TRACKING CONTROL OF NONSMOOTH LAGRANGIAN SYSTEMS WITH TIME CONSTRAINTS

Irinel-Constantin Morărescu
INRIA
BIPOP Research Team
France
constantin.morarescu@inrialpes.fr

Bernard Brogliato
INRIA
BIPOP Research Team
France
bernard.brogliato@inrialpes.fr

Abstract

In this study one considers the tracking control problem of a class of nonsmooth fully actuated Lagrangian systems subject to frictionless unilateral constraints. The task under consideration consists of a succession of free and constrained phases. The transition from a constrained to a free phase is monitored via a Linear Complementarity Problem (LCP). On the other hand during the transition from a free to a constrained phase the dynamics contains some impacts that hamper the asymptotic stability. Nevertheless we have proved the *practical weak stability* of the system with an *almost decreasing* Lyapunov function. One numerical example illustrates the methodology described in the paper.

Key words

Lagrangian systems, complementarity problem, impacts, stability, tracking control.

1 Introduction

This paper focuses on the problem of tracking control of complementary Lagrangian systems (Moreau, 1988) subject to frictionless unilateral constraints whose dynamics may be expressed as:

$$\begin{cases} M(X)\ddot{X} + C(X,\dot{X})\dot{X} + G(X) = U + \nabla F(X)\lambda_X \\ 0 \le \lambda_X \perp F(X) \ge 0, \\ \text{Collision rule} \end{cases} \tag{1}$$

where $X \in \mathbb{R}^n$ is the vector of generalized coordinates, $M(X) = M^T(X) \in \mathbb{R}^{n\times n}$ is the positive definite inertia matrix, $F(X) \in \mathbb{R}$ represents the distance to the constraints, $C(X,\dot{X})$ is the matrix containing Coriolis and centripetal forces, $G(X)$ contains conservative forces, $\lambda_X \in \mathbb{R}$ is the Lagrangian multiplier associated to the constraint and $U \in \mathbb{R}^n$ is the vector of generalized torque inputs. For the sake of completeness we precise that $\nabla F(X) \in \mathbb{R}^n$ represents the vector of partial derivatives of F with respect to the components of X. We assume that the function F is continuously differentiable and that $\nabla F(X(t_\ell)) \neq 0$ for all impact times t_ℓ. For any function f the limit to the right at the instant t will be denoted by $f(t^+)$ and the limit to the left will be denoted by $f(t^-)$.

The admissible domain associated to the system (1) is the closed set Φ where the system can evolve and it is described as:

$$\Phi = \{X \mid F(X) \ge 0\}$$

In order to have a well posed problem with a physical meaning we consider that Φ contains at least a closed ball of positive radius. More precisely, the boundary of Φ (denoted $\partial\Phi$) might be a wall or something like, which restricts the evolution of the system in a precise domain. The presence of $\partial\Phi$ can induce some impacts that must be included in the dynamics of the system. The collision (or restitution) rule in (1), is a relation between the post-impact velocity and the pre-impact velocity. In this paper the collision rule is given by the Newton's law of impact: $\dot{X}_N(t_\ell^+) = -e_n\dot{X}_N(t_\ell^-)$, where $\dot{X}_N = \nabla F^T(X)\dot{X}$ denotes the normal component of the velocity and $e_n \in [0,1]$ is the restitution coefficient. The kinetic energy loss at the impact t_ℓ will be denoted by $T_L(t_\ell)$.

The tracking control problem under consideration was studied in (Brogliato *et al.*, 1997) mainly in the 1-dof (degree-of-freedom) case and in (Bourgeot and Brogliato, 2005) in the n-dof case. Both of these papers consider the problem of tracking a desired path without imposing from the beginning a desired velocity. Here we not only consider the case when the systems' task must be accomplished in a given period (see Section 4) but the results in Section 8 relax some very hard to verify condition imposed in (Bourgeot and Brogliato, 2005).

2 Stability analysis criteria

The system (1) is a complex nonsmooth dynamical system which involves continuous and discrete time phases. A stability criterion for this type of systems has been proposed in (Brogliato *et al.*, 1997). This criterion is an extension of the Lyapunov second method adapted to closed-loop mechanical systems with unilateral constraints. Next, let us clarify the framework and introduce some definitions.

