# SYNTHESIS OF ROBUST STABILIZING CONTROL FOR NONLINEAR SYSTEMS

**I.E. Zuber**
Department of Mathematics and Mechanics
Saint Petersburg State University
Russia

**A.Kh. Gelig**
Department of Mathematics and Mechanics
Saint Petersburg State University
Russia
a@ag1050.spb.edu

**Abstract**
Systems described by differential equations are considered for which only bounds of coefficients variation are known. For such systems a linear scalar control is formed which stabilizes the system. In the case when the system is controlled by a pulse modulator the lower bound of a sampling frequency is established for which the system retains stability.

**Key words**
Global asymptotical stability, nonlinear systems, pulse modulation.

## 1 Introduction

Consider a system

$$\dot{x} = A(x)x + b(x)u, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$, $b \in \mathbb{R}^{m\times 1}$, $u \in \mathbb{R}^1$. Suppose that the matrix $A(x)$ and the vector $b(x)$ have sufficiently smooth elements.

An exact feedback linearization method for synthesis of stabilizing control is known, see [Isidori, 1995; Miroshnik, Nikiforov and Fradkov, 2000; Zak, 2002; Khalil, 2002]. This method is based on construction of a transformation

$$y = \Phi(x), \tag{2}$$

which reduces the matrix $A$ to the Frobenius form with a functional lower row and vector $b$ to the last unit vector. With an appropriate control the considered system becomes a linear asymptotically stable system with constant coefficients. In [Isidori, 1995; Zak, 2002] the necessary and sufficient conditions for the existance of transformation (2) are proposed. These conditions set a rather narrow class of systems. Moreover, the control formed with this method is not robust, because its design requires a full knowledge of equations coefficients (1).

In [Gelig, Zuber and Churilov, 2006] a robust stabilizing control for system (1) is constructed without a supposition about smoothness of coefficients (1) in the following cases:

1. Coefficients $\alpha_{ij}(x)$ of matrix $A(x)$ and $\beta_i(x)$ of vector $b(x)$ have the properties

$$\begin{aligned}
&\alpha_{i,i+1} = 1 && (i = 1,\dots,m-1),\\
&\alpha_{ij} = 0 && (i = 1,\dots,m-1;\ j > i+1),\\
&\beta_i = 0 && (i = 1,\dots,m-1),\quad \beta_m = 1,\\
&|\alpha_{ij}(x)| < \text{const} && (i = 1,\dots,m;\ j < i+1).
\end{aligned}$$

2. $$\alpha_{i,i+1} = 1, \qquad \alpha_{ij} = 0,$$
$$(j < i-1,\ j > i+1,\ i = 1,\dots,m-1),$$
$$|\beta_i(x)| < \text{const},\ |\alpha_{m,i}| < \text{const}\ (i = 1,\dots,m).$$

In this paper a robust stabilizing control for system (1) is constructed without supposition about equality to zero of the coefficients indicated above. Continuous and pulse-modulated systems are considered.

## 2 Continuous Systems

Consider a system

$$\dot{x} = A(\cdot)x + (e_m + b(\cdot))u, \tag{3}$$

where $A(\cdot) \in \mathbb{R}^{m\times m}$, $b(\cdot) \in \mathbb{R}^{m\times 1}$, $e_m^* = (0,\dots,0,1)$. Suppose that coefficients of $A(\cdot)$ and of $b(\cdot)$ are arbitrary functionals. For example, $A(\cdot) = A(t, x(t), x(t-\tau), \psi(t))$, where $\psi(t)$ is an external perturbation. For these coefficients only bounds of variations are known:

$$|\alpha_{ij}(\cdot)| < \alpha_0,\ |\beta_i(\cdot)| < \beta_0\ (i,j = 1,\ldots,m). \quad (4)$$

Moreover, suppose that evaluation

$$|\alpha_{i,i+1}(\cdot)| > \alpha_* \quad (i = 1,\ldots,m-1) \quad (5)$$

is fulfilled.

