# ON STABLE SOLUTIONS OF TIME DELAY SYSTEM CONTAINING HYSTERESIS NONLINEARITIES

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**Abstract**
Sufficient conditions are obtained for existing of stable periodic solutions of time delay control systems containing hysteresis nonlinearities.

**Key words**
relay hysteresis, periodic oscillations, stability

## 1 Introduction

The question on existence of periodic modes in nonlinear control systems and problem of exact construction of such modes are among central problems for automatic control theory. Special difficulties appear when dealing with systems containing the so-called essential nonlinearities, which are nonanalytic functions of phase (for example, relay nonlinearities). In this article one system with hysteresis nonlinearity will be considered. Nonlinearities of this sort, for example, may describe spatial delay of control mechanism, e.g. autopilot or stabilizer [V.I. Zubov]. Furthermore, study of physical processes in systems containing springing or magnetic elements, electrical relays etc. on certain assumptions gives rise to different mathematical models of hysteresis nonlinearity [Krasnoselsky; Visitin]. Numerous works are devoted to the analysis of problem of periodic oscillations presence in such systems [Astrom; S.V. Zubov; Pokrovsky; Nelepin; Kamachlin]. A wide variety of questions concerned to sliding modes in nonlinear systems considered in [Utkin].

Some results of [S.V. Zubov] concerning control systems containing hysteresis nonlinearities will be extrapolated below in case of time delay presence in such systems.

## 2 Models under consideration

Let us consider following systems

$$\dot{x} = Ax + cu(t-\tau), \qquad (1)$$

here $x = x(t) \in \mathbb{E}^n$, $t \geq t_0 \geq \tau$, $A$ is real $n \times n$ matrix, $c \in \mathbb{E}^n$, vector $x(t)$, $t \in [t_0 - \tau, t_0]$, is considered to be known. Quantity $\tau > 0$ describes time delay of actuator or observer.

Control statement is $u$ defined in the following way:

$$u(t-\tau) = f\left(\sigma(t-\tau)\right), \quad \sigma(t-\tau) = \gamma' x(t-\tau),$$

where $\gamma \in \mathbb{E}^n$, $\|\gamma\| \neq 0$, $f$ describes a nonlinear element of hysteresis sort and has one of the following forms:

$$f(\sigma) = \begin{cases} m_1, \ \sigma < l_2, \\ m_2, \ \sigma > l_1, \end{cases} \quad l_1 < l_2, \quad m_1 < m_2; \qquad (2)$$

or

$$f(\sigma) = \begin{cases} f_1(\sigma) = \alpha_1\left(\sigma_{0,\,1} - \sigma\right), & \sigma < l_2, \\ f_2(\sigma) = \alpha_2\left(\sigma_{0,\,2} - \sigma\right), & \sigma > l_1 \end{cases} \qquad (3)$$

( here $|f_1(\sigma) - f_2(\sigma)| > 0$ when $\sigma \in (l_1; l_2)$); or

$$f(\sigma(t)) = $$
$$= \begin{cases} m_1, & \left[\begin{array}{c} \sigma(t) \leq \frac{m_1}{\kappa} + l_1, \\ \begin{cases} l_1 < \sigma(t) - \frac{m_1}{\kappa} < l_2, \\ f(\sigma(t-0)) = m_1, \end{cases} \end{array}\right. \\ m_2, & \left[\begin{array}{c} \sigma(t) \geq \frac{m_2}{\kappa} + l_2 \\ \begin{cases} l_1 < \sigma(t) - \frac{m_2}{\kappa} < l_2, \\ f(\sigma(t-0)) = m_2, \end{cases} \end{array}\right. \\ \kappa(\sigma(t) - l_1), & \begin{cases} m_1 < \kappa(\sigma(t) - l_1) \leq m_2, \\ f(\sigma(t-0)) > m_1, \end{cases} \\ \kappa(\sigma(t) - l_2), & \begin{cases} m_1 \leq \kappa(\sigma(t) - l_2) < m_2, \\ f(\sigma(t-0)) < m_2. \end{cases} \end{cases} \qquad (4)$$

Suppose that all nonlinearities introduced above are walked in counterclockwise direction.