First of all we split the time axis into intervals Ω_k and I_k corresponding to specific phases of motion. Precisely, Ω_{2k} corresponds to free-motion phases ($F(X) > 0$) and Ω_{2k+1} corresponds to constrained-motion phases ($F(X) = 0$). Therefore, during the Ω_k phases no impact can occur. Between a free phase and a constrained phase the dynamical system always passes into a transition phase I_k containing some impacts. Since the dynamics of the system does not change during the transition between a constrained and a free-motion phase, in the time domain one gets the following typical task representation (see (Brogliato *et al.*, 1997)):

$$\mathbb{R}^+ = \Omega_0 \cup I_0 \cup \Omega_1 \cup \Omega_2 \cup I_1 \cup \ldots \cup \Omega_{2k} \cup I_k \cup \Omega_{2k+1} \cup \ldots \tag{2}$$

Throughout the paper, the sequence $\Omega_{2k} \cup I_k \cup \Omega_{2k+1}$ will be referred as the cycle k of the system evolution. *The central issues of the paper are the design of transition phases and the study of stability of the trajectory evolving along (2)* (i.e. an infinity of cycles). Although the simplest way to stabilize the system on $\partial\Phi$ is to impose a tangential contact, this is not a good idea since a bad estimation of the constraint position may result in no stabilization at all. Therefore, the best strategy for stabilization on $\partial\Phi$ is to impose a closed-loop dynamics (containing impacts) which mimics the bouncing-ball dynamics (see e.g. (Brogliato, 1999)). In order to simplify the discussion we introduce the following trajectories (more details can be found in (Bourgeot and Brogliato, 2005)):

1) $X^{nc}(\cdot)$ denotes the desired trajectory of the unconstrained system. We suppose that $F(X^{nc}(t)) < 0$ for some t ($\in \Omega_{2k+1}$), otherwise the problem reduces to the tracking control of a system with no constraints.

2) $X_d^*(\cdot)$ denotes the signal entering the control input and plays the role of the desired trajectory during some parts of the motion.

3) $X_d(\cdot)$ represents the signal entering the Lyapunov function. This function is set on the boundary $\partial\Phi$ after the first impact of each cycle.

These signals may coincide on some time intervals as we can see in figure 1.

Let us define Ω as the complement of $I = \bigcup_{k \geq 0} I_k$ and assume that the Lebesgue measure of Ω, denoted $\lambda(\Omega)$, equals infinity. Consider $x(\cdot)$ the state of the closed-loop system in (1) with some feedback controller $U(X, \dot{X}, X_d^*, \dot{X}_d^*, \ddot{X}_d^*)$.

Definition 1 (Weakly Stable System). *The closed loop system is called weakly stable if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $||x(0)|| \leq \delta(\epsilon) \Rightarrow ||x(t)|| \leq \epsilon$ for all $t \geq 0$, $t \in \Omega$. The system is asymptotically weakly stable if it is weakly stable and $\lim_{t \in \Omega,\, t \to \infty} x(t) = 0$. Finally, the practical weak stability holds if there exists $0 < R < +\infty$ and $t^* < +\infty$ such that $||x(t)|| < R$ for all $t > t^*$, $t \in \Omega$.*

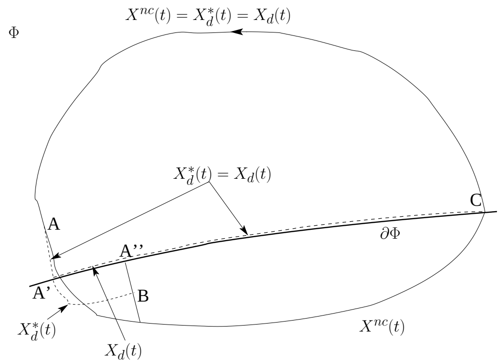

Figure 1. The closed-loop desired trajectory and control signals

Throughout the paper we consider $I_k = [\tau_0^k, t_f^k]$, where τ_0^k is chosen by the designer as the start of the transition phase I_k and t_f^k is the end of I_k. We note that all superscripts $(\cdot)^k$ will refer to cycle k of the system motion. We also use the following notations:

• t_0^k is the first impact during the cycle k,

• t_∞^k is the accumulation point of the sequence $\{t_\ell^k\}_{\ell \geq 0}$ of the impact instants during the cycle k (obviously $t_f^k \geq t_\infty^k$),

Consider $V(\cdot)$ such that there exists strictly increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying the conditions $\alpha(0) = 0, \beta(0) = 0$ and $\alpha(||x||) \leq V(x,t) \leq \beta(||x||)$.

