# Nonlinear Observer-based Synchronization of Neuron Models

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***Abstract*— Multiple subsystems are required to behave synchronously or cooperatively in many areas. For example, synchronous behaviors are common in networks of (eletro-)mechanical systems, cell biology, coupled neurons, and cooperating robots. This paper presents an observer-based nonlinear feedback scheme for synchronization among Hindmarsh-Rose models which have polynomial vector fields. To this end, first we show that the problem is equivalent to finding an asymptotically stabilizing control for error dynamics which is also a polynomial system. On the basis of the previous result which uses full state information of the other model, we propose a certainty-equivalence control for the error dynamics. In other words, it is shown that the observer error linearization method can be applied to the error dynamics and that the error variable converges to zero due to the stable observer error dynamics and the globally asymptotically stable error dynamics.**

## I. Introduction

Synchronization is the asymptotic coincidence of the state vectors of two (or more) systems [5]. Recently, synchronization phenomena among multiple subsystems have received much attention in various research fields. For example, neuroscience [21], [29], [16], biology [28], physics [17], [33], [8], [4], coordinated motion and consensus problems [10], [20], control and dynamical systems theory [18], [5], [22], [19], and mechanical engineering [25], [23]. In these areas, synchronous behavior among subsystems plays an important role. For details of synchronization in particular fields, see these papers and the references therein.

To investigate synchronization in various research areas, mostly, five models are popularly used: Hodgkin-Huxley [13], [34], Kuramoto [1], [6], [28], Lorenz [15], [32], [11], Fitzhugh-Nagumo [16], [27], and Hindmarsh-Rose [21], [7] model. Note that the first two models are not in polynomial form and the others in polynomial form. In our previous result, a synthesis problem for synchronization is concerned for coupled oscillators modelled in polynomial systems [12]. In particular, a nonlinear feedback synchronization scheme for coupled Hindmarsh-Rose models is proposed. To do that, it is assumed that the full state information is exchanged between models. This means the full state information must be exchanged between models in order to employ the scheme. However, this is not desirable if the synchronization method is intended to be applied to the multiple models case ultimately. Therefore, it is necessary to develop a feedback synchronization scheme which allows less information exchange between models. The objective of this paper is to propose an observer-based feedback synchronization scheme for Hindmarsh-Rose model based on the previously proposed state feedback scheme. In our result using state feedback, it turned out that the error dynamic stabilization is the most important step in designing coupling function for synchronization. So this paper presents an observer-based stabilization method for the error dynamics.

In the previous results, there are several common properties. Firstly, the resulting feedback laws are of the form of linear feedback or linear feedback with variable gain [32], [15], [11], [8], [17], [33], [2], [4]. Secondly, the feedback interaction is unidirectional [30], [31] in the sense that a reference model without input is employed for synchronization. Thirdly, many papers handle synchronization between two models or assume a particular interconnection structure [16], [32], [15], [11]. Lastly, no efficient computational tools are employed for designing the feedback law.

Unlike these, the proposed scheme can improve the previous approaches. Namely, the resulting feedback is nonlinear, bidirectional and devised using efficient computational tools: SOS and SDP. Most of all, the presented scheme allows less information exchange.

## II. Problem Setup and Preliminary

Before proceeding further, problem setup and some preliminaries are presented in this section.

### A. Problem Setup

Let us consider two Hindmarsh-Rose models

$$\begin{aligned} \dot{x}_1 &= ax_2 + bx_1^2 - cx_1^3 - dx_3 + I + u_1 \\ \dot{x}_2 &= e - fx_1^2 - x_2 + gx_4 \\ \dot{x}_3 &= \mu[-x_3 + S(x_1 + h)] \\ \dot{x}_4 &= \nu[-kx_4 + r(x_2 + l)] \end{aligned} \tag{1}$$

and

$$\begin{aligned} \dot{y}_1 &= ay_2 + by_1^2 - cy_1^3 - dy_3 + I + u_2 \\ \dot{y}_2 &= e - fy_1^2 - y_2 + gy_4 \\ \dot{y}_3 &= \mu[-y_3 + S(y_1 + h)] \\ \dot{y}_4 &= \nu[-ky_4 + r(y_2 + l)] \end{aligned} \tag{2}$$

where $a,\ b,\ c,\ I,\ d,\ e,\ f,\ g,\ \mu,\ \nu,\ S,\ r,\ l,\ h$ and $k$ are parameters. See Appendix for the details of the parameters.

**Definition** *1:* The synchronization problem is to design each coupling function $u_i$ so as to satisfy the following two conditions.