## 3 Main results

By the analogy with [Kamachkin] following result may be formulated:

**Lemma 1.** *If Re* $\lambda < 0$*, for any eigenvalue* $\lambda$ *of matrix* $A$*, and*

$$-\gamma' A^{-1} c m_1 > l_2, \quad -\gamma' A^{-1} c m_2 < l_1,$$

*then system (1), (2) has at least one non-trivial periodic solution.*

Suppose that exists the periodic solution of system (1), (2) having two switchig points $s_{1,\,2} \in \mathbb{E}^n$ such as $\gamma' s_1 = l_1$, $\gamma' s_2 = l_2$. In that case there exists a pair of points $\hat{s}_{1,\,2} \in \mathbb{E}^n$ ("virtual" switching points) and real constants $\tau_{1,\,2}$, $\tau_i > \tau$ such as

$$\begin{aligned}
\hat{s}_1 &= e^{A\tau} s_1 + \int_0^{\tau} e^{A(\tau - s)} c m_2\, ds, \\
s_2 &= e^{A(\tau_1 - \tau)} \hat{s}_1 + \int_0^{\tau_1 - \tau} e^{A(\tau_1 - \tau - s)} c m_1\, ds, \\
\hat{s}_2 &= e^{A\tau} s_2 + \int_0^{\tau} e^{A(\tau - s)} c m_1\, ds, \\
s_1 &= e^{A(\tau_2 - \tau)} \hat{s}_2 + \int_0^{\tau_2 - \tau} e^{A(\tau_2 - \tau - s)} c m_2\, ds.
\end{aligned}$$

Note that hereafter "swithing points" are only points lying on switching hyperplane, but in fact switching of control action occurs in virtual switching points.

Let us generalize one of the results cited in [S.V. Zubov].

**Theorem 1.** *Let*

$$\gamma' \left( A s_1 + c m_2 \right) \neq 0, \quad \gamma' \left( A s_2 + c m_1 \right) \neq 0.$$

*Denote*

$$\begin{aligned}
A_1 &= \left( E - \frac{\left( A s_2 + c m_1 \right) \gamma'}{\gamma' \left( A s_2 + c m_1 \right)} \right) e^{A\tau_1}, \\
A_2 &= \left( E - \frac{\left( A s_1 + c m_2 \right) \gamma'}{\gamma' \left( A s_1 + c m_2 \right)} \right) e^{A\tau_2}.
\end{aligned}$$

*If*

$$A = \| A_2\, A_1 \| < 1,$$

*then concerned periodic mode of system (1), (2) is asymptotically orbitally stable.*

*Proof.* Since

$$\begin{aligned}
s_2 &= e^{A\tau_1} s_1 + \int_0^{\tau} e^{A(\tau_1 - s)} c m_2\, ds + \\
&\quad + \int_{\tau}^{\tau_1} e^{A(\tau_1 - s)} c m_1\, ds, \\
s_1 &= e^{A\tau_2} s_2 + \int_0^{\tau} e^{A(\tau_2 - s)} c m_1\, ds + \\
&\quad + \int_{\tau}^{\tau_2} e^{A(\tau_2 - s)} c m_2\, ds,
\end{aligned}$$

then

$$\begin{aligned}
\frac{\partial s_1}{\partial s_2} &= e^{A\tau_2}, \quad \frac{\partial s_1}{\partial \tau_2} = A s_1 + c m_2, \\
\frac{\partial s_2}{\partial s_1} &= e^{A\tau_1}, \quad \frac{\partial s_2}{\partial \tau_1} = A s_2 + c m_1.
\end{aligned}$$

So, as

$$d \left( \gamma'\, s_1(s_2,\, \tau_2) \right) = 0, \quad \left( \gamma'\, s_2(s_1,\, \tau_1) \right) = 0,$$

then

$$\begin{aligned}
d\tau_2 &= - \left( \gamma' \left( A s_1 + c m_2 \right) \right)^{-1} \gamma' e^{A\tau_2}\, ds_2, \\
d\tau_1 &= - \left( \gamma' \left( A s_2 + c m_1 \right) \right)^{-1} \gamma' e^{A\tau_1}\, ds_1.
\end{aligned}$$

