Assignable polynomials to a weakly reachable single-input system over a Bézout domain

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Abstract

We describe how can we obtain all assignable polynomials to a weakly reachable linear system over a Bézout domain in normal form by solving a system of linear diophantine equations. We also point out some applications.

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1 Introduction

A single input linear system over a commutative domain R (commutative ring with no zerodivisors) is just a pair $\Sigma = (A, \underline{b})$ where $A \in R^{n\times n}$ is a square matrix and $\underline{b} \in R^{n\times 1} = R^n$ is a column vector. For control theoretic affairs, this pair summarizes the sequential dynamical equation

$$\underline{x}(t+1) = A\underline{x}(t) + \underline{b}u(t)$$

where $\underline{x}(t)$ is the internal state at discrete time t and $u(t)$ is the scalar input we introduce in the system.

The design of $u = \underline{f}^t\underline{x}$ as a R-linear function of the states is the celebrated ”Feedback Action” which allows to stabilize linear systems in some cases: Closed loop $u = \underline{f}^t\underline{x}$ yields to the finite difference equation

$$\underline{x}(t+1) = (A + \underline{b}\underline{f}^t)\underline{x}(t)$$

and the behavior is defined by the characteristic polynomial

$$\chi(A + \underline{b}\underline{f}^t) = \det(z\mathbf{1} - (A + \underline{b}\underline{f}^t))$$

It is interesting to research what are the assignable polynomials to a given system $\Sigma = (A, \underline{b})$; that is to say, to describe the family

$$\text{Pols}((A, \underline{b})) = \{\chi(A + \underline{b}\underline{f}^t) : \underline{f}^t \in R^{1\times n}\}$$

Classical results show that every monic polynomial of degree n is in Pols$((A,\underline{b}))$ if and only the reachability matrix of system $\Sigma = (A,\underline{b})$

$$(A * \underline{b}) = (\underline{b}, A\underline{b}, A^2\underline{b}, ..., A^{n-1}\underline{b})$$

is invertible (see [1], Theorem 3.2). That is to say, if $\det(\underline{b}, A\underline{b}, A^2\underline{b}, ..., A^{n-1}\underline{b})$ is an unit of R.

A very few work is done if system $\Sigma = (A,\underline{b})$ is not reachable. In particular we are interested in the case of weakly reachable systems; which which is the case of $\det(\underline{b}, A\underline{b}, A^2\underline{b}, ..., A^{n-1}\underline{b}) \neq 0$. Weakly reachable linear systems are interesting because they are reachable in the field of fractions of domain R. For instance, if $R = \mathbb{Z}$ then weakly reachable linear systems over $\mathbb{Z}$ are reachable if we consider them over $\mathbb{Q}$.

Now we recall the property of that Pols(Σ) is a feedback invariant associated to system Σ. Recall that feedback actions on system $\Sigma = (A,\underline{b})$ are finite composition of basis changes $\underline{x}' = P\underline{x}$, $P \in GL_n(R)$ and closed loops $\underline{u} = \underline{u}' + \underline{f}^t\underline{x}$. Hence a general feedback action brings linear system $\Sigma = (A,\underline{b})$ to system $\Sigma = (P(A + \underline{b}\underline{f}^t)P^{-1}, P\underline{b})$.

But it is clear that

$$\text{Pols}((P(A + \underline{b}\underline{f}^t)P^{-1}, \underline{b})) = \{\chi(P(A + \underline{b}\underline{f}^t)P^{-1} + \underline{b}\underline{\phi}^t) : \underline{\phi}^t \in R^{1\times n}\} =$$

$$= \{\chi(A + \underline{b}\underline{f}^t + \underline{b}\underline{\phi}^t P) : \underline{\phi}^t \in R^{1\times n}\} = \text{Pols}((A,\underline{b}))$$

Hence we have proven the following result:

Lemma 1.1. *Pols*(Σ) *is a feedback invariant associated to system* Σ.

Note that for reachable single input systems we have that every reachable single input linear system is feedback equivalent to the Canonical Controller Form (see [1] Theorem 3.2):

$$\Sigma^\flat = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and it is now clear that every monic polynomial can be assigned to a reachable single input system, because every monic polynomial can be easily assigned to the canonical controller form $\Sigma^\flat$.

The paper is organized as follows:

2 The canonical form

With the Canonical Controller Form in mind, a Canonical Form for the weakly reachable case over a Bézout domain is introduced in [2]. For reader's convenience we recall that a

Bézout domain is a domain such that every finitely generated ideal is principal. Essentially we may think in a Bezout domain as a principal ideal domain that fails to be Noetherian. Of course every principal ideal domain (a field, $\mathbb{Z}$, $k[t]$,...) is a Bezout domain but the converse is not true (see Note 4.1). The Canonical Form is the following.

Theorem 2.1. *(cf. [2]Theorem 2.5) Let R be a Bezout domain. Let $\Sigma = (A, \underline{b})$ be a single input n-dimensional linear system over R. If Σ is weakly reachable then there exist nonzero elements $d_1, ..., d_n$ of R such that Σ is feedback equivalent to the system Σ^Δ given by:*

$$\Sigma^\Delta = (A^\Delta, \underline{b}^\Delta) = \left(\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ d_2 & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & d_3 & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{n-1} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & d_n & a_{nn} \end{pmatrix}, \begin{pmatrix} d_1 \\ 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix} \right)$$

This canonical form is obtained by performing $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$ Bezout identities in R (see [2] for details).

Example 2.2. Running Algorithms in [2] on the linear system over $\mathbb{Z}$ given by

$$\Sigma = \begin{pmatrix} -4 & 9 & 5 \\ 7 & 1 & -16 \\ -2 & 4 & -9 \end{pmatrix}, \begin{pmatrix} 20 \\ 0 \\ 8 \end{pmatrix}$$

one obtains the canonical form

$$\Sigma^\Delta = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

3 Effective calculation of the determinants

In this section we give a recursive procedure in order to obtain the characteristic polynomial of square matrix A and all polynomials reached by feedback from the linear system $\Sigma = (A, \underline{b})$ in canonical form.

Let A be the square $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ d_2 & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & d_3 & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{n-1} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & d_n & a_{nn} \end{pmatrix}$$

In order to obtain the above family $\mathrm{Pols}(A, \underline{b})$ of assignable polynomials to system $(A, \underline{b})$, denote by $A_{i:n}$ the square $(n-i+1) \times (n-i+1)$ submatrix of A containing the last $(n-i+1)$ rows and $(n-i+1)$ columns. That is to say:

$$A_{i:n} = \begin{pmatrix} a_{ii} & a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,n-1} & a_{in} \\ d_{i+1} & a_{i+1,i+1} & a_{i+1,i+2} & \cdots & a_{i+1,n-1} & a_{i+1,n} \\ 0 & d_{i+2} & a_{i+2,i+2} & \cdots & a_{i+2,n-1} & a_{i+2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{n-1} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & d_n & a_{nn} \end{pmatrix}$$

With this notations one has that

Theorem 3.1. *With the above notations we have:*

$$\chi(A + \underline{b}\underline{f}^t) = \chi(A^\Delta) - f_1 d_1 \chi(A^\Delta_{2:n}) + f_2 d_1 d_2 \chi(A^\Delta_{3:n}) - \cdots$$

$$\cdots + (-1)^{n-1} f_{n-1} d_1 \cdots d_{n-1} \chi(A^\Delta_{n:n}) + (-1)^n f_n d_1 \cdots d_n$$

Proof.-

$$\chi(A+\underline{b}\underline{f}^t) = \begin{vmatrix} z-(a_{11}+d_1 f_1) & -(a_{12}+d_1 f_2) & \cdots & -(a_{1,n-1}+d_1 f_{n-1}) & -(a_{1n}+d_1 f_n) \\ -d_2 & z-a_{22} & \cdots & -a_{2,n-1} & -a_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & -d_{n-1} & z-a_{n-1,n-1} & -a_{n-1,n} \\ 0 & \cdots & 0 & -d_n & z-a_{nn} \end{vmatrix} =$$

$$= \chi(A) - d_1 \cdot \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ -d_2 & z-a_{22} & -a_{23} & \cdots & -a_{2,n-1} & -a_{2n} \\ 0 & -d_3 & z-a_{33} & \cdots & -a_{3,n-1} & -a_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d_{n-1} & z-a_{n-1,n-1} & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 & -d_n & z-a_{nn} \end{vmatrix} =$$

$$= \chi(A) - d_1 f_1 \chi(A_{2:n}) + d_1 d_2 \cdot \begin{vmatrix} f_2 & f_3 & f_4 & \cdots & f_{n-1} & f_n \\ -d_3 & z-a_{33} & -a_{34} & \cdots & -a_{3,n-1} & -a_{3n} \\ 0 & -d_4 & z-a_{44} & \cdots & -a_{4,n-1} & -a_{4n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d_{n-1} & z-a_{n-1,n-1} & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 & -d_n & z-a_{nn} \end{vmatrix}$$

and the result follows from induction □

Now the set of assignable polynomials for a single input is given in terms of some indeterminates freely chosen by the controller (the f_i's) and of some fixed polynomials (the $\chi(A_{i:n})$). All these polynomials can be obtained by a recursive method from $\chi(A_{n:n})$ to $\chi(A_{2:n})$ and $\chi(A)$.

Theorem 3.2. *With the above notations we have*

$$\chi(A_{i:n}) = (z - a_{i,i})\chi(A_{i+1:n}) - d_{i+1}a_{i,i+1}\chi(A_{i+2:n}) + d_{i+1}d_{i+2}a_{i,i+2}\chi(A_{i+3:n}) - \cdots$$
$$\cdots + (-1)^{n-i-1}d_{i+1}\cdots d_{n-1}a_{i,n-1}\chi(A_{n:n}) + (-1)^{n-i}d_{i+1}\cdots d_n a_{i,n}$$

Proof.-

$$\chi(A_{i:n}) = \begin{vmatrix} z-a_{ii} & -a_{i,i+1} & -a_{i,i+2} & \cdots & -a_{i,n-1} & -a_{in} \\ -d_{i+1} & z-a_{i+1,i+1} & -a_{i+1,i+2} & \cdots & -a_{i+1,n-1} & -a_{i+1,n} \\ 0 & -d_{i+2} & z-a_{i+2,i+2} & \cdots & -a_{i+2,n-1} & -a_{i+2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d_{n-1} & z-a_{n-1,n-1} & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 & -d_n & z-a_{nn} \end{vmatrix} =$$

$$= \begin{vmatrix} z-a_{ii} & 0 & 0 & \cdots & 0 & 0 \\ -d_{i+1} & z-a_{i+1,i+1} & -a_{i+1,i+2} & \cdots & -a_{i+1,n-1} & -a_{i+1,n} \\ 0 & -d_{i+2} & z-a_{i+2,i+2} & \cdots & -a_{i+2,n-1} & -a_{i+2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d_{n-1} & z-a_{n-1,n-1} & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 & -d_n & z-a_{nn} \end{vmatrix} +$$

$$+ \begin{vmatrix} 0 & -a_{i,i+1} & -a_{i,i+2} & \cdots & -a_{i,n-1} & -a_{in} \\ -d_{i+1} & z-a_{i+1,i+1} & -a_{i+1,i+2} & \cdots & -a_{i+1,n-1} & -a_{i+1,n} \\ 0 & -d_{i+2} & z-a_{i+2,i+2} & \cdots & -a_{i+2,n-1} & -a_{i+2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d_{n-1} & z-a_{n-1,n-1} & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 & -d_n & z-a_{nn} \end{vmatrix} = \cdots$$

and the result follows from induction □

Note that it would be possible to write down explicit formulae for $\chi(A)$ and for $\chi(A + \underline{b}\underline{f}^t)$ by recursive substitution of $\chi(A_{i:n})$ as a function of $\chi(A_{i+1:n}), \ldots, \chi(A_{n:n})$. However we think this method is not adequate and consequently it is recommended to perform the recursive calculation. Next we solve the problem for system in Example 2.2:

Example 3.3. Put the Canonical form over $\mathbb{Z}$:

$$\Sigma = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

Then, by Theorem 3.2:

$$\begin{array}{rcccc} \chi(A_{3:3}) & = & (z-1) & & \\ \chi(A_{2:3}) & = & (z-1)\chi(A_{3:3}) - 2\cdot 1 & = & z^2 - 2z - 1 \\ \chi(A) & = & (z-2)\chi(A_{2:3}) - 3\cdot 3\cdot \chi(A_{3:3}) & = & z^3 - 4z^2 - 6z + 11 \end{array}$$

and by Theorem 3.1:

$$\text{Pols}(A, \underline{b}) =$$

$$= \{(z^3 - 4z^2 - 6z + 11) - 4f_1(z^2 - 2z - 1) + 12f_2(z-1) - 24f_3 : f_i \in R\}$$

Thus a polynomial $z^3 + a_1 z^2 + a_2 z + a_3$ is assignable to system $(A, \underline{b})$ by feedback if and only if the following system of diophantine equations is solvable

$$\begin{pmatrix} a_1 + 4 \\ a_2 + 6 \\ a_3 - 11 \end{pmatrix} = \begin{pmatrix} (-4) & 0 & 0 \\ (-4)(-2) & 12 & 0 \\ (-4)(-1) & 12(-1) & (-24) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Note that all columns in the system of equations of above Example are obtained from $\chi(A_{i:3}))$ in a straightforward way. On the other hand, diagonal elements in coefficient matrix are $d_1, d_1 d_2, ..., d_1 \cdots d_n$; i.e. the sequence of diagonal invariants obtained in [2].