The problem is to construct a stabilizing control

$$u = s^* x, \quad (6)$$

where $s$ is a constant vector depending only on parameters $\alpha_0, \beta_0$ and $\alpha_*$. If one choose a Lyapunov function in the form

$$V(x) = x^* H x,$$

where $H$ is positive definite constant matrix, the synthesis problem of stabilizing control $u = s^* x$ reduce to finding solution $H$, $s$ for bilinear matrix inequalities

$$\begin{aligned} &A^*(\cdot)H + HA(\cdot) + H(e_n + b(\cdot))s^* + \\ &+ s(e_n + b(\cdot))^* H < 0. \end{aligned} \quad (7)$$

In this paper a class of uncertain systems is found for which as distinction from [Collins, Sadhukhan and Watson, 1999; Arzelier, Peaucelle and Salhi, 2002] the solution of (7) has an explicit form. To solve the problem we represent system (3) in the form

$$\dot{x} = (A_1(\cdot) + A_2(\cdot))x + (e_m + b(\cdot))u, \quad (8)$$

where coefficients $\alpha^{(1)}_{ij}(\cdot)$ and $\alpha^{(2)}_{ij}(\cdot)$ of matrices $A_1(\cdot)$ and $A_2(\cdot)$ have properties:

$$\alpha^{(1)}_{ij}(\cdot) = \begin{cases} \alpha_{ij}(\cdot)\ for\ j \le i+1, \\ \qquad\qquad i = 1,\ldots,m, \\ 0 \quad for\ j > i+1, \\ \qquad\qquad i = 1,\ldots,m-2, \end{cases}$$

$$\alpha^{(2)}_{ij}(\cdot) = \begin{cases} \alpha_{ij}(\cdot)\ for\ j > i+1, \\ \qquad\qquad i = 1,\ldots,m-2, \\ 0 \quad for\ j \le i+1, \\ \qquad\qquad i = 1,\ldots,m. \end{cases}$$

Consider a Lyapunov function

$$V = x^* H^{-1} x,$$

where $H$ is a triple-band matrix with coefficients $h_{ij}\ :\ h_{ii} = h_i > 0,\ \ h_1 = 1,\ \ h_{i,i+1} = h_{i+1,i} = -\sqrt{h_i h_{i+1}}/2$ for $i = 1,\ldots,m-1,\ \ h_{ij} = 0$ for $j > i+1$ and $j < i-1$. This matrix is positive definite for any $h_i > 0\ \ (i = 2,\ldots,m)$, see [Gantmaher and Krein, 1950]. The derivative $\dot{V}$ along the solutions of system (8) has a form

$$\begin{aligned} \dot{V} = x^*\{&[A_1(\cdot) + A_2(\cdot)]^* H^{-1} + H^{-1}[A_1(\cdot) + A_2(\cdot)] + \\ &+ s e_m^* H^{-1} + H^{-1} e_m s^* + s b^*(\cdot) H^{-1} + H^{-1} b(\cdot) s\} x. \end{aligned}$$

Our aim is to receive evaluation

$$\dot{V} \le -\alpha x^* H^{-2} x, \quad (9)$$

which is equivalent to a matrix inequality

$$L(\cdot) + P(\cdot) + R(\cdot) < -\alpha I, \quad (10)$$

where

$$\begin{aligned} L(\cdot) &= Q(\cdot) + H s e_m^* + e_m s^* H, \\ Q(\cdot) &= H A_1^*(\cdot) + A_1(\cdot) H, \\ P(\cdot) &= H A_2^*(\cdot) + A_2(\cdot) H, \\ R(\cdot) &= H s b^*(\cdot) + b(\cdot) s^* H. \end{aligned}$$