Hence

$$\begin{aligned}
ds_1 &= e^{A\tau_2} ds_2 - \frac{\left( A s_1 + c m_2 \right) \gamma' e^{A\tau_2}}{\gamma' \left( A s_1 + c m_2 \right)}\, ds_2 = \\
&= A_2\, ds_2, \\
ds_2 &= e^{A\tau_1} ds_1 - \frac{\left( A s_2 + c m_1 \right) \gamma' e^{A\tau_1}}{\gamma' \left( A s_2 + c m_1 \right)}\, ds_1 = \\
&= A_1\, ds_1,
\end{aligned}$$

and

$$ds_1^{(i+1)} = A_2\, A_1\, ds_1^{(i)}, \quad i = 0,\, 1,\, 2,\, \ldots$$

Application of the principle of fixed point completes the proof. ■

Let us pass onto the system (1), (3). Denote

$$\hat{A}_i = A - \alpha_i c \gamma', \quad \hat{c}_i = \alpha_i \sigma_{0,\, i} c, \quad i = 1, 2.$$

**Lemma 2.** *If Re* $\lambda < 0$*, for any eigenvalue* $\lambda$ *of matrices* $\hat{A}_{1,\,2}$*, and*

$$-\gamma' \hat{A}_1^{-1} \hat{c}_1 > l_2, \quad -\gamma' \hat{A}_2^{-1} \hat{c}_2 < l_1,$$

*then system (1), (3) has at least one non-trivial periodic solution.*

Suppose that exists the periodic solution of system (1), (3) having two switchig points $s_{1,\,2} \in \mathbb{E}^n$ such as $\gamma' s_1 = l_1$, $\gamma' s_2 = l_2$. In that case there exists a pair of virtual switching points $\hat{s}_{1,\,2} \in \mathbb{E}^n$ and real constants $\tau_{1,\,2}$, $\tau_i > \tau$ such as

$$\begin{gathered}
\hat{s}_1 = e^{\hat{A}_2 \tau} s_1 + \int_0^{\tau} e^{\hat{A}_2(\tau - s)} \hat{c}_2 \, ds, \\
s_2 = e^{\hat{A}_1(\tau_1 - \tau)} \hat{s}_1 + \int_0^{\tau_1 - \tau} e^{\hat{A}_1(\tau_1 - \tau - s)} \hat{c}_1 \, ds, \\
\hat{s}_2 = e^{\hat{A}_1 \tau} s_2 + \int_0^{\tau} e^{\hat{A}_1(\tau - s)} \hat{c}_1 \, ds, \\
s_1 = e^{\hat{A}_2(\tau_2 - \tau)} \hat{s}_2 + \int_0^{\tau_2 - \tau} e^{\hat{A}_2(\tau_2 - \tau - s)} \hat{c}_2 \, ds.
\end{gathered}$$

**Theorem 2.** *Let*

$$\gamma' \left( \hat{A}_1 s_2 + \hat{c}_1 \right) \neq 0, \quad \gamma' \left( \hat{A}_2 s_1 + \hat{c}_2 \right) \neq 0.$$

*Denote*

$$\begin{aligned}
A_1 &= \left( E - \frac{\left( \hat{A}_1 s_2 + \hat{c}_1 \right) \gamma'}{\gamma' \left( \hat{A}_1 s_2 + \hat{c}_1 \right)} \right) e^{\hat{A}_1 \tau_1 + \left( \hat{A}_2 - \hat{A}_1 \right) \tau}, \\
A_2 &= \left( E - \frac{\left( \hat{A}_2 s_1 + \hat{c}_2 \right) \gamma'}{\gamma' \left( \hat{A}_2 s_1 + \hat{c}_2 \right)} \right) e^{\hat{A}_2 \tau_2 + \left( \hat{A}_1 - \hat{A}_2 \right) \tau}.
\end{aligned}$$

*If*

$$A = \| A_2 \, A_1 \| < 1,$$

*then concerned periodic mode of system (1), (3) is asymptotically orbitally stable.*

*Proof.* Since

$$\begin{gathered}
s_1 = e^{\hat{A}_2(\tau_2 - \tau) + \hat{A}_1 \tau} s_2 + e^{\hat{A}_2(\tau_2 - \tau)} \times \\
\times \int_0^{\tau} e^{\hat{A}_1(\tau_2 - s)} \hat{c}_1 \, ds + \int_{\tau}^{\tau_2} e^{A(\tau_2 - s)} c m_2 \, ds, \\
s_2 = e^{\hat{A}_1(\tau_1 - \tau) + \hat{A}_2 \tau} s_1 + e^{\hat{A}_1(\tau_1 - \tau)} \times \\
\times \int_0^{\tau} e^{\hat{A}_2(\tau_1 - s)} \hat{c}_2 \, ds + \int_{\tau}^{\tau_1} e^{A(\tau_1 - s)} c m_1 \, ds,
\end{gathered}$$

then

$$\begin{aligned}
\frac{\partial s_1}{\partial s_2} &= e^{\hat{A}_2 \tau_2 + \left( \hat{A}_1 - \hat{A}_2 \right) \tau}, \quad & \frac{\partial s_1}{\partial \tau_2} &= \hat{A}_2 s_1 + \hat{c}_2, \\
\frac{\partial s_2}{\partial s_1} &= e^{\hat{A}_1 \tau_1 + \left( \hat{A}_2 - \hat{A}_1 \right) \tau}, \quad & \frac{\partial s_2}{\partial \tau_1} &= \hat{A}_1 s_2 + \hat{c}_1.
\end{aligned}$$

So, as

$$d \left( \gamma' \, s_1(s_2, \, \tau_2) \right) = 0, \quad d \left( \gamma' \, s_2(s_1, \, \tau_1) \right) = 0,$$

then

$$\begin{aligned}
ds_2 = \left( E - \left( \gamma' \, \frac{\partial s_2}{\partial \tau_1} \right)^{-1} \frac{\partial s_2}{\partial \tau_1} \, \gamma' \right) \frac{\partial s_2}{\partial s_1} \, ds_1 = \\
= A_1 ds_1.
\end{aligned}$$

Similarly, $ds_1 = A_2 ds_2$, and

$$ds_1^{(i+1)} = A_2 \, A_1 \, ds_1^{(i)}, \quad i = 0, \, 1, \, 2, \, \ldots$$

Application of the principle of fixed point completes the proof. ■

Now let us turn to the system (1), (4). Denote

$$\hat{A} = A + \kappa \, c \, \gamma'; \quad \hat{c}_i = -\kappa \, l_i \, c, \quad i = 1, \, 2.$$

**Lemma 3.** *If Re $\lambda < 0$, for any eigenvalue $\lambda$ of matrices $A$, $\hat{A}$, and*

$$\begin{gathered}
-\gamma' A^{-1} c m_1 > \frac{m_1}{\kappa} + l_2, \quad -\gamma' \hat{A}^{-1} \hat{c}_2 > \frac{m_2}{\kappa} + l_2, \\
-\gamma' A^{-1} c m_2 < \frac{m_2}{\kappa} + l_1, \quad -\gamma' \hat{A}^{-1} \hat{c}_1 < \frac{m_1}{\kappa} + l_1,
\end{gathered}$$

*then system (1), (4) has at least one non-trivial periodic solution.*

Suppose that exists the periodic solution of system (1), (4) having four switchig points $s_{1,\,2,\,3,\,4} \in \mathbb{E}^n$ such as

$$\begin{gathered}
\gamma' s_1 = l_1 + m_2/\kappa, \quad \gamma' s_2 = l_1 + m_1/\kappa, \\
\gamma' s_3 = l_2 + m_1/\kappa, \quad \gamma' s_4 = l_2 + m_2/\kappa.
\end{gathered}$$