C1. The differences between states of the subsystems converge to zero, i.e., $x(t) - y(t) \to 0$ as $t \to \infty$.

C2. The whole states are bounded ($\|x(t)\| < \infty$ and $\|y(t)\| < \infty$) for $t \geq 0$. □

Later, for notational simplicity, we denote the system in (1) by

$$\dot{x} = f(x) + Gu_1$$

where $G = [1, 0, 0, 0]^T$. Although the two models are identical, the trajectories are different from each other when $u_1 = u_2 = 0$ because of the different initial conditions. In order to solve the synchronization problem, we want to force the states of the two HR models in (1) and (2) to converge to each other by designing the coupling functions $u_1$ and $u_2$. To do this, we consider the error dynamics

$$\begin{aligned} \dot{e}_1 &= ae_2 + bt_1e_1 - ct_2e_1 - de_3 + u_{12} \\ \dot{e}_2 &= -ft_1e_1 - e_2 + ge_4 \\ \dot{e}_3 &= \mu(-e_3 + Se_1) \\ \dot{e}_4 &= \nu(-ke_4 + re_2) \end{aligned} \tag{3}$$

where $e_i = x_i - y_i$, $e = [e_1, \cdots, e_4]$, $t_1 = y_1 + x_1$, $t_2 = y_1^2 + y_1x_1 + x_1^2$, and $u_{12} = u_1 - u_2$. Again, for notational simplicity, the error dynamics is written in a vector form

$$\dot{e} = A(t_1, t_2)e + Gk(e, t_1, t_2). \tag{4}$$

where

$$A(t_1, t_2) = \begin{bmatrix} bt_1 - ct_2^2 & a & -d & 0 \\ -ft_1 & -1 & 0 & 0 \\ \mu S & 0 & -\mu & 0 \\ 0 & r\nu & 0 & -k\nu \end{bmatrix}.$$

Now, the synchronization problem boils down to designing a stabilizing feedback law for the error dynamics in (3). In other words, the task for the synchronization problem is to find a feedback law $u = k(e, t_1, t_2)$ that is a globally asymptotically stabilizing control for the error dynamics. Note that since the HR model is a polynomial model, so is error dynamics (4).

### B. Stabilization of Polynomial Systems

In view of the previous result, it turned out that the most important part in designing a coupling function for synchronization is to devise a stabilizing input for the error dynamics.

Since the resulting error dynamics are polynomial systems, it is nice to consider a tailored stabilization method for polynomial systems. To this end, a brief overview is given on stabilization method for polynomial systems.

Consider the following system

$$\dot{x} = f(x) + G(x)u \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input, and $f$ and $G$ sufficiently smooth polynomial functions. A globally asymptotically stabilizing control is designed for the system in (5) using a dissipation inequality and SOS in [7] with a state feedback of the form $u = K(x)x$. To do that, the system in (5) is rewritten as a linear-like system as

$$\dot{x} = A(x)x + G(x)u. \tag{6}$$

Note that the matrix $A(x)$ is not unique, i.e. there are different matrices $A(x)$ to write $f(x) = A(x)x$. As a control Lyapunov function candidate, a Krasovskii-like Lyapunov function of the form

$$V(x) = w(x)^T P w(x) \tag{7}$$

is assumed, where $P$ is a positive definite matrix and $w(\cdot)$ is defined later. To design a stabilizing feedback control, we want to determine the function $K(x)$ such that it satisfies

$$\dot{V}(x) = \nabla V(x)[A(x)x + G(x)K(x)x] < 0.$$

Note that this is a bilinear dissipation inequality and just the derivative of the Lyapunov function candidate $V$ with respect to the trajectories of (6) with $u = K(x)x$. To solve this bilinear problem, it is assumed that $w(x) = W(x)x$, where $W = (w_{ij})$ is a prespecified quadratic, lower triangular, and polynomial matrix with all its diagonal elements being 1:

$$W(x) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ w_{11}(x) & 1 & & \vdots \\ \vdots & & & 0 \\ w_{n1}(x) & w_{n2}(x) & \cdots & 1 \end{bmatrix}. \tag{8}$$

Note that the inverse of $W(x)$ is polynomial thanks to $det(W(x)) = 1$. The next theorem provides a solution to this problem.