Find $H$ and $s$ to satisfy the inequality

$$L(\cdot) < -3\alpha I. \quad (11)$$

Denote the main diagonal minors of $Q(\cdot)$ (beginning from above) by $\Delta_i(\cdot)$. It cam be easily shown that

$$\begin{aligned} \Delta_1(\cdot) &= 2(\alpha_{11}(\cdot)h_1 + \alpha_{12}(\cdot)h_{12}) + 3\alpha = \\ &= 2\alpha_{11}(\cdot) - \alpha_{12}(\cdot)\sqrt{h_2} + 3\alpha. \end{aligned} \quad (12)$$

We find $h_2$ from condition $\Delta_1(\cdot) < 0$. It is easy to check that representation

$$\Delta_k(\cdot) = \begin{vmatrix} Q_{k-1}(\cdot) & q_k(\cdot) \\ q_k^*(\cdot) & 2\alpha_{k,k+1}(\cdot)h_{k,k+1} + 3\alpha \end{vmatrix}$$

is valid, where column $q_k(\cdot)$ depends only on $h_j$ and $\alpha_{ij}(\cdot)$ for $j \le k,\ i \le k$. By the Schur's lemma we arrive at a representation

$$\begin{aligned} \Delta_k(\cdot) = \Delta_{k-1}(\cdot)[&-\alpha_{k,k+1}(\cdot)\sqrt{h_k h_{k+1}} + 3\alpha - \\ &- q_k^*(\cdot) Q_{k-1}^{-1}(\cdot) q_k(\cdot)], \end{aligned} \quad (13)$$

where $q_k(\cdot)$ and $Q_{k-1}(\cdot)$ do not depend on $h_{k+1}$. Without lack of generality suppose that $\alpha_{k,k+1}(\cdot) > \alpha_*$ for $k = 1, \ldots, m-1$. (If it is not so, one can multiply the equation by $-1$ beginning from below, and replace corresponding $x_i$ by $-x_i$.) The expression standing in square brackets (13) is negative for sufficiently large $h_{k+1}$. So we conclude that $\Delta_k(\cdot)\Delta_{k-1}(\cdot) < 0$ for $k \leq m-1$ and for sufficiently large $h_2, \ldots, h_m$, which can be selected monotonically increasing:

$$h_1 \leq h_2 \leq \ldots \leq h_m. \tag{14}$$

Let

$$s = \lambda H^{-1} e_m \tag{15}$$

and represent matrix $L(\cdot) + 3\alpha I$ in form

$$L(\cdot) + 3\alpha I = \begin{pmatrix} Q_{m-1}(\cdot) & q(\cdot) \\ q^*(\cdot) & 2\lambda + \varkappa(\cdot) \end{pmatrix} \tag{16}$$

where $Q_{m-1}(\cdot)$ and $q(\cdot)$ do not depend on $\lambda$, and $\varkappa(\cdot)$ has a form

$$\varkappa(\cdot) = 2[\alpha_{m-1,m}(\cdot)h_{m-1,m} + \alpha_{mm}(\cdot)h_m] + 3\alpha h_m. \tag{17}$$

By the Schur's lemma

$$\det(L + 3\alpha I) = \det Q_{m-1}(\cdot) \times (\varkappa - q^*(\cdot)Q_{m-1}^{-1}(\cdot)q(\cdot)).$$

Choosing $\lambda$ to meet the inequality

$$\lambda < -\sup[\alpha_{m-1,m}(\cdot)h_{m-1,m} + \alpha_{mm}(\cdot)h_m + \tfrac{3}{2}\alpha h_m - q^*(\cdot)Q_{m-1}^{-1}(\cdot)q(\cdot)]$$

one gets

$$\det(L(\cdot) + 3\alpha I) \det Q_{m-1}(\cdot) < 0$$

and hence (11).

