MEAN SQUARE STABILITY AND STABILIZATION FOR STOCHASTIC NONLINEAR OSCILLATIONS

Lev Ryashko *

* *Department of Mathematics, Ural State University, Ekaterinburg, Russia*

Abstract: An exponential mean square stability for the limit cycles of nonlinear stochastic systems is considered. The first approximation linear systems are introduced and a notion of P-stability (projective) is proposed. A spectral criterion for P-stability is obtained. Mean square stabilization of periodic solutions of stochastically forced nonlinear systems is considered. The necessary and sufficient conditions of stabilizability are presented. The possibilities of constructive design of stabilizing regulator for 2D limit cycles are demonstrated.

Keywords: Control, limit cycles, stochastic stability, stabilization

1. INTRODUCTION

Many nonlinear phenomena of mechanics observed under transition from the order to chaos are frequently connected with a chain of bifurcations: a stationary regime (equilibrium point) – periodic regime (limit cycle) – chaotic regime (strange attractor). Each such transition is accompanied by the loss of stability of simple attractor and new more complicated stable attractor birth. Stability analysis of appropriate invariant manifolds is key for understanding of the complex behavior of nonlinear dynamical systems. The stability investigation and control of stochastic systems are attractive from theoretical and engineering points of view. Even weak noise can result in qualitative changes in the system's dynamics. We consider the mean square stability problem for limit cycles of stochastic differential equations. One of the most important directions of stability analysis is Lyapunov function technique (LFT) (Krasovskii (1963); Kats and Krasovskii (1960); Khasminskii (1980); Kushner (1967)). LFT in research of a stationary point stochastic stability has been widely studied by many authors (see Arnold (1974); Arnold (1998); Mao (1994); Khasminskii (1980)).

The orbital Lyapunov functions were used in stability and sensitivity analysis via a quasipotential of stochastic forced limit cycles (Ryashko (1996); Bashkirtseva and Ryashko (2002); Bashkirtseva and Ryashko (2004)).

The aim of this work is to present foundations of stability analysis for stochastically forced nonlinear oscillations and its applications to stabilization problem.

The first approximation linear systems for limit cycles are introduced and a notion of P-stability (projective) is proposed. A general criterion for P-stability is obtained. The stochastic stability analysis is reduced to the estimation of the spectral radius of some positive operator.

For important case of limit cycle in 2-dimensional stochastic system the parametric criterion is given.

[1] Partially supported by the RFBR grants 06-01-00625, 06-08-00396

This stochastic stability criterion allows to solve relevant stabilization problem effectively. The necessary and sufficient stabilizability conditions are presented. The example of constructive solving of stabilization problem for stochastic periodic regime is demonstrated. As shown, this approach gives the useful analytical tool for analysis and control of thin effects observed in nonlinear stochastic models.

2. STOCHASTIC STABILITY OF LIMIT CYCLES

Consider a deterministic nonlinear system

$$dx \; = \; f(x)\, dt \tag{1}$$

where x is n-vector, $f(x)$ is sufficiently smooth vector-function of the appropriate dimension. It is assumed that system (1) has a T-periodic solution $x = \xi(t)$ – limit cycle M.

A standard model for random forced deterministic system (1) is a system of Ito's stochastic differential equations

$$dx \; = \; f(x)dt \; + \; \sum_{r=1}^{m} \sigma_r(x) dw_r(t), \tag{2}$$

where $w_r(t)\,(r = 1, ..., m)$ are independent standard Wiener processes, $\sigma_r(x)$ are sufficiently smooth vector-functions of the appopriate dimension. To ensure cycle M is an invariant of stochastic system (2) we assume

$$\sigma_r|_M \; = \; 0 \tag{3}$$

Definition 1. The cycle M is called exponentially stable in the mean square sense (EMS-stable) for the system (2) in neighborhood U if there exist $K > 0$, $l > 0$ such that

$$E||\Delta(x(t))||^2 \; \leq \; Ke^{-lt} E||\Delta(x_0)||^2$$

where $x(t)$ is a solution of system (2) with initial condition $x(0) = x_0 \in U$.

Here $\Delta(x) \; = \; x - \gamma(x)$ is a deviation of a point x from a cycle M, $\gamma(x)$ is the point on cycle M that is nearest to x. It is assumed that a neighborhood U is invariant for systems (1), (2).

Consider for (1), (2) the corresponding first approximation systems

$$dz = F(t)zdt \tag{4}$$

$$dz = F(t)zdt + \sum_{r=1}^{m} S_r(t) z dw_r(t) \tag{5}$$

with T-periodic coefficients

$$F(t) = \frac{\partial f}{\partial x}(\xi(t)), \quad S_r(t) = \frac{\partial \sigma_r}{\partial x}(\xi(t)).$$

Due to (3), matrix functions $S_r(t)$ are singular: $S_r(t) f(\xi(t)) \equiv 0$.

Solution $z = 0$ of systems (4), (5), because of presence of the solution $z = f(\xi(t))$ can not be exponentially mean square stable in standard sense. Here more weak analog of exponential stability defined with the help of a projector $P(t) = P_{f(\xi(t))}$, $P_r = I - \frac{rr^\top}{r^\top r}$ is considered.

Definition 2. The solution $z = 0$ of system (5) is called exponentially P-stable in the mean square sense (system (5) is P-stable for short) if there exist $K > 0$, $l > 0$ such that

$$\mathrm{E}\|P(t)z(t)\|^2 \; \leq \; Ke^{-lt}\, \mathrm{E}\|P(0)z_0\|^2$$

for any solution $z(t)$ of system (5) with initial conditions $z(0) = z_0 \in R^n$.

Consider a space Σ of symmetrical $n \times n$-matrix functions $V(t)$ defined and sufficiently smooth on R^1 and satisfying periodicity $V(t+T) = V(t)$ and singularity $V(t) f(\xi(t)) = 0$ conditions.

Definition 3. A matrix $V(t) \in \Sigma$ is called P-positive definite if

$$\forall\, t \in R^1 \quad \forall\, z \in R^n \quad P(t)z \neq 0 \Rightarrow (z, V(t)z) > 0.$$

In space Σ we shall consider a cone $\mathcal{K}$ of nonnegative definite matrices and set

$$\mathcal{K}_P = \{V \in \Sigma | V \text{ is } P - \text{positive definite}\}.$$

Theorem 1. The following statements are equivalent:

(a) Cycle M for system (2) is EMS-stable;
(b) System (5) is P-stable;
(c) For any matrix $W \in \mathcal{K}_P$ equation

$$\mathcal{L}[V] \; = \dot{V} + F^\top V + VF + \sum_{r=1}^{m} S_r^\top V S_r = \; -W \tag{6}$$

has unique matrix solution $V \in \mathcal{K}_P$.

2.1 A spectral stability criterion

Theorem 1 reduces a problem of cycle M stability to analysis of equation $\mathcal{L}[V] \; = \; -W$ decision problem in the space of $P-$positive definite matrices $\mathcal{K}_P$.

It is difficult to analyze the system stability by direct investigation of decision problem for matrix Lyapunov equation especially in cases close to critical. Here we shall consider an extension of the effective criteria (Ryashko (1979); Ryashko (1981); Ryashko (1999)) based on positive operators spectral theory (Krasnosel'skii *et al.* (1985)).

Represent the operator $\mathcal{L}$ from (6) in the form

$$\mathcal{L} \; = \; \mathcal{A} + \mathcal{S},$$

where

$$\mathcal{A}[V] = \dot{V} + F^\top V + VF,$$

$$S[V] \; = \; \sum_{r=1}^{m} S_r^\top V S_r.$$

Consider the operator $\mathcal{P} = -\mathcal{A}^{-1}\mathcal{S}$.

Theorem 2. The stochastic system (5) is P-stable if and only if it holds that
(a) The deterministic system (5) is P-stable,
(b) The inequality $\rho(\mathcal{P}) < 1$ holds.

Remark 1. Spectral radius $\rho = \rho(\mathcal{P}) \neq 0$ defines bifurcation value $\varepsilon^* = \sqrt{1/\rho}$ of random noises intensity $\varepsilon \geq 0$ for a system

$$dx \; = \; f(x)dt \; + \; \varepsilon \sum_{r=1}^{m} \sigma_r(x) dw_r(t).$$

The cycle M for this system is EMS-stable for any $\varepsilon < \varepsilon^*$ and is unstable for any $\varepsilon \geq \varepsilon^*$. Case $\rho = 0$ means this system is stable for any $\varepsilon \geq 0$.

Remark 2. If one can not find spectral radius ρ exactly then its estimations $\rho_1 < \rho < \rho_2$ may be useful. Actually, the inequality $\rho_2 < 1$ gives sufficient and $\rho_1 < 1$ gives necessary stability condition.

3. STABILITY OF 2D LIMIT CYCLE

In the case $n = 2$ one can find for spectral radius of operator $\mathcal{P}$ the following explicit representation

$$\rho(\mathcal{P}) = -\frac{\langle \beta \rangle}{\langle \alpha \rangle}$$

Here

$$\alpha(t) = p(t)^\top [F^\top(t) + F(t)] p(t),$$

$$\beta(t) = p^\top(t) \left(\sum_{r=1}^{m} S_r(t) S_r^\top(t) \right) p(t),$$

$p(t)$ is a vector orthonormal to limit cycle M at a point $\xi(t)$, brackets $\langle \cdot \rangle$ mean time averaging

$$\langle \alpha \rangle \; = \; \frac{1}{T} \int_0^T \alpha(t) dt.$$

Inequality (famous Poincare criterion)

$$\langle \alpha \rangle < 0$$

is necessary and sufficient condition of exponential stability of limit cycle M for the deterministic system (1). Thus, the inequality $\rho(\mathcal{P}) < 1$ written as

$$\langle \alpha + \beta \rangle < 0. \tag{7}$$

is necessary and sufficient condition of EMS-stability of cycle M for stochastic system (2) in 2D-case (Ryashko (1996)).

4. STABILIZATION

Consider controlled deterministic system

$$dx \; = \; f(x,u) \; dt, \tag{8}$$

and corresponding stochastic system

$$dx \; = \; f(x,u)dt \; + \; \sum_{r=1}^{m} \sigma_r(x,u) dw_r(t), \tag{9}$$

where x is n-dimensional state variable, u is l-dimensional vector of control functions, $f(x,u)$, $\sigma_r(x,u)$ are vector functions of the appropriate dimension, $w_r(t)$ $(r = 1, ..., m)$ are independent standard Wiener processes. It is supposed that for $u = 0$ cycle M is invariant for system (8). Under condition

$$\sigma_r(x,0)|_M \; = \; 0. \tag{10}$$

for $u = 0$ cycle M is invariant for system (9) too. Our stabilization problem is to design a regulator guaranteeing EMS-stability of invariant cycle M for system (9).

The stabilizing regulator we shall select from the class $\mathcal{F}$ of admissible feedbacks $u = u(x)$ satisfying conditions:
(a) $u(x)$ is sufficiently smooth and $u|_M = 0$;
(b) for the deterministic system

$$dx \; = \; f(x,u)dt$$

the cycle M is exponentially stable in the neighborhood U of M.

The analysis of stabilization problem of cycle M for $u \in \mathcal{F}$ is connected with investigation of the first approximation systems

$$dz = (A(t) + B(t)K(t))zdt, \tag{11}$$

$$\begin{aligned} dz = {} & (A(t) + B(t)K(t))zdt + \\ & + \sum_{r=1}^{m} (C_r(t) + H_r(t)K(t))zdw_r(t), \end{aligned} \tag{12}$$

where

$$A(t) = \frac{\partial f}{\partial x}(\xi(t), 0), \quad B(t) = \frac{\partial f}{\partial u}(\xi(t), 0),$$