In that case there exists virtual switching points $\hat{s}_{1,\,2,\,3,\,4} \in \mathbb{E}^n$ and real constants $\tau_{1,\,2,\,3,\,4}$, $\tau_i > \tau$ such as

$$\hat{s}_1 = e^{A\tau} s_1 + \int_0^{\tau} e^{A(\tau - s)} c m_2 \, ds,$$
$$s_2 = e^{\hat{A}(\tau_1 - \tau)} \hat{s}_1 + \int_0^{\tau_1 - \tau} e^{\hat{A}(\tau_1 - \tau - s)} \hat{c}_1 \, ds,$$
$$\hat{s}_2 = e^{\hat{A}\tau} s_2 + \int_0^{\tau} e^{\hat{A}(\tau - s)} \hat{c}_1 \, ds,$$
$$s_3 = e^{A(\tau_2 - \tau)} \hat{s}_2 + \int_0^{\tau_2 - \tau} e^{A(\tau_2 - \tau - s)} c m_1 \, ds.$$
$$\hat{s}_3 = e^{A\tau} s_3 + \int_0^{\tau} e^{A(\tau - s)} c m_1 \, ds,$$
$$s_4 = e^{\hat{A}(\tau_3 - \tau)} \hat{s}_3 + \int_0^{\tau_3 - \tau} e^{\hat{A}(\tau_3 - \tau - s)} \hat{c}_2 \, ds,$$
$$\hat{s}_4 = e^{\hat{A}\tau} s_4 + \int_0^{\tau} e^{\hat{A}(\tau - s)} \hat{c}_2 \, ds,$$
$$s_1 = e^{A(\tau_4 - \tau)} \hat{s}_4 + \int_0^{\tau_4 - \tau} e^{A(\tau_4 - \tau - s)} c m_2 \, ds.$$

**Theorem 3.** *Let*

$$\gamma' \left( \hat{A} s_2 + \hat{c}_1 \right) \neq 0, \quad \gamma' \left( A s_3 + c m_1 \right) \neq 0,$$
$$\gamma' \left( \hat{A} s_4 + \hat{c}_2 \right) \neq 0, \quad \gamma' \left( A s_1 + c m_2 \right) \neq 0.$$

*Denote*

$$A_1 = E - \frac{\left( \hat{A} s_2 + \hat{c}_1 \right) \gamma'}{\gamma' \left( \hat{A} s_2 + \hat{c}_1 \right)} \, e^{\hat{A}\tau_1 + \left(A - \hat{A}\right)\tau},$$
$$A_2 = E - \frac{\left( A s_3 + c m_1 \right) \gamma'}{\gamma' \left( A s_3 + c m_1 \right)} \, e^{A\tau_2 + \left(\hat{A} - A\right)\tau},$$
$$A_3 = E - \frac{\left( \hat{A} s_4 + \hat{c}_2 \right) \gamma'}{\gamma' \left( \hat{A} s_4 + \hat{c}_2 \right)} \, e^{\hat{A}\tau_3 + \left(A - \hat{A}\right)\tau},$$
$$A_4 = E - \frac{\left( A s_1 + c m_2 \right) \gamma'}{\gamma' \left( A s_1 + c m_2 \right)} \, e^{A\tau_4 + \left(\hat{A} - A\right)\tau}.$$

*If*

$$A = \| A_4 \, A_3 \, A_2 \, A_1 \| < 1,$$

*then concerned periodic mode of system (1), (4) is asymptotically orbitally stable.*

*Proof.* Since

$$s_1 = e^{A(\tau_4 - \tau) + \hat{A}\tau} s_4 + \int_0^{\tau} e^{A(\tau_4 - \tau) + \hat{A}(\tau - s)} \hat{c}_2 \, ds +$$
$$+ \int_0^{\tau_4 - \tau} e^{A(\tau_4 - \tau - s)} c m_2 \, ds,$$

then

$$\frac{\partial s_1}{\partial s_4} = e^{A\tau_4 + \left(\hat{A} - A\right)\tau},$$
$$\frac{\partial s_1}{\partial \tau_4} = A s_1 + c m_2;$$

and

$$d s_1 = \left( E - \left( \gamma' \, \frac{\partial s_1}{\partial \tau_4} \right)^{-1} \frac{\partial s_1}{\partial \tau_4} \, \gamma' \right) \, \frac{\partial s_1}{\partial s_4} \, d s_4 =$$
$$= A_4 d s_4.$$

Similarly,

$$d s_{i+1} = A_i d s_i, \quad i = 1, \, 2, \, 3,$$

and

$$d s_1^{(i+1)} = A_4 \, A_3 \, A_2 \, A_1 \, d s_1^{(i)}, \quad i = 0, \, 1, \, 2, \, \ldots$$

Applying the principle of fixed point, the proof reachs the end. ■

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