Note 3.4. Of course the main task here is to obtain a procedure to obtain the linear system of diophantine equations directly from a linear system (perhaps without obtaining the Canonical Form). This is our next research challenge in the field.

4 Concluding Remarks

Note 4.1. Bézout domains. A Bézout Domain is a domain such that every finitely generated ideal is principal. The more usual examples of Bézout domains are Principal Ideal Domains as $\mathbb{Z}$ or $\mathbb{K}[x]$. These Bézout domains are noetherian and hence principal ideal domains, moreover they are also Euclid domains and hence the usual division algorithm allows us to perform computations in order to obtain canonical forms.

The ring $\mathcal{H}(\Omega)$ of holomorphic complex functions defined in a domain $\Omega \subseteq \mathbb{C}$ is also a Bézout domain (see [2, 15.3.3]) hence our method works in this case. But $\mathcal{H}(\Omega)$ is not an Euclid domain (it is not even a principal ideal domain because is not noetherian), thus a Division Theorem is needed in order to obtain efficiently the canonical form and to get the assignable polynomials.

Note 4.2. Change of scalars. Let R be any commutative ring and $\Sigma = (A, \underline{b})$ a single input linear system. A ring homomorphism $f : R \to S$ defines a new system $f^*\Sigma = (f(A), f(\underline{b}))$ as extension of scalars from R to S via f, where $f(A) = f(a_{ij})$ if $A = (a_{ij})$.

If $R = \mathbb{Z}$ and $f : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the canonical quotient map then the extension of scalars is just the residual study of linear systems modulo m. Some results are easily given; for instance if m is coprime with all invariants d_i in the Canonical Form of Σ then every monic polynomial can be assigned modulo m. Otherwise, chances of assign polynomials decrease. In the critical case of d_1 is a divisor of m then no polynomial can be assigned.

If $R = \mathbb{C}[x]$ is the ring of complex polynomials in a single indeterminate then residual study at a point $\omega \in \mathbb{C}$ is performed by using the extension of scalars given by the homomorphism $ev_\omega : \mathbb{C}[x] \to \mathbb{C}$ sending $p(x) \mapsto p(\omega)$. Local study around a point $\omega \in \mathbb{C}$ may be performed by using the canonical inclusion $f : \mathbb{C}[x] \to \mathbb{C}[x]_{(x-\omega)}$ where $\mathbb{C}[x]_{(x-\omega)}$ is the localization of $\mathbb{C}[x]$ at maximal ideal $< (x-\omega) >$. Here if $d_i(\omega) \neq 0$ then every polynomial can be assigned to Σ in a *small* neighborhood of $\omega \in \mathbb{C}$.

In the case $R = \mathcal{H}(\Omega)$ this covers the study at a neighborhood of a point $\omega \in \Omega$. From this point of view it is clear that if

$$\prod_{i=1}^{n} d_i(\omega) \neq 0$$

then every polynomial can be assigned to Σ.

On the other hand, if z is a zero of some d_i then not every polynomial can be assigned. But even in this case we will be able to estimate the set of assignable polynomials.

The extremal case is of course when $d_1(z) = 0$: In this case no polynomial (different from $\chi(A)$) can be assigned.

Example 4.3. Consider the Canonical Form

$$\Sigma = \begin{pmatrix} 0 & 0 \\ (x-1) & 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}$$

over $\mathbb{C}[x]$ (or over $\mathcal{H}(\mathbb{C})$):

- Residual study:
  - The evaluation homomorphism $ev_\omega : \mathbb{C}[x] \to \mathbb{C}$ at any point $\omega neq 0, 1$ yields that we can (residually) assign any monic polynomial.
  - At point $\omega = 0$ no polynomial different from z^2 can be assigned.
  - At point $\omega = 1$ only polynomials on the form $z^2 + \alpha z$ may be assigned.
- Local study:
  - The localization homomorphism $f_\omega : \mathbb{C}[x] \to \mathbb{C}[x]_{(x-\omega)}$ at any point $\omega neq 0, 1$ yields that we can (locally) assign any monic polynomial. In fact the change of basis $P = \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{x(x-1)} \end{pmatrix}$ brings system to the Canonical Controller Form

- Localization at $\omega = 0$ yields that x is not invertible in $\mathbb{C}[x]_{(x)}$. However any polynomial on the form $z^2 + x(f_1 z + f_2)$ can be assigned.
- Localization at $\omega = 1$ yields that $x - 1$ is not invertible in $\mathbb{C}[x]_{(x)}$, but x is and hence any polynomial on the form $z^2 + f_1 z + f_2(x-1)$ may be assigned.

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