Proposition 1 (Weak Stability). *(Bourgeot and Brogliato, 2005) Assume that the task admits the representation (2) and that*

a) $\lambda[I_k] < +\infty, \quad \forall k \in \mathbb{N}$,

b) *outside the impact accumulation phases* $[t_0^k, t_\infty^k]$ *one has* $\dot{V}(x(t),t) \leq -\gamma V(x(t),t)$ *for some constant* $\gamma > 0$,

c) $V(t_{\ell+1}^-) - V(t_\ell^+) \leq 0, \quad \forall \ell \geq 0$,

d) *the system is initialized on* Ω_0 *with* $V(\tau_0^0) \leq 1$,

e) $\sum_{\ell \geq 0} \sigma_V(t_\ell) \leq K V^p(\tau_0^k) + \epsilon$ *for some* $p \geq 0$, $K \geq 0$ *and* $\epsilon \geq 0$.

Then for $p < 1$ one has $V(\tau_0^k) \leq \delta(\gamma)$, where $\delta(\gamma)$ is a function that can be made arbitrarily small by increasing the value of γ. The system is practically weakly stable with $R = \alpha^{-1}(\delta(\gamma))$.

So $V(x(\cdot).\cdot)$ is a kind of "almost decreasing" Lyapunov function as illustrated in the numerical example section (see figure 5).

Proposition 1 is very useful because attaining asymptotic stability is not an easy task for the unilaterally constrained systems described by (1) especially when $n \geq 2$ and $M(q)$ is not a diagonal matrix (i.e. there are inertial couplings, which is the general case).

3 Controller design

In order to overcome some difficulties that can appear in the controller definition, the dynamical system (1) will be expressed in the generalized coordinates introduced in (McClamroch and Wang, 1988). The new coordinates will be $q = Q(X) \in \mathbb{R}^n$, with $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, $q_1 \in \mathbb{R}$, such that $\Phi = \{q_1(t) \geq 0\}$ and then the set of complementary relations can be rewritten as $0 \leq \lambda \perp Dq \geq 0$ with $D = (1, 0, \ldots, 0) \in \mathbb{R}^{1\times n}$. The controller used here consists of different low-level control laws for each phase of the system. More precisely, the controller can be expressed as

$$T(q)U = \begin{cases} U_{nc} & \text{for } t \in \Omega_{2k} \\ U_t & \text{for } t \in I_k \\ U_c & \text{for } t \in \Omega_{2k+1} \end{cases} \tag{3}$$

where $T(q) = \begin{pmatrix} T_1(q) \\ T_2(q) \end{pmatrix} \in \mathbb{R}^{n\times n}$.

The switching controller used in the following is based on the fixed-parameter scheme presented in (Slotine and Li, 1988) and the closed-loop stability analysis of the system is based on Proposition 1. First, let us introduce some notations: $\tilde{q} = q - q_d$, $\bar{q} = q - q_d^*$, $s = \dot{\tilde{q}} + \gamma_2 \tilde{q}$, $\bar{s} = \dot{\bar{q}} + \gamma_2 \bar{q}$, $\dot{q}_r = \dot{q}_d - \gamma_2 \tilde{q}$ where $\gamma_2 > 0$ is a scalar gain and q_d, q_d^* will be explicitly defined in the next section. Using the above notations the controller is given by

$$\begin{cases} U_{nc} = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - \gamma_1 s \\ U_t = U_{nc} \;\; \text{before the first impact} \\ U_t = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - \gamma_1 \bar{s} \\ \qquad\qquad\qquad \text{after the first impact} \\ U_c = U_{nc} - P_d + K_f(P_q - P_d) \end{cases} \tag{4}$$

where $\gamma_1 > 0$ is a scalar gain, $K_f > 0$, $P_q = D^T\lambda$ and $P_d = D^T\lambda_d$ is the desired contact force during constraint motion.

