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FLATNESS-BASED ADAPTIVE FUZZY CONTROL OF CHAOTIC FINANCE DYNAMICS

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Abstract

A flatness-based adaptive fuzzy control is applied to the problem of stabilization of the dynamics of a chaotic finance system, describing interaction between the interest rate, the investment demand and the price exponent. First it is proven that the system is differentially flat. This implies that all its state variables and its control inputs can be expressed as differential functions of a specific state variable, which is a so-called flat output. It also implies that the flat output and its derivatives are differentially independent which means that they are not connected to each other through an ordinary differential equation. By proving that the system is differentially flat and by applying differential flatness diffeomorphisms, its transformation to the linear canonical (Brunovsky) is performed. For the latter description of the system, the design of a stabilizing state feedback controller becomes possible. A first problem in the design of such a controller is that the dynamic model of the finance system is unknown and thus it has to be identified with the use of nonlinear regressors, among which neurofuzzy approximators are known to be very accurate. The estimated dynamics provided by the approximators is used in the computation of the control input, thus establishing an indirect adaptive control scheme. The learning rate of the approximators is chosen from the requirement the system's Lyapunov function to have always a negative first-order derivative. Another problem that has to be dealt with is that the control loop is implemented only with the use of output feedback. To estimate the non-measurable state vector elements of the finance system, a state observer is implemented in the control loop. The computation of the feedback control signal requires the solution of two algebraic Riccati equations at each iteration of the control algorithm. Lyapunov stability analysis demonstrates first that an H-infinity tracking performance criterion is satisfied. This signifies elevated robustness against modelling errors and external perturbations. Moreover, the global asymptotic stability is proven for the control loop.

Key words
Chaotic finance dynamics, flatness-based control, adaptive control, global linearization, output feedback, neurofuzzy approximators, asymptotic stability.

1 Introduction

The problem of control and synchronization of chaotic dynamics is a non-trivial one [Andrievsky, 2016], [Guzenko et al., 2013]. Chaotic dynamics is often apparent in finance and results in random-like variations of the parameