{\chi^4 + 4\lambda^4}}{2}} \end{array} \right. \tag{9}$$

Imposing the boundary conditions leads to the characteristic equation:

$$\gamma\phi \sinh(\phi)\sin(\psi\alpha)\sin[\psi(\alpha-1)] - \gamma\psi \sin(\psi)\sinh(\phi\alpha)\sinh[\phi(\alpha-1)] = (\phi^2 + \psi^2)\sin(\psi\alpha)\sin[\psi(\alpha-1)]\sinh(\phi\alpha)\sinh[\phi(\alpha-1)] \tag{10}$$

When γ tends towards an infinite value, the structural model is reduced to the classical two-span continuous column. Eq. (10) can be simplified for the case without axial force $(\gamma \to \infty, \chi = 0)$ - see also Karnovsky and Lebed (2001).

$$\sinh(\lambda)\sin(\lambda\alpha)\sin[\lambda(\alpha-1)] = \sin(\lambda)\sinh(\lambda\alpha)\sinh[\lambda(\alpha-1)] \tag{11}$$

Introducing the smallest solution λ_1 of Eq. (10) yields the fundamental vibration mode:

$$\left| \begin{array}{l} \dfrac{w_1(\xi)}{D_1} = \sin(\psi_1\alpha)\left[-\dfrac{\sinh(\phi_1\xi)}{\sinh(\phi_1\alpha)} + \dfrac{\sin(\psi_1\xi)}{\sin(\psi_1\alpha)} \right] \\ \dfrac{w_2(\xi)}{D_1} = \sin(\psi_1\alpha)\left[-\dfrac{\sinh(\phi_1(1-\xi))}{\sinh(\phi_1(1-\alpha))} + \dfrac{\sin(\psi_1(1-\xi))}{\sin(\psi_1(1-\alpha))} \right] \end{array} \right.$$

$$\text{if } \sin(\psi_1\alpha) \neq 0 \text{ and } \sin(\psi_1(1-\alpha)) \neq 0 \tag{12}$$

3. ASYMPTOTIC ANALYSIS

It would be more convenient to introduce the dimensionless parameter ε as :

$$\alpha = \frac{1}{2}(1-\varepsilon) \text{ with } \varepsilon << 1 \tag{13}$$

ε is a dimensionless misplacement with respect to the symmetrical configuration. An asymptotic analysis is now performed in order to check closed-form approximation of the smallest parameter λ_1 (see for instance Bush, 1992):

$$\lambda_1 = \lambda_1^{(0)} + \varepsilon\lambda_1^{(1)} + \varepsilon^2\lambda_1^{(2)} + O(\varepsilon^3) \tag{14}$$

For symmetrical reasons, the first order term $\lambda_1^{(1)}$ is vanishing ($\lambda_1^{(1)} = 0$). The unperturbed solution $\lambda_1^{(0)}$ $(\varepsilon = 0)$, is given by:

$$\lambda_1^{(0)} = \left(16\pi^4 - 4\pi^2\chi^2\right)^{\frac{1}{4}} \tag{15}$$

Introducing Eq. (14) into the terms ϕ_1 and ψ_1 of Eq. (9) leads to the following result:

$$\left| \begin{array}{l} \psi_1 = 2\pi\left(1 + \lambda_1^{(2)} \dfrac{\left(16\pi^4 - 4\pi^2\chi^2\right)^{\frac{3}{4}}}{2\pi^2\left(8\pi^2 - \chi^2\right)}\varepsilon^2\right) + O(\varepsilon^3) \\ \phi_1 = \sqrt{4\pi^2 - \chi^2}\left(1 + 2\lambda_1^{(2)} \dfrac{\left(16\pi^4 - 4\pi^2\chi^2\right)^{\frac{3}{4}}}{\left(4\pi^2 - \chi^2\right)\left(8\pi^2 - \chi^2\right)}\varepsilon^2\right) + O(\varepsilon^3) \end{array} \right. \tag{16}$$

Introducing Eq. (16) into Eq. (10) leads to the calculation of $\lambda_1^{(2)}$ for $\gamma \neq 0$:

$$\lambda_1^{(2)} = -\frac{\pi^2\left(8\pi^2 - \chi^2\right)^2}{2\gamma\left(16\pi^4 - 4\pi^2\chi^2\right)^{\frac{3}{4}}} - \frac{\pi^2\sqrt{4\pi^2 - \chi^2}\left(8\pi^2 - \chi^2\right)\cosh\left(\dfrac{\sqrt{4\pi^2 - \chi^2}}{2}\right)}{\left(16\pi^4 - 4\pi^2\chi^2\right)^{\frac{3}{4}}\sinh\left(\dfrac{\sqrt{4\pi^2 - \chi^2}}{2}\right)} \tag{17}$$

The parameter $\lambda_1^{(2)}$ quantifies the curvature at origin. The curvature grows as the stiffness parameter γ decreases. This curvature is singular when γ tends towards zero, leading to the curve veering effect highlighted by Pierre (1988) for another mechanical system with an external spring instead of the internal spring considered in this paper. It is clear however that the second-order perturbation (Eq. (14)) has to be questioned when γ tends towards zero, thereby indicating that the vibration modes undergo a dramatic change (the relevancy of asymptotic methods with respect to the connection parameter is also analysed by Andrianov et al, 2005).

As suggested by Pierre (1988) for a similar problem with an external spring, a modified perturbation approach (with fixed ε) is probably more readable when γ tends towards zero.

$$\lambda_1 = \lambda_1^{(0)} + \gamma\lambda_1^{(1)} + \mathbf{O}\left(\gamma^2\right) \tag{18}$$

The unperturbed case $(\gamma = 0)$ leads to the fundamental frequency:

$$\lambda_1^{(0)} = \left(\frac{16\pi^4}{(1+\varepsilon)^4} - \frac{4\pi^2\chi^2}{(1+\varepsilon)^2}\right)^{\frac{1}{4}} \tag{19}$$

Introducing Eq. (18) into Eq. (10) leads to the identification of the following parameters:

$$\lambda_1^{(1)} = \frac{4\pi^2}{(1+\varepsilon)^3}\frac{1}{\left(\frac{16\pi^4}{(1+\varepsilon)^4} - \frac{4\pi^2\chi^2}{(1+\varepsilon)^2}\right)^{\frac{3}{4}}} \tag{20}$$

4. LOCALIZATION ANALYSIS

The localization phenomenon can be quantified by comparing the deflection values in each span. The localization parameter δ can be introduced as in Pierre et *al* (1987):

$$\delta = \frac{\max\left[\max_{\xi\in[0;\alpha]}\left|w_1(\xi)\right|;\ \max_{\xi\in[\alpha;1]}\left|w_2(\xi)\right|\right]}{\min\left[\max_{\xi\in[0;\alpha]}\left|w_1(\xi)\right|;\ \max_{\xi\in[\alpha;1]}\left|w_2(\xi)\right|\right]} \tag{21}$$

For symmetrical arguments, δ is equal to 1 without misplacement $(\varepsilon = 0)$. The maximum of the vibration mode in each unsymmetrical span is calculated from the derivative of Eq. (12):

$$\left|\begin{array}{l} -\phi_1\sin(\psi_1\alpha)\cosh(\phi_1\xi_1) + \psi_1\sinh(\phi_1\alpha)\cos(\psi_1\xi_1) = 0 \\ -\phi_1\sin[\psi_1(\alpha-1)]\cosh[\phi_1(\xi_2-1)] + \psi_1\sinh[\phi_1(\alpha-1)]\cos[\psi_1(\xi_2-1)] = 0 \end{array}\right. \tag{22}$$

Finally, for positive values of ε $(\varepsilon > 0)$, the localization parameter can be written as:

$$\delta = \frac{\left|w_2(\xi_2)\right|}{\left|w_1(\xi_1)\right|} \tag{23}$$

The localization parameter δ can be also derived by using an asymptotic method. In this case, one needs to develop a two-parameter (ε,γ) asymptotic method (see also Matkowsky and Reiss, 1977; Happawana et al, 1991). It is first assumed that the two parameters (ε,γ) are of the same order of magnitude *(* $\varepsilon \sim \gamma$ *)*. The first step consists in calculating the fundamental frequency, from the asymptotic expansion. It is assumed that:

$$\lambda_1 = \left(16\pi^4 - 4\pi^2\chi^2\right)^{\frac{1}{4}} + f_1(\varepsilon,\gamma) + \ldots \text{ with } f_1(\varepsilon,\gamma) \sim \varepsilon \sim \gamma \tag{24}$$

Introducing Eq. (24) into the terms ϕ_1 and ψ_1 of Eq. (9) leads to the following result:

$$\left|\begin{array}{l} \psi_1 = 2\pi + \dfrac{f_1}{\pi}\dfrac{\left(16\pi^4 - 4\pi^2\chi^2\right)^{\frac{3}{4}}}{8\pi^2 - \chi^2} + \ldots \\ \phi_1 = \sqrt{4\pi^2 - \chi^2} + 4\pi f_1 \dfrac{\left(16\pi^4 - 4\pi^2\chi^2\right)^{\frac{1}{4}}}{8\pi^2 - \chi^2} + \ldots \end{array}\right. \tag{25}$$

Introducing Eq. (25) into Eq. (10) leads to the second-order polynomial expression of f_1 :

$$f_1^2 - 8\pi^2\gamma\left(16\pi^4 - 4\pi^2\chi^2\right)^{-\frac{3}{4}} f_1 - 4\pi^4\left(8\pi^2 - \chi^2\right)^2\left(16\pi^4 - 4\pi^2\chi^2\right)^{-\frac{3}{2}}\varepsilon^2 = 0 \tag{26}$$

Two solutions are obtained from the resolution of Eq. (48) (the smallest solution is chosen for the fundamental vibration mode):

$$f_1 = 4\pi^2\left(16\pi^4 - 4\pi^2\chi^2\right)^{-\frac{3}{4}}\left[\gamma - \sqrt{\gamma^2 + \left(8\pi^2 - \chi^2\right)^2\frac{\varepsilon^2}{4}}\right] \tag{27}$$

The calculation of δ needs the computation of ξ_i $(i \in \{1;2\})$, from the non-linear equation Eq. (22). An asymptotic analysis may also be performed from:

$$\xi_i = \xi_i^{(0,0)} + g_i + ... \text{ with } \xi_1^{(0,0)} = \frac{1}{4} \text{ and } \xi_2^{(0,0)} = \frac{3}{4} \quad (28)$$

The characteristic coordinates ξ_i $(i \in \{1;2\})$ can now be introduced into the modal deflection given by Eq. (12), to obtain the asymptotic expansion of the deflection in the first span. The maximum of the normalized deflection in the first span is close to unity:

$$\frac{w_1(\xi_1)}{D_1} = 1 - \frac{\pi\varepsilon - \frac{2\pi}{8\pi^2 - \chi^2}\left(\gamma - \sqrt{\gamma^2 + \left(8\pi^2 - \chi^2\right)^2 \frac{\varepsilon^2}{4}}\right)}{2\cosh\left(\frac{\sqrt{4\pi^2 - \chi^2}}{4}\right)} + ... \quad (29)$$

For similar reasons, it can be shown that the minimum of the normalized deflection in the second span is equal to:

$$\frac{w_2(\xi_2)}{D_1} = -\frac{\pi\varepsilon - \frac{2\pi}{8\pi^2 - \chi^2}\left(\gamma - \sqrt{\gamma^2 + \left(8\pi^2 - \chi^2\right)^2 \frac{\varepsilon^2}{4}}\right)}{2\cosh\left(\frac{\sqrt{4\pi^2 - \chi^2}}{4}\right)} - \frac{\pi\varepsilon - \frac{2\pi}{8\pi^2 - \chi^2}\left(\gamma - \sqrt{\gamma^2 + \left(8\pi^2 - \chi^2\right)^2 \frac{\varepsilon^2}{4}}\right)}{\pi\varepsilon + \frac{2\pi}{8\pi^2 - \chi^2}\left(\gamma - \sqrt{\gamma^2 + \left(8\pi^2 - \chi^2\right)^2 \frac{\varepsilon^2}{4}}\right)} + ... \quad (30)$$

This means that the localization factor evolves as the minimum of the deflection in the second span, which is controlled by the second term of Eq. (30):

$$\delta = \frac{\pi\varepsilon - \frac{2\pi}{8\pi^2 - \chi^2}\left(\gamma - \sqrt{\gamma^2 + \left(8\pi^2 - \chi^2\right)^2 \frac{\varepsilon^2}{4}}\right)}{\pi\varepsilon + \frac{2\pi}{8\pi^2 - \chi^2}\left(\gamma - \sqrt{\gamma^2 + \left(8\pi^2 - \chi^2\right)^2 \frac{\varepsilon^2}{4}}\right)} + ... \quad (31)$$

Moreover, some simplifications may occur from the fundamental assumption that $\varepsilon \sim \gamma$:

$$\delta = \left(8\pi^2 - \chi^2\right)\frac{\varepsilon}{\gamma} + ... \quad (32)$$

Two particular cases can be deduced (vibrations without normal force, and buckling problem):

$$\chi = 0 \Rightarrow \delta = \frac{8\pi^2\varepsilon}{\gamma} + ...;\ \chi = 2\pi \Rightarrow \delta = \frac{4\pi^2\varepsilon}{\gamma} + ... \quad (33)$$

The asymptotic expansion can also be obtained when the two parameters (ε, γ) are not of the same order of magnitude:

$$\gamma \sim \varepsilon^p \text{ with } p \geq 2 \Rightarrow \delta = \left(8\pi^2 - \chi^2\right)\frac{\varepsilon}{\gamma} + ... \quad (34)$$

It appears that Eq. (32) is also valid when γ is small before ε. The localization phenomenon is clearly highlighted in Figure 2 and Figure 3.

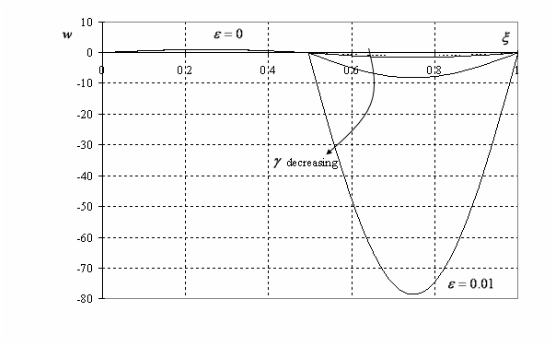

Figure 2 – Influence of the parameter γ on the localization phenomenon; $\gamma \in \{0.01;0.1;1\}$; $\chi = 0$ *and* $\varepsilon = 0.01$

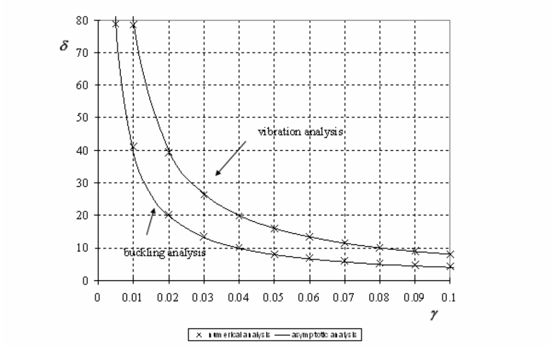

Figure 3 – Comparison of the numerical analysis with the asymptotic analysis; $\varepsilon = 0.01$

On the opposite, it can be shown that the localization factor δ tends towards unity when ε is sufficiently small before γ.

$$\varepsilon \sim \gamma^p \text{ with } p \geq 2 \Rightarrow \delta = 1 + \frac{\left(8\pi^2 - \chi^2\right)}{2}\frac{\varepsilon}{\gamma} + ... \quad (35)$$

5. CONCLUSIONS

- The dynamic analysis of a two-span weakened column with minor perturbations of the length of each span has been studied in this paper. The point of reduced bending stiffness represented by a rotational spring may result from a crack located at the intermediate support. This rotational spring can also be associated to a semi-rigid connection in structural engineering. The buckling problem is treated as a particular case of the axially loaded vibration problem.

- A quantitative criterion of localization is established. An asymptotic analysis is performed in order to get closed-form solution of this localization parameter. Moreover, from the asymptotic point of view, there is a correspondence between the buckling problem and the free vibration problem (without axial loading).

- Such a result is quite encouraging as localization is strongly associated to the flexibility values of the rotational spring, that is, when considering the open crack analogy, to the size of the crack. Localization only appears for severely damaged column. It can then be understood as an indicator of the damage level of the global structure. In this paper, it is shown that the concept of mode localization can be also used to passively control the structural integrity (as anticipated by Nayfeh and Hawwa, 1994 for systems with external springs).

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