Introduce the following notation. Let $|A|$ be Euclidean norm of matrix $A$, $\|A\| = \sqrt{\lambda_{max}(AA^*)}$ be spectral norm of matrix $A$. Evaluate $\|P(\cdot)\|$ to ensure the inequality

$$P(\cdot) < \alpha I. \tag{18}$$

Because of symmetry of matrices $H$ and $A_2(\cdot)A_2^*(\cdot)$ inequality

$$\lambda_{max}[HA_2(\cdot)A_2^*(\cdot)H] \leq \leq [\lambda_{max}(H)]^2 \lambda_{max}[A_2(\cdot)A_2^*(\cdot)]$$

is valid, see [Gantmaher, 1967]. Since matrix $H$ is a normal Jacobi's symmetrical matrix,

$$\lambda_{max}(H) \leq \max_i h_i + 2\cos\frac{\pi}{m+1} \max_i \frac{1}{2}\sqrt{h_i h_{i+1}}$$

is valid, see [Gantmaher and Krein, 1950]. So (14) implies

$$\lambda_{max}(H) \leq 2h_m.$$

Therefore the inequality

$$\|HA_2^*(\cdot)\| \leq 2h_m \|A_2(\cdot)\|$$

is valid. In a similar way one comes to

$$\|A_2(\cdot)H\| \leq 2h_m \|A_2(\cdot)\|.$$

So the evaluation

$$\|P(\cdot)\| \leq 4h_m \|A_2(\cdot)\| \tag{19}$$

is valid. According to the Euler's inequality the relationship (18) is valid if

$$\lambda_{max}(P(\cdot)) < \alpha. \tag{20}$$

Since matrix $P(\cdot)$ is symmetrical

$$\lambda_{max}(P(\alpha)) = \|P(\cdot)\|.$$

So (18) is valid if either inequality (20) is fulfilled, or $4h_m\|A_2(\cdot)\| < \alpha$ . Since evaluation $\|A_2(\cdot)\| \leq |A_2(\cdot)|$ is valid [Wilkinson, 1970], for inequality (18) to hold it suffices that

$$|A_2(\cdot)| < \frac{\alpha}{4h_m}. \tag{21}$$

Evaluate now $\|R(\cdot)\|$ in such a way, that inequality

$$R(\cdot) < \alpha I \tag{22}$$

is valid. Reasoning in the same way as for deduction of (18) we receive

$$\|R(\cdot)\| \leq 4h_m \|b(\cdot)s^*\|.$$

So (22) is fulfilled if

$$|b(\cdot)s^*| < \frac{\alpha}{4h_m}.$$

Hence

$$|b(\cdot)| < \frac{\alpha}{4h_m|s|} \tag{23}$$

implies inequality (22). From (11), (18), (22) evaluation (9) follows, that guarantees global asymptotical stability of equilibrium for system (4). So the following result is received.

**Theorem 1.** If (4), (5), (21), (23) are fulfilled and control (6), (15) is chosen, the zero equilibrium $x = 0$ of system (3) is globally asymptotically stable.

## 3 Pulse-Modulated Systems

Consider a system

$$\begin{aligned} &\dot{x} = A(\cdot)x + (e_m + b)\xi, \\ &\xi = \mathcal{M}\zeta, \quad \zeta = \psi(\sigma), \quad \sigma = s^*x, \end{aligned} \tag{24}$$

where the matrix $A(\cdot)$ is the same as in system (3), $b$ is a constant $m$-dimensional vector, $\xi(t)$ is a signal at the output of pulse modulator, $\zeta(t)$ is a signal at its input, $\mathcal{M}$ is an operator ($G$-modulator [Gelig, Zuber and Churilov, 2006]). It is required to find a function $\psi(\sigma)$ and a constant vector $s$ such that system (24) is globally asymptotically stable.

The operator $\mathcal{M}$ maps each function $\zeta(t)$ continuous on $[t_0, +\infty)$ to a sequence of numbers $t_0 < t_1 < t_2 < \dots$ and to a function $\xi(t)$ with the following properties:

1. $\delta_0 T \leq t_{n+1} - t_n \leq T, \ n = 0, 1, 2, \dots$;
2. $\xi(t)$ is piecewise-continuous in each interval $[t_n, t_{n+1})$ and does not change sign on it;
3. the operator $\mathcal{M}$ is causal, i.e., $t_n$ depends only on $\zeta(t)$ for $t \leq t_n$, $\xi(t)$ depends only on $\zeta(\tau)$ for $\tau \leq t$;
4. there exists a function ("an equivalent nonlinearity") $\varphi(\zeta) \in \mathbb{C}(-\infty, +\infty)$ such that for each $n$ there is $\tilde{t}_n \in [t_n, t_{n+1})$ for which the average value of the $n$-th pulse

$$v_n = \frac{1}{t_{n+1} - t_n} \int_{t_n}^{t_{n+1}} \xi(t)dt$$

satisfies the equality

$$v_n = \varphi(\zeta(\tilde{t}_n)). \tag{25}$$

The majority of known forms of pulse modulation has properties 1–4 (pulse-amplitude modulation, pulse-frequency modulation, pulse-width modulation, combined pulse modulation and others [Tsypkin and Popkov, 1973; Gelig and Churilov, 1998]). It is supposed that function $\varphi(\sigma)$ is continuous and is increasing monotonically on $(-\infty, +\infty)$, $\varphi(0) = 0$, $\varphi(\pm\infty) = \pm\infty$.

For stabilization of system (24) let $\psi = \varphi^{-1}$, where $\varphi^{-1}$ is reciprocal function for $\varphi$. Then we have equality

$$v_n = \sigma(\tilde{t}_n). \tag{26}$$

To use the method of averaging [Gelig and Churilov, 1998] we introduce functions $v(t) = v_n$ for $t_n \leq t < t_{n+1}$ ($n = 0, 1, 2, \dots$) and

$$w(t) = \int_{t_0}^{t} [\xi(\tau) - v(\tau)]d\tau.$$

Changing in (24) variables

$$y = x - (e_m + b)w(t),$$

we come to system

$$\begin{aligned} &\dot{y} = B(\cdot)y + f(\cdot), \\ &\sigma = s^*y + \varkappa w, \quad \varkappa = s^*(e_m + b), \end{aligned} \tag{27}$$

where

$$\begin{aligned} &B(\cdot) = A(\cdot) + (e_m + b)s^*, \\ &f(\cdot) = A(\cdot)(e_m + b)w + (e_m + b)(v - s^*y). \end{aligned}$$

Obviously an equation (27) is different from (3) only by member $f(\cdot)$. So choosing $V = y^*H^{-1}y$ and $s$ by formula (15) we come to the following evaluation for derivative $\dot{V}$ in respect to system (27):

$$\dot{V} < -\alpha y^*H^{-2}y + p(\cdot), \tag{28}$$

where $p(\cdot) = f^*(\cdot)H^{-1}y + y^*H^{-1}f(\cdot)$. An evaluation

$$\begin{aligned} |p(\cdot)| \leq 2|H^{-1}y| \cdot |f(\cdot)| \leq \\ \leq \mu|f(\cdot)|^2 + \frac{1}{\mu}|H^{-1}y|^2 \end{aligned} \tag{29}$$

is obvious, where $\mu$ — positive parameter, which will be constructed later.

By traditional for method of averaging arguments [Gelig and Churilov, 1998], based on evaluation $|w| \le T|v|$ and Wirtinger inequality

$$\int_a^b \sigma(t)^2 dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b \left[\frac{d\sigma}{dt}\right]^2 dt,$$

($\sigma(c) = 0,\ a \le c \le b,\ \frac{d\sigma}{dt} \in L_2[a,b]$) we make sure of validity of evaluation

$$\int_{t_n}^{t} |f|^2 dt \le \gamma_1 T^2 \int_{t_n}^{t} |y|^2 dt \tag{30}$$

for

$$T < \gamma_2. \tag{31}$$

Here and later $\gamma_i$ depend only on numbers $m$, $\alpha_0$, $\alpha_*$, $\beta_0$. From (28), (29), (30) relation

$$\dot{V} \le [(\frac{1}{\mu} - \alpha)|H^{-1}|^2 + \gamma_3 T^2 \mu]|y|^2 \tag{32}$$

follows. It is easy to find such an evaluation

$$T < \gamma_4. \tag{33}$$

That if (33) is fulfilled, such $\mu > 0$ exists, for which inequality (32) has a form

$$\dot{V} \le -\gamma_5 |y|^2.$$

There follows

$$V(y(t_{n+1})) < V(y(t_n)) - \gamma_5 \int_{t_n}^{t_{n+1}} |y|^2 dt$$
$$(n = 0, 1, 2, \ldots).$$

Summing these inequalities by $n$ we receive an evaluation

$$V(y(t_N)) + \gamma_5 \int_{t_0}^{t_N} |y|^2 dt < V(y(t_0)).$$

From here in respect to arbitrariness of $N$ relation $|y| \in L_2[t_0, +\infty)$ follows. Further by way of standard arguments following properties are proved sequentially:

$$v \in L_2[t_0,+\infty), \quad w \in L_2[t_0,+\infty),$$
$$|\dot{y}| \in L_2[t_0,+\infty), \quad \lim_{t\to+\infty} |y(t)| = 0,$$
$$\lim_{t\to+\infty} w(t) = 0, \quad \lim_{t\to+\infty} |x(t)| = 0,$$
$$\sup_{t\ge t_0} |x(t)| \to 0 \quad \text{for } |x(t_0)| \to 0.$$

Thus the following result was received:

**Theorem 2.** Let suppositions of theorem 1 are fulfilled, $b$ — constant vector, $\psi = \varphi^{-1}$ and vector $s$ is set by formula (15). If evaluations (31), (33) are fulfilled state of equilibrium $x = 0$ of pulse-modulated system (24) is globally stable.

## 4 Example

Consider the problem of synthesis of robust stabilizing control for system

$$\begin{aligned}
\dot{x}_1 &= a_{11}(\cdot)x_1 + a_{12}(\cdot)x_2 + a_{13}(\cdot)x_3 = \beta_1(\cdot)u,\\
\dot{x}_2 &= a_{21}(\cdot)x_1 + a_{22}(\cdot)x_2 + a_{23}(\cdot)x_3 = \beta_2(\cdot)u,\\
\dot{x}_3 &= a_{31}(\cdot)x_1 + a_{32}(\cdot)x_2 + a_{33}(\cdot)x_3 = (1+\beta_3(\cdot))u,
\end{aligned} \tag{34}$$

where

$$|a_{ij}(\cdot)| \le \alpha_0,\ |\beta_i(\cdot)| \le \beta_0,\ a_{12}(\cdot) > \alpha_*,\ a_{23}(\cdot) > \alpha_*. \tag{35}$$

For constructing the control $u(x)$ we will use theorem 1. It is evident that for system (34) the matrices $A_1(\cdot)$ and $A_2(\cdot)$ have the forms

$$A_1(\cdot) = \begin{pmatrix} a_{11}(\cdot) & a_{12}(\cdot) & 0 \\ a_{21}(\cdot) & a_{22}(\cdot) & a_{23}(\cdot) \\ a_{31}(\cdot) & a_{32}(\cdot) & a_{33}(\cdot) \end{pmatrix},$$

$$A_2(\cdot) = \begin{pmatrix} 0 & 0 & a_{13}(\cdot) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a matrix $H$ we consider

$$H = \begin{pmatrix} 1 & -\frac{\sqrt{h_2}}{2} & 0 \\ -\frac{\sqrt{h_2}}{2} & h_2 & -\frac{\sqrt{h_2 h_3}}{2} \\ 0 & -\frac{\sqrt{h_2 h_3}}{2} & h_3 \end{pmatrix}.$$

It is evident

$$\Delta_1(\cdot) = 2(a_{11}(\cdot) - \frac{\sqrt{h_2}}{2}a_{12}(\cdot)) + 3\alpha.$$