In order to prove the stability of the closed-loop system (1)- (4) we will use the following positive definite function:

$$V(t, s, \tilde{q}) = \frac{1}{2} s^T M(q) s + \gamma_1 \gamma_2 \tilde{q}^T \tilde{q} \tag{5}$$

4 Tracking control: two different formulations

In the sequel we also use the notations:

- t^{*k} is the time corresponding to $q_{1d}^*(t^{*k}) = 0$ (A' on figure 1),
- τ_1^k is such that $q_{1d}^*(\tau_1^k) = -\varphi V(\tau_0^k)$ and $\dot{q}_{1d}^*(\tau_1^k) = 0$, where $\varphi > 0$ is chosen by the designer in order to impose a closed-loop dynamics with impacts during the transient,
- t_d^k is the desired detachment instant, therefore $\Omega_{2k+1} = [t_f^k, t_d^k]$ (t_d^k is such that $(0, q_{2d}(t_d^k))$ is a point of the unconstrained desired trajectory $q^{nc}(\cdot)$).

The rationale behind the choice for $q_d^*(\cdot)$ is to improve the robustness of the impact phase by mimicking the bouncing-ball dynamics. However, the presence of impacts and velocity jumps hampers to obtain asymptotic stability along a path like (2) for systems with inertial couplings.

It is noteworthy that t_0^k, t_∞^k, t_d^k are state dependent whereas t^{*k}, τ_1^k and τ_0^k are exogenous and imposed by the designer. The points A, A'' and C in figure 1 correspond to the moments τ_0, t_f and t_d respectively. Obviously during the free-motion phase the three trajectories are the same and the discrepancy appears on the transient and constraint-motion phases.

The main difficulties here, consist of

- stabilizing the system on $\partial\Phi$ during the transition phases I_k;
- deactivating the constraint at the moment when the desired trajectory re-enters the admissible domain;
- maintaining a constraint movement between the moment when the system was stabilized on $\partial\Phi$ and the detachment moment.

Time-unconstrained case

In this case we want to solve the tracking control problem for the closed-loop dynamical system (1)-(4) with the complete desired path constructed taking into account a priori the unilateral constraint. The desired trajectory will be defined as a twice differentiable signal that coincides with the unconstrained trajectory in the admissible domain and allows a smooth passage between the constraint-motion phase and the free-motion phase.

Time-constrained case

This case might be motivated by some practical application in which a desired path must be followed in a given period. Additional difficulties related to the motion-synchronization are introduced in the dynamics.Let us consider the motion of a virtual particle perfectly following a given trajectory (represented by $X^{nc}(\cdot)$ on figure 1) that leaves the admissible domain Φ for a given period. Let $t_d^{ref,k}$ be the instant when the virtual particle re-enters the admissible domain (point C of the orbit on figure 1). The time-constraint formulation will refer to the tracking control strategy with a desired trajectory constructed such that:

- on Ω_{2k} it coincides with the trajectory of the virtual particle (the desired path and velocity are defined by the path and velocity of the virtual particle, respectively),
- on Ω_{2k+1} its orbit coincides with the projection of the outer part of the virtual particle's orbit on the constraint surface (X_d between A'' and C in Figure 1),
- the desired detachment moment and the moment when the virtual particle re-enters the admissible domain are synchronized.

5 Desired trajectory on phases I_k

In both of the previous two cases on $[\tau_0^k, t_0^k)$ we impose that $q_d^*(\cdot)$ is twice differentiable and $q_{1d}^*(t)$ decreases towards $-\varphi V(\tau_0^k)$ on $[\tau_0^k, \tau_1^k]$. For the sake of simplicity, in order to satisfy the previous requirements we define on $[\tau_0^k, \tau_1^k]$ the signal $q_{1d}^*(\cdot)$ as a degree 3 polynomial function with limit conditions ($t_{ini} = \tau_0^k$ and $t_{end} = \tau_1^k$).

The procedure presented above allows two different situations. The first one is given by $t_0^k > \tau_1^k$, which means that all the jumps of the Lyapunov function are negative. In this case the system is said strongly stable (for further details on this notion of stability see (Bourgeot and Brogliato, 2005; Brogliato *et al.*, 1997)). The second situation was pointed out in (Bourgeot and Brogliato, 2005) and is given by $t_0^k < \tau_1^k$. In this situation there exist positive jumps in the variation of the Lyapunov function, therefore the system can be only weakly stable.

In order to have a better control of the stabilization process the signal $q_{2d}^*(t)$ is frozen during the transition phase:

$$q_{2d}^*(t) = q_{2d}^*,\ \dot{q}_{2d}^*(t) = 0 \text{ on } [\tau_0^k, t_\infty^k],$$
$$q_{2d}^*(t) \text{ is defined such that } \dot{q}_{2d}^*(t^{*k}) = 0.$$

On $(t_0^k, t_f^k]$ we set q_d and q_d^* as follows:

$$q_d = \begin{pmatrix} 0 \\ q_{2d}^* \end{pmatrix}, \quad q_d^* = \begin{pmatrix} -\varphi V(\tau_0^k) \\ q_{2d}^* \end{pmatrix}, \tag{6}$$

and on $[t_f^k, t_d^k]$ we set

$$q_d = \begin{pmatrix} 0 \\ q_2^{nc}(t) \end{pmatrix}, \quad q_d^* = q_d. \tag{7}$$

Assuming a finite accumulation period, the impact process can be considered in some way equivalent to a plastic impact. Therefore, q_{1d} and $\dot{q}_{1d}$ are set to zero at t_0^{k+}.

6 Design of the desired contact force during constraint phases

The contact force P_d in (4) has to be sufficiently large at the beginning of Ω_{2k+1} and then to continuously decrease in order to assure a smooth passage to the unconstrained phase at the end of Ω_{2k+1}. Otherwise, a detachment is not possible and the system will remain on $\partial\Phi$. On the other hand, if the desired force decreases too much a detachment can take place before the end of the constraint phase. This can generate other impacts. Therefore we need a lower bound of the desired force which assures the contact during the constraint phases.

Dropping the time argument, the dynamics of the system on Ω_{2k+1} can be written as

$$\begin{cases} M(q)\ddot{q} + F(q,\dot{q}) = U_c + D^T\lambda \\ 0 \le q_1 \perp \lambda \ge 0 \end{cases} \tag{8}$$

where $F(q,\dot{q}) = C(q,\dot{q})\dot{q} + G(q)$. On $[t_f^k, t_d^k)$ the system is permanently constrained which implies $q_1(\cdot) = 0$ and $\dot{q}_1(\cdot) = 0$. In order to assure these conditions it is sufficient to have $\lambda > 0$. In the following let us denote

$M^{-1}(q) = \begin{pmatrix} [M^{-1}(q)]_{11} & [M^{-1}(q)]_{12} \\ [M^{-1}(q)]_{21} & [M^{-1}(q)]_{22} \end{pmatrix}$ and

$C(q,\dot{q}) = \begin{pmatrix} C(q,\dot{q})_{11} & C(q,\dot{q})_{12} \\ C(q,\dot{q})_{21} & C(q,\dot{q})_{22} \end{pmatrix}$.

Proposition 2. *On Ω_{2k+1} the constraint motion of the closed-loop system (8)-(4) is assured if the desired force is defined by*

$$\lambda_d = \big[-\big([M^{-1}(q)]_{11}C_{12}(q,\dot{q}) + \gamma_1[M^{-1}(q)]_{12} + [M^{-1}(q)]_{12}C_{22}(q,\dot{q})\big)s_2 - \beta\big]\frac{\bar{M}_{11}(q)}{1+K_f} \tag{9}$$

where $\bar{M}_{11}(q) = \big([M^{-1}(q)]_{11}\big)^{-1}$ and $\beta > 0$.

7 Strategy for take-off at the end of constraint phases Ω_{2k+1}

Now, we are interested in finding the conditions on the control signal U_c that assures the take-off at the end of constraint phases Ω_{2k+1}. As we have already seen before, the phase Ω_{2k+1} can be expressed as the time interval $[t_f^k, t_d^k)$. The dynamics on $[t_f^k, t_d^k)$ is given by (8) and the system is permanently constrained which means $q_1(\cdot) = 0$ and $\dot{q}_1(\cdot) = 0$. Thus, the detachment takes place at t_d^k if $\ddot{q}(t_d^{k+}) > 0$ which implies $\lambda(t_d^{k-}) = 0$.