$$C_r(t) = \frac{\partial \sigma_r}{\partial x}(\xi(t), 0), \; H_r(t) = \frac{\partial \sigma_r}{\partial u}(\xi(t), 0)$$

and $K(t) = \dfrac{\partial u}{\partial x}(\xi(t))$ are T-periodic matrices. Here condition (10) looks like

$$K(t)P(t) \equiv K(t), \;\; P(t) = P_{f(\xi(t))}. \tag{13}$$

Consider Taylor's expansion of control function $u(x)$ at a point γ

$$u(x) = u(\gamma) + \frac{\partial u}{\partial x}(\gamma)(x - \gamma) + O(\|x - \gamma\|^3).$$

For $\gamma = \gamma(x) \in M$ we get

$$u(x) = \frac{\partial u}{\partial x}(\gamma(x))\Delta(x) + O(\|\Delta(x)\|^3).$$

As we see, a first approximation control function near M for small deviations $\Delta(x) = x - \gamma(x)$ is the feedback

$$u_1(x) = \frac{\partial u}{\partial x}(\gamma(x))\Delta(x).$$

As it follows from (12), the stabilization capabilities of control u are completely determined by first approximation $u_1(x)$ of a function $u(x)$ and are independent on higher order terms. It allows to restrict our consideration by simpler regulators in the following form

$$u(x) = K(t(\gamma(x)))\Delta(x), \tag{14}$$

where $t(x)$ is inverse function for $x = \xi(t)$. Regulator (14) is feedback for deviation $\Delta(x) = x - \gamma(x)$ of state x for system (9) from cycle M.

Stabilizing possibilities of this regulator are defined by $l \times n$-matrix T-periodic function $K(t)$, satisfying (13). If for some matrix function $K(t)$ the cycle M of system (9), (14) is EMS-stable then system (9) is called *stabilized,* and regulator (14) is called *stabilizing regulator*.

Consider set of feedback matrices

$$\mathbf{K} = \{K(t) |\ \text{system (11) is P} - \text{stable}\}$$

and operators

$$\mathcal{A}_K[V] = \dot{V} + (A + BK)^\top V + V(A + BK),$$

$$\mathcal{S}_K[V] = \sum_{r=1}^{m} (C_r + H_r K)^\top V (C_r + H_r K),$$

$$\mathcal{P}_K = -\mathcal{A}_K^{-1}\mathcal{S}_K.$$

Theorem 3. The system (9) with feedback (14) is stabilized if and only if it holds that

(a) $\mathbf{K} \neq \emptyset$,

(b) The inequality $\inf_{K \in \mathbf{K}} \rho(\mathcal{P}_K) < 1$ holds.

The feedback (14) with $K \in \mathbf{K}$ stabilizes the stochastic system (9) if inequality $\rho(\mathcal{P}_K) < 1$ holds.

This Theorem reduces stabilization problem to minimization of operator $\mathcal{P}_K$ spectral radius.

4.1 Stabilization of 2D cycle.

For the case of cycle on a plane ($n = 2$) from (13) it follows that $\operatorname{rank} K(t) \leq 1$. It gives us factorization: $K(t) = k(t)p^\top(t)$. Here $k(t)$ is l-vector-function, $p(t)$ is a vector orthonormal to limit cycle M at a point $\xi(t)$. So the regulator (14) can be written as

$$u = k(t(\gamma(x)))\delta(x), \tag{15}$$

where $\delta(x) = p^\top(t(\gamma(x)))\Delta(x)$.

Criterion (7) for system (12) with $K(t) = k(t)p^\top(t)$ can be written in the following form

$$J_k < 0, \tag{16}$$

where

$$J_k = \langle \alpha_k(t) + \beta_k(t) \rangle. \tag{17}$$

Here

$$\alpha_k(t) = p^\top(t)\left(A^\top(t) + A(t) + p(t)k^\top(t)B^\top(t) + \right.$$
$$\left. + B(t)k(t)p^\top(t)\right)p(t) = \alpha(t) + 2b^\top(t)k(t),$$

$$\beta_k(t) = \sum_{r=1}^{m} p^\top(t)\left(C_r(t) + H_r(t)k(t)p^\top(t)\right) \times$$
$$\times \left(p(t)k^\top(t)H_r^\top(t) + C_r^\top(t)\right)p(t) =$$
$$= \sum_{r=1}^{m} \left(p^\top(t)C_r(t)C_r^\top(t)p(t) + \right.$$
$$+ 2p^\top(t)C_r(t)p(t)p^\top(t)H_r(t)k(t) +$$
$$\left. + k^\top(t)H_r^\top(t)p(t)p^\top(t)H_r(t)k(t)\right) =$$
$$= \beta(t) + 2c^\top(t)k(t) + k^\top(t)H(t)k(t),$$

$$\alpha(t) = p^\top(t)\left(A^\top(t) + A(t)\right)p(t), \quad b(t) = B^\top(t)p(t),$$

$$\beta(t) = \sum_{r=1}^{m} p^\top(t)C_r(t)C_r^\top(t)p(t),$$

$$c(t) = \sum_{r=1}^{m} p^\top(t)C_r(t)p(t)H_r^\top(t)p(t),$$

$$H(t) = \sum_{r=1}^{m} H_r^\top(t)p(t)p^\top(t)H_r(t).$$

Thus the functional J_k from (17) has an explicit representation

$$J_k = \langle \alpha(t) + \beta(t) + 2(b(t) + c(t))^\top k(t) + k^\top(t)H(t)k(t) \rangle. \tag{18}$$

It allows from inequality (16) to choose coefficient $k(t)$ of stabilizing regulator (15) constructively.

5. CONCLUSION

Mean square stability analysis of the limit cycles of nonlinear stochastic systems was developed. Criterion of P-stability allows (see Theorem 1) to investigate nonlinear systems stability using the first approximation linear systems. A spectral variant of P-stability criterion (see Theorem 2) is useful tool for constructive analysis for stabilization problem of limit cycles (see Theorem 3).

Acknowledgment. This work was partially supported by grants RFBR 06-01-00625, 06-08-00396.

REFERENCES

Arnold, L. (1974). *Stochastic Differential Equations: Theory and Applications.* New York: Wiley.

Arnold, L. (1998). *Random Dynamical Systems.* Berlin: Springer.

Bashkirtseva, I. A. and L. B. Ryashko (2002). Sensitivity analysis of stochastically forced Lorenz model cycles under period doubling bifurcations. *Dynamic Systems and Applications*, 11, 293–310.

Bashkirtseva, I. A. and L. B. Ryashko (2004). Stochastic sensitivity of 3D-cycles. *Mathematics and Computers in Simulation*, 66, 55–67.

Kats, I. Ja. and N. N. Krasovskii (1960). The stability of systems with random parameters. *Prikl. Mat. Mech*, 24, 809–23.

Khasminskii, R. Z. (1980). *Stochastic stability of differential equations.* Sijthoff & Noordhoff, Alpen aan den Rijn.

Krasnosel'skii, M. A., E. A. Lifshits and A. V. Sobolev (1985). *Positive Linear Systems. The method of positive operators.* Nauka, Moscow.

Krasovskii, N. N. (1963) *Stability of Motion.* CA:Stanford University Press, Stanford.

Kushner, H. J. (1967). *Stochastic stability and control.* Academic Press, New York.

Mao, X. (1994). *Exponential stability of stochastic differential equations.* Marcel Dekker.

Ryashko, L. B. (1979). Stabilization of linear stochastic systems with state and control-dependent perturbations. *Prikl. Mat. Mekh.*, 43, 612–620.

Ryashko, L. B. (1981). Stabilization of linear systems with multiplicative noise under incomplete information. *Prikl. Mat. Mekh.*, 45, 778–786.

Ryashko, L. B. (1996). The stability of stochastically perturbed orbital motions. *J.Appl.Math.Mech.*, 60, 579–90.

Ryashko, L. B. (1999). Stability and stabilization of SDEs with periodic coefficients. *Dynamic Systems and Applications*, 8, 21–34.