It is easily to prove that respect to (35) we have

$$\Delta_1(\cdot) \leq -1, \tag{36}$$

if

$$\sqrt{h_2} > \frac{2\alpha_0 + 3\alpha + 1}{\alpha_*}.$$

Consider

$$h_2 = \max\Big\{1,\ \frac{(2\alpha_0 + 3\alpha + 1)^2}{\alpha_*^2}\Big\}. \tag{37}$$

Then (36) is fulfilled. Besides an evaluation

$$|\Delta_1(\cdot)| \leq \alpha_0(2 + \sqrt{h_2}) + 3\alpha \triangleq \Delta_1^+ \tag{38}$$

is evident. Find $h_3$ from condition

$$\Delta_2(\cdot) \geq 1. \tag{39}$$

By Schur's lemma

$$\Delta_2(\cdot) = \Delta_1(\cdot)[-a_{23}(\cdot)\sqrt{h_2 h_3} + 3\alpha - \mu^2(\cdot)\Delta_1^{-1}(\cdot)],$$

where

$$\mu = a_{12}(\cdot)h_2 + a_{21}(\cdot) + 0,5(a_{11}(\cdot) + a_{22}(\cdot))\sqrt{h_2}.$$

In respect to (36) for fulfillment (39) it is sufficiently that an expression in square brackets were less or equal to $-1$. For this is sufficiently of fulfillment of inequality

$$a_{23}(\cdot)\sqrt{h_2 h_3} \geq 3\alpha + \mu^2(\cdot) + 1. \tag{40}$$

As $\mu^2(\cdot) \leq \alpha_0^2(1 + h_2 + \sqrt{h_2})^2$ then an evaluation (40) is fulfilled by

$$h_3 = \max\Big\{h_2,\ \frac{[3\alpha + 1 + \alpha_0^2(1 + h_2 + h_2^2)^2]^2}{\alpha_*^2 h_2}\Big\}. \tag{41}$$

It is easy to receive an evaluation

$$|\Delta_2(\cdot)| \leq \Delta_1^+(0,5\alpha_0\sqrt{h_2 h_3} + 3\alpha) + \\ + \alpha_0^2(1 + \sqrt{h_2} + h_2^2) \triangleq \Delta_2^+. \tag{42}$$

We will find $\lambda$ from condition

$$\det(L(\cdot) + 3\alpha I) < 0. \tag{43}$$

So the next formula is valid

$$\det(L(\cdot) + 3\alpha I) = \Delta_3(\cdot) + 2\lambda\Delta_2(\cdot), \tag{44}$$

where

$$\Delta_3(\cdot) = \begin{vmatrix} Q_2(\cdot) & q(\cdot) \\ q^*(\cdot) & \varkappa(\cdot) \end{vmatrix}, \quad q = \begin{pmatrix} \varkappa_1(\cdot) \\ \varkappa_2(\cdot) \end{pmatrix},$$

$$\begin{aligned} \varkappa(\cdot) &= -a_{23}(\cdot)\sqrt{h_2 h_3} + 2a_{33}(\cdot)h_3 + 3\alpha h_3, \\ \varkappa_1(\cdot) &= a_{31}(\cdot) - 0,5(a_{32}(\cdot)\sqrt{h_2} + a_{12}(\cdot)\sqrt{h_2 h_3}), \\ \varkappa_2(\cdot) &= -0,5a_{31}(\cdot) + h_2 a_{32}(\cdot) - \\ &\quad -0,5(a_{33}(\cdot) + a_{22}(\cdot))\sqrt{h_2 h_3} + a_{23}(\cdot)h_3, \end{aligned}$$

$$Q_2(\cdot) = \begin{pmatrix} \mu_1(\cdot) & \mu_2(\cdot) \\ \mu_2(\cdot) & \mu_3(\cdot) \end{pmatrix},$$

$$\begin{aligned} \mu_1(\cdot) &= 2a_{11}(\cdot) - a_{12}(\cdot)\sqrt{h_2} + 3\alpha, \\ \mu_2(\cdot) &= -(a_{11}(\cdot) + a_{22}(\cdot))\sqrt{h_2} + a_{12}(\cdot)h_2 + a_{21}(\cdot), \\ \mu_3(\cdot) &= -a_{21}(\cdot)\sqrt{h_2} + 2a_{22}(\cdot)h_2 - a_{23}(\cdot)\sqrt{h_2 h_3} + 3\alpha. \end{aligned}$$