Remark 1. *In (Bourgeot and Brogliato, 2005) the necessary conditions for detachment are stated from the complementarity relation in (8). However a precise definition of the desired signals q_{1d} and λ_d that guarantee a smooth detachment is not given.*

To simplify the notation we drop the time argument in many equations of this paragraph. Straightforward computation leads to the following result:

Proposition 3. *For the closed-loop system (8)(4) the passage from the constraint-motion phase to the free-motion phase can take place if*

$$b(q,\dot{q},U_{nc},\lambda_d) \triangleq DM^{-1}(q)[U_{nc} - F(q,\dot{q}) - D^T(1+K_f)\lambda_d] \ge 0$$

Proposition 4. *The closed-loop system (8)(4) is permanently constrained on $[t_f^k, t_d^k)$ and a smooth detachment is guaranteed on $[t_d^k, t_d^k+\epsilon)$ (ϵ is a small positive real number chosen by the designer) if*

(i) *λ_d is defined on $[t_f^k, t_d^k)$ by (9) where β is replaced by a time dependent function that decreases to 0 as the time t approaches t_d^k (we can choose a linear function as $\beta(t_d^k - t)$ where β is a real strictly positive number).*

(ii) *On $[t_d^k, t_d^k+\epsilon)$*

$$q_d(t) = \begin{pmatrix} q_{1d}^*(t) \\ q_2^{nc}(t) \end{pmatrix},$$

where $q_{1d}^*(t)$ is a twice differentiable function such that

$$q_{1d}^*(t_d^k) = 0,\, q_{1d}^*(t_d^k+\epsilon) = q_1^{nc}(t_d^k+\epsilon),$$
$$\dot q_{1d}^*(t_d^k) = 0,\, \dot q_{1d}^*(t_d^k+\epsilon) = \dot q_1^{nc}(t_d^k+\epsilon) \quad (10)$$

and $\ddot q_{1d}^*(t_d^k) = a$ *with*
$a > \max\big(0, -[M^{-1}(q)]_{11}(1+K_f)\lambda_d(t_d^{k-})\big)$.

Proof: For the sake of brevity and due to the space constraint we give here only the idea to prove Propositions 2, 3 and 4. All these results are based on the solution of the LCP derived by combining (8) and (4), which is:

$$0 \le DM^{-1}(q)\big[-F(q,\dot q) + U_{nc} - (1+K_f)D^T\lambda_d\big] + (1+K_f)DM^{-1}(q)D^T\lambda \perp \lambda \ge 0. \quad (11)$$

8 Closed-loop stability analysis

To simplify the notation $V(t, s(t), \tilde q(t))$ is denoted as $V(t)$.

Assumption 1. *The controller U_t in (4) assures that the sequence $\{t_\ell^k\}_{\ell\ge 0}$ of the impact times possesses a finite accumulation point t_∞^k i.e. $\lim_{\ell\to\infty} t_\ell^k = t_\infty^k < +\infty$, $\forall k \ge 0$.*

Theorem 1. *Let Assumption 1 hold, $e_n \in [0,1)$ and q_{1d}^* be defined as in Section 5. The closed-loop system (1)(4) initialized on Ω_0 with $V(\tau_0^0) \le 1$ is practically weakly stable with the closed loop state $x(\cdot) = [s(\cdot), \tilde q(\cdot)]$.*

Proof idea: Without entering into details one can show that the function V satisfies all the conditions in Proposition 1 with $p = \frac{3}{4}$. This Theorem extends the results presented in (Bourgeot and Brogliato, 2005) as follows. On one hand it generalizes Claim 7 of (Bourgeot and Brogliato, 2005) dropping the unnecessary and very hard to verify condition $||\tilde q_2(t_{\ell+1})|| \le ||\tilde q_2(t_\ell)||$ for all t_ℓ on $[t_0, t_\infty)$. On the other hand, it extends Claim 9 of (Bourgeot and Brogliato, 2005) from plastic to non-elastic impacts.

The closed-loop stability analysis of the system in the time-constrained case is assured by Theorem 1 replacing Assumption 1 with a stronger condition and changing the desired trajectory on the transition and constrained motion phases. In order to be more precise, Assumption 1 is replaced with $t_f^0 < t_d^{ref,0}$ which means $t_f^k < t_d^{ref,k}$, $\forall k \ge 0$ since $V(\tau_0^{k+1}) \le V(\tau_0^k)$, $\forall k \ge 0$ (in other words we suppose that the transition phase is short enough to allow the motion synchronization presented in Section 4).

Since on the transient q_{2d} was frozen we have to reduce the length of Ω_{2k+1} defining on $[t_f^k, t_d^{ref,k}]$ a desired trajectory $q_d = \begin{pmatrix} 0 \\ q_{2d} \end{pmatrix}$ such that

$$q_{2d}(t_f^{k+}) = q_{2d}^*,\, q_{2d}(t_d^{ref,k-}) = q_2^{nc}(t_d^{ref,k}),$$
$$\dot q_{2d}(t_f^{k+}) = 0,\, \dot q_{2d}(t_d^{ref,k-}) = \dot q_2^{nc}(t_d^{ref,k}).$$

For the sake of simplicity we consider again that $q_{2d}(t)$ is given by a polynomial function of degree 3 (i.e. the minimum degree that allows to satisfy all the conditions).

Finally, in order to assure the smoothness of the desired trajectory before the first impact one has to reduce the velocity $\dot q_{2d}^*(\cdot)$ before freezing $q_{2d}^*(\cdot)$. Therefore, on $[\tau_0^k, \tau_0^k+\delta]$ we define

$$q_{2d}^*(t) = q_2^{nc}\left(\tau_0^k + \frac{(t-\tau_0^k-\delta)^2(t-\tau_0^k)}{\delta^2}\right).$$

9 Two-link Manipulator

Let us test the control scheme (4) in simulation on a two-link planar manipulator moving in the XOY plane. We consider the following notations (see figure 2): θ_i represents the joint angle of the joint i, m_i is the mass of link i, I_i denotes the moment of inertia of link i about the axis that passes through the center of mass and is parallel to the Z axis, l_i is the length of link i, and g denotes the gravitational acceleration.

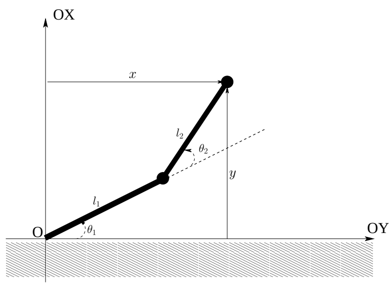

Figure 2. Two-link planar manipulator

Let us consider the tracking control problem of the trajectory described by the end point of the manipulator's second link with the constraint given by the ground (i.e. $y = 0$). Obviously the associated admissible domain is $\Phi = \{(x,y) \mid y \ge 0\}$. One introduces the generalized coordinates $q = \begin{bmatrix} y \\ x \end{bmatrix}, y \ge 0$ where (x,y) are the Cartesian coordinates of the end point. One assumes that the system must accomplish a cyclic task consisting of tracking a circle that violates the constraint. In order to track the trajectory the manipulator must follow the ground line from the point where the circle leaves the admissible domain to the point where the circle re-enters in it. Thus, the task has a time representation given by (2). Here we consider only the situation of the time-constrained case and we use the Moreau's time-stepping simulation algorithm of the SICONOS software platform (Acary and Pérignon, 2007). The choice of a time-stepping algorithm was mainly dictated by the presence of accumulations of impacts which render the use of event-driven methods difficult. The numerical values used for the dynamical model are $l_1 = l_2 = 0.5m$, $I_1 = I_2 =$

$1kg.m^2$ and $m_1 = m_2 = 1kg$. The restitution coefficient is set to $e_n = 0.7$ and the gains $\gamma_1 = 35$, $\gamma_2 = 20$ (see (4)). In figure 3 is plotted the trajectory of the system when the period of each cycle is set to 5 seconds and the final simulation time is 300 seconds. Obviously the trajectory is a little bit deformed with respect to the half circle since the desired trajectory on the transition between the free-motion and constraint-motion phases is changed.

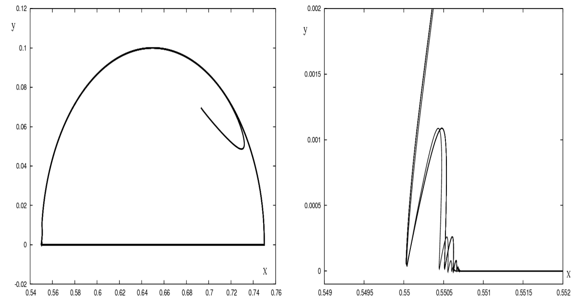

Figure 3. Left: The trajectory of the end point in the Oxy-plane for $\gamma_1 = 35, \gamma_2 = 20$ and the period of each cycle fixed at $T = 5$ seconds; Right: Zoom on the transition phases.

The variation of the generalized coordinates q_1 and q_2 during 6 cycles is depicted in figure 4. On this figure one can see that during the transition phases I_k the system mimics the bouncing ball motion.

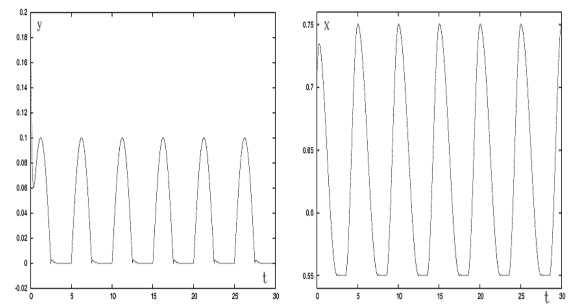

Figure 4. The variation of $q_1(t)$ (left) and $q_2(t)$ (right) for $\gamma_1 = 35, \gamma_2 = 20$ and the period of each cycle fixed at $T = 5$ seconds.

As we can see in figure 5 the tracking error rapidly decreases and remains close to 0. In other words the practical weak stability is guaranteed. Due to the impacts during each cycle the asymptotic stability is impossible to obtain. On the zoom made in figure 5 one can also observe the stabilization of the system on $\partial\Phi$ during the transition phases.

10 Conclusions

In this paper we have proposed a methodology to study the tracking control of fully actuated Lagrangian systems subject to frictionless unilateral constraints. The main contribution of the work is threefold: first, it extends the stability analysis results obtained in (Bourgeot and Brogliato, 2005), second, it presents an explicit expression of the desired contact force that assures both the constrained motion on Ω_{2k+1} and the take-off at the end of Ω_{2k+1}, and third, it considers the case of tracking control in presence of some time-constraints. A numerical example validates the methodology. A study regarding the robustness with respect to the joint flexibilities is in preparation.

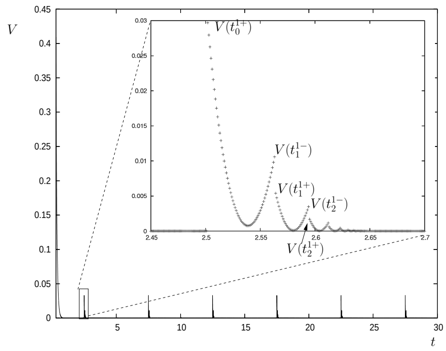

Figure 5. Variation of the Lyapunov function; Zoom: Variation of the Lyapunov function during the phase I_1

References

Acary, V. and F Pérignon (2007). An introduction to siconos. *INRIA Technical report* . **RT 340**, SICONOS open source software available at: http://siconos.gforge.inria.fr/.

Bourgeot, J.-M. and B. Brogliato (2005). Tracking control of complementary lagrangian systems. *International Journal of Bifurcation and Chaos* **15**(6), 1839–1866.

Brogliato, B. (1999). *Nonsmooth Mechanics.* Springer CCES, 2nd Ed.. London.

Brogliato, B., S.-I. Niculescu and P. Orhant (1997). On the control of finite-dimensional mechanical systems with unilateral constraints. *IEEE Trans. Autom. Contr.* **42**(2), 200–215.

McClamroch, N. and D. Wang (1988). Feedback stabilization and tracking of constrained robots. *IEEE Trans. Autom. Contr.* **33**(5), 419–426.

Moreau, J. J. (1988). *Unilateral contact and dry friction in finite freedom dynamics*. Nonsmooth Mechanics and Applications, CISM Courses and Lectures Vol. **302** Springer-Verlag. London.

Slotine, J.J. and W. Li (1988). Adaptive manipulator control: A case study. *IEEE Trans. Autom. Contr.* **33**(5), 995–1003.