**Theorem** *1:* [7] Let $W(x)$ be of the form as defined in (8). If there exist a polynomial matrix function $M : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ and a positive definite constant matrix $Q$ such that

$$\theta^T \frac{\partial w}{\partial x}[A(x)W(x)^{-1}Q + G(x)M(x)]\theta < 0 \tag{9}$$

for all nonzero $x$ and $\theta$, then $u = K(x)x$ with $K(x) = M(x)Q^{-1}W(x)$ is a globally asymptotically stabilizing state feedback for the system in (6). ■

It seems to be difficult to choose $W$ appropriately. However, sometimes we can do that easily by combining analytical reasoning with efficient computation as shown in [7]. Note that inequality (9) is a bilinear inequality. As mentioned previously, semidefinite programming and SOS tools can effectively solve this inequality [7], [24].

### C. integral Input-to-State Stability (iISS)

In order to develop an observer-based synchronization scheme, the following stability concept is useful.

**Definition** *2:* ([26]) Consider the system $\dot{x} = f(x, w)$ where $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^m$ the external input. The system is said to be integral-input-to-state stable (iISS) if there exist a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that the solution $x(t)$ of the system satisfies

$$\alpha(\|x(t)\|) \leq \beta(\|x(0)\|, t) + \int_{s=t_0}^{t} \gamma(\|w(s)\|)ds \tag{10}$$

where $\|\cdot\|$ denotes the standard Euclidean norm.

Similar to Lyapunov stability, Lyapunov function characterization for iISS is also possible.

**Theorem** *2:* The system $\dot{x} = f(x, w)$ is iISS if and only if there exists an iISS Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for class $\mathcal{K}_\infty$ functions $\bar{\alpha}_1, \bar{\alpha}_2, \gamma$ and a positive definite function $\rho$, $V$ satisfies

$$\bar{\alpha}_1(\|x\|) \leq V(x) \leq \bar{\alpha}_2(\|x\|) \tag{11}$$

and

$$\dot{V}(x(t)) \leq -\rho(\|x\|) + \gamma(\|w\|). \tag{12}$$

The next lemma plays an important role in deriving the main result.

**Lemma** *1:* ([26]) If the system $\dot{x} = f(x, w)$ is iISS and the external input satisfies

$$\int_0^\infty \gamma(\|w(s)\|)ds < \infty,$$

then, the state converges to the origin.

## III. Nonlinear Synchronization Based on Complete State Information

Before presenting the proposed observer-based synchronization, a nonlinear synchronization scheme is introduced in this section under the assumption that the full state information of a model is exchanged between models [12].

If the feedback design presented in Theorem 1 is applied to error dynamics (4), we obtain the following design dissipation inequality

$$\theta^T[A(t_1, t_2)Q + GM(t_1, t_2)]\theta < 0$$

where the stabilizing input is

$$u_{12} = k(e, t_1, t_2) = M(t_1, t_2)Q^{-1}e.$$

Note that in this case a quadratic Lyapunov function is used with $W = I$. Finally, solving this inequality using semidefinite programming and SOS tools leads to the following stabilizing input for the error dynamics.

$$u_{12} = [-0.0197+1.738t_{12}]e_2+[0.969-0.04t_{12}]e_3. \tag{13}$$

Note that this feedback law is nonlinear and goes to zero as synchronization is achieved. To design coupling functions $u_1$ and $u_2$ using stabilizing input $u_{12}$ , we revisit the definition of $u_{12}$. We recall that if $u_1$ and $u_2$ are designed to satisfy

$$\begin{aligned} & u_1 - u_2 = u_{12} \\ & = [-0.0197+1.738t_{12}]e_2+[0.969-0.04t_{12}]e_3, \end{aligned} \tag{14}$$

condition C1 is fulfilled. Here, the coupling functions $u_1$ and $u_2$ are unknowns and $u_{12}$ is known. Therefore, we can think of equation (14) as an underdetermined equation since there are two variables ($u_1$ and $u_2$) and one equation. As a result, we can see that if the coupling functions $u_1$ and $u_2$ are determined such that they satisfy the relation in (14) and go to zero as the error does, then C1 is fulfilled. To meet these two conditions, we distribute the feedback $u_{12}$ to the two coupling functions as follows:

$$u_1 = \frac{1}{2}u_{12}, \quad u_2 = -\frac{1}{2}u_{12}. \tag{15}$$

In fact, by designing the coupling functions like this, we can be sure that the state $x$ and $y$ converge to each other. However, it is also possible for both states to diverge to infinity simultaneously. This is why the second condition C2 has to be fulfilled in addition. See [12] for the proof of C2.

**Remark** *1:* The coupling functions in (15) are nonlinear and go to zero as the error $e$ does. In order to implement the coupling function, $x$ model has to know three pieces of information $y_1, y_2, y_3$ of $y$ model. The objective of this paper is to devise a nonlinear synchronization method which uses only the partial state information of the other model. Considering the ultimate objective in which synchronization problem for multiple models is considered, it is beneficial to develop a synchronization method to use only partial state information.

## IV. Nonlinear Observer-based Synchronization

As mentioned in the previous section, it is important to design the stabilizing input to the error dynamics. Therefore, the observer-based synchronization between two subsystems boils down to designing an observer-based stabilizing output feedback control for the error dynamics defined in (4). In case the full state information is available, it is shown that the state feedback law $u_{12}$ is a globally asymptotically stabilizing control for the error dynamics. Hereafter, it is discussed how to design certainty-equivalence type feedback for stabilization of the error dynamics using the state feedback law $u_{12}$. For this purpose, a nonlinear observer is designed using a structural property of the error dynamics at first. Then, a robustness property of the feedback in (13) against the estimation error is proved to derive a nonlinear separation principle. The error dynamics in (4) can be rewritten as

$$\begin{aligned} \dot{e} &= Ae + \phi(e_1, t_1, t_2) + G\hat{u}_{12} && (16) \\ z &= [1\ 0\ 0\ 0]e =: Ce && (17) \end{aligned}$$

where $z$ is the output and $A$, $\phi(e_1, t_1, t_2)$, and $\hat{u}_{12}$ are

$$A = \begin{bmatrix} 0 & a & -d & 0 \\ 0 & -1 & 0 & g \\ \mu S & 0 & -\mu & 0 \\ 0 & \nu r & 0 & -\nu k \end{bmatrix},$$

$$\phi(e_1, t_1, t_2) = \begin{bmatrix} bt_1e_1 - ct_2e_1 \\ -ft_1e_1 \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{u}_{12} = [-0.0197+1.738t_1]\hat{e}_2 + [0.969-0.04t_2]\hat{e}_3,$$

with $(C, A)$ being observable when the parameter values in the appendix are used, and the term $\phi(e_1, t_1, t_2)$ is measurable since $x_1$ and $y_1$ are. This means that all nonlinearities are measurable. It is well-known that the observer error linearization method ([9], [14]) can be applied to such a nonlinear system in order to design an observer, and that the resulting estimation error exponentially goes to zero. For the error dynamics in (16)-(17), we can design an observer as follows

$$\begin{aligned}\dot{\hat{e}} &= A\hat{e}+\phi(e_1,t_1,t_2)+G\hat{u}_{12}+L(\hat{z}-z) \quad (18)\\ \hat{z} &= C\hat{e} \quad (19)\end{aligned}$$

where the observer gain $L$ is determined such that $A+LC$ is Hurwitz. The estimation error dynamics[1] becomes

$$\dot{e}_o = (A+LC)e_o$$

where $e_o = \hat{e} - e$. This means that the estimation error resulting from the observer exponentially converges to the origin.

Unlike the linear system case in which the separation principle holds, designing such a convergent observer is not the end of the design for output feedback stabilization since the system is nonlinear. In other words, the combination of the stabilizing control and the convergent observer may not result in a stable closed-loop. To deal with this, it is required to either prove that the resulting closed-loop system is robust against the estimation error or to redesign a robust control against the estimation error. A popular approach to accomplish this in the literature is to prove that the closed-loop consisting of the observer and the certainty-equivalence type control is ISS (Input-to-State Stable [3]) or iISS (integral-ISS [26]) with respect to the estimation error. Then due to ISS or iISS of the closed-loop, the exponentially stable estimation error, and the definition of the estimation error, convergence of the state to the origin is proved. The next lemma is instrumental in deriving the main result.

**Lemma** *2:* Suppose that the unforced system $\dot{x} = f(x)$ is globally asymptotically stable. Then, the perturbed system

$$\dot{x} = f(x) + \sigma_1(w) \quad (20)$$

is iISS from the external disturbance $w$ to the state $x$ where $w \in \mathbb{R}^m$ is an external input and the function $\sigma_1(\cdot) : \mathbb{R}^m \to \mathbb{R}^n$ satisfies $\|\sigma_1(w)\| \le \sigma(\|w\|)$ with $\sigma(\cdot) \in \mathcal{K}$.

Proof. In view of Theorem 2, to prove the lemma, it is sufficient to show that there exists an iISS Lyapunov function to satisfy the inequalities in (11) and (12). From the converse Lyapunov theorem, there exists a global Lyapunov function $U : \mathbb{R}^n \to \mathbb{R}^+$ such that

$$\begin{gathered}\alpha_1(\|x\|) \le U(x) \le \alpha_2(\|x\|)\\ \dot{U} = \tfrac{\partial U}{\partial x} f(x) \le -\alpha_3(\|x\|), \qquad \|\tfrac{\partial U}{\partial x}\| \le \alpha_4(\|x\|)\end{gathered}$$

for some class $\mathcal{K}_\infty$ functions $\alpha_i$ ($i = 1, \cdots, 4$). The derivative of this Lyapunov function along $\dot{x} = f(x)+\sigma_1(w)$ has the following upper bound

$$\begin{aligned}\dot{U} &= \frac{\partial U}{\partial x}\cdot f(x) + \frac{\partial U}{\partial x}\sigma_1(w)\\ &\le -\alpha_3(\|x\|) + \|\frac{\partial U}{\partial x}\|\|\sigma_1(w)\|\\ &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|)\sigma(\|w\|).\end{aligned}$$

As an attempt to find an iISS Lyapunov function of system (20), consider a weighted function

$$V(r) = (\pi \circ U)(r), \quad \pi(r) = \int_0^r \frac{ds}{1+\chi(s)} \quad (21)$$

and $\chi(r) = (\alpha_4 \circ \alpha_1^{-1})(r)$. Then, the upper bound of the derivative of the weighted function is

$$\begin{aligned}\dot{V} = \frac{\frac{\partial U}{\partial x}(f(x)+\sigma_1(w))}{1+\chi(U(x))} &\le \frac{-\alpha_3(\|x\|)}{1+\chi(U(x))} + \frac{\alpha_4(\|x\|)\sigma(\|w\|)}{1+\chi(U(x))}\\ \le \quad & \frac{-\alpha_3(\|x\|)}{1+\alpha_4(\alpha_1^{-1}(\alpha_1(\|x\|)))} + \frac{\alpha_4(\|x\|)\sigma(\|w\|)}{1+\alpha_4(\|x\|)}\\ \le \quad & -\rho(\|x\|) + \sigma(\|w\|). \quad (22)\end{aligned}$$

Due to the monotone property of the function $\pi$, it is easy to see that the function $V$ is upper- and lower-bounded by some class $\mathcal{K}_\infty$ functions like in (11). Therefore, the inequality in (22) implies that the function $V$ in (21) is an iISS Lyapunov function. This completes the proof. □

The main theorem of this section is as follows.

**Theorem** *3:* The state of the closed-loop system consisting of the error dynamics for synchronization in (16)-(17), the observer in (18)-(19), and the feedback law $\hat{u}_{12}$ converges to the origin as $t \to \infty$.

Proof: From Lemma 2, the closed-loop system

$$\dot{\hat{e}} = A\hat{e} + \phi(e_1,t_1,t_2) + G\hat{u}_{12} + LCe_o$$

is iISS with respect to $LCe_o$. In other words, we have

$$\|\hat{e}(t)\| \le \beta(\|\hat{e}(0)\|, t) + \int_{t_0}^{t} \gamma(\|LCe_o(s)\|)ds$$

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$. Moreover, the estimation error exponentially converges to the origin and therefore satisfies

$$\int_{t_0}^{\infty} \|e_o(s)\|ds < \infty.$$

Thanks to this and Lemma 1, the observer state $\hat{e}$ goes to zero as the observer error $e$ does. The convergence of the error $e$ to the origin follows from the relation $e = \hat{e} - e_o$. This completes the proof. □

This theorem implies that the certainty-equivalence type control $\hat{u}_{12}$ is a globally asymptotically stabilizing control for the error dynamics and goes to zero as $e$ does. Therefore, $u_1 = \frac{1}{2}\hat{u}_{12}$ and $u_2 = -\frac{1}{2}\hat{u}_{12}$ solve the synchronization problem between two HR models with only the first state information of the HR model.

[1] In this paper, there are two kinds of error variables. Error or error dynamics denote the synchronization error ($e_i = x_i - y_i$). On the other hand, estimation error implies observer error between the error dynamics and its observer ($e_o = \hat{e} - e$).

## V. Conclusion

In this paper, we proposed a nonlinear feedback synchronization scheme for Hindmarsh-Rose models using stabilization techniques for polynomial systems; dissipation inequalities for those systems and SOS tools. Based on the previous result in which the full state information has to be exchanged between models, we develop an observer-based scheme which uses only the output information.

In view of the previous result, it is very important to design a stabilizing input to the error dynamics to develop a synchronization scheme. To do that, we designed a nonlinear observer using the fact that all nonlinearities in the error dynamics are measurable. Due to this property, stable error dynamics are obtained after applying observer error linearization. Finally, we proved a nonlinear separation principle which implies that the proposed certainty-equivalence control is a stabilizing input to the error dynamics. Finally, the coupling function for synchronization is derived using the observer-based stabilizing input for the error dynamics.

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