It is evident the validness of formula

$$Q_2^{-1}(\cdot) = \frac{1}{\Delta_2(\cdot)}M(\cdot),$$

where

$$M(\cdot) = \begin{pmatrix} \mu_3(\cdot) & -\mu_2(\cdot) \\ -\mu_2(\cdot) & \mu_1(\cdot) \end{pmatrix}.$$

By Schur's lemma

$$\Delta_3(\cdot) = \Delta_2(\cdot)[\varkappa(\cdot) - q^*(\cdot)Q_2^{-1}(\cdot)q(\cdot)].$$

Here in respect to (42) next evaluation follows

$$|\Delta_3(\cdot)| \leq |\Delta_2(\cdot)|\,|\varkappa(\cdot)| + \|q(\cdot)\|^2\|M(\cdot)\| \leq \\ \leq \Delta_2^+|\varkappa(\cdot)| + (\varkappa_1^2(\cdot) + \varkappa_2^2(\cdot))\sqrt{\mu_1^2(\cdot) + \mu_3^2(\cdot) + 2\mu_2^2(\cdot)}.$$

It is evidently that evaluations

$$
\begin{aligned}
|\varkappa(\cdot)| &\le \alpha_0(\sqrt{h_2h_3}+2h_3)+3\alpha h_3 \triangleq \varkappa^+,\\
|\varkappa_1(\cdot)| &\le \alpha_0+0,5\alpha_0(\sqrt{h_2}+\sqrt{h_2h_3}) \triangleq \varkappa_1^+,\\
|\varkappa_2(\cdot)| &\le \alpha_0(0,5\sqrt{h_2}+h_2+\sqrt{h_2h_3}+h_3) \triangleq \varkappa_2^+,
\end{aligned}
$$

$$
\begin{aligned}
|\mu_1(\cdot)| &\le \alpha_0(2+\sqrt{h_2})+3\alpha \triangleq \mu_1^+,\\
|\mu_2(\cdot)| &\le \alpha_0(2+\sqrt{h_2}+h_2) \triangleq \mu_2^+,\\
|\mu_3(\cdot)| &\le \alpha_0(\sqrt{h_2}+2h_2+\sqrt{h_2h_3}) \triangleq \mu_3^+
\end{aligned}
$$

are valid. So

$$
\begin{aligned}
|\Delta_3(\cdot)| \le \Delta_2^+\varkappa^+ + ((\varkappa_1^+)^2+(\varkappa_2^+)^2)\times\\
\times\sqrt{(\mu_1^+)^2+(\mu_3^+)^2+2(\mu_2^+)^2} \triangleq \Delta_3^+.
\end{aligned}
$$

From this evaluation and (44), (39) it is follows that for validness (43) it is sufficiently to choise

$$\lambda < -\Delta_3^+. \tag{45}$$

So evaluations (21) and (23) received a form

$$
\begin{aligned}
|a_{13}(\cdot)| &\le \frac{\alpha}{4h_3},\\
\beta_1^2(\cdot)+\beta_2^2(\cdot)+\beta_3^2(\cdot) &\le \frac{\alpha^2}{16h_3\|s\|}.
\end{aligned} \tag{46}
$$

In this way by theorem 1 the system (34) is globally stable if $u = s^*x$, where $s$ is set by formula (15) and coefficients $a_{ij}(\cdot)$ and $\beta_i(\cdot)$ comply with evaluations (35), (46).

## 5 Conclusion

We considered the stabilization problem of such nonlinear continuous and impulse-modulator systems for which only the bound of variation of its coefficients were known. The linear scalar control is constructed which provides global stability of closed-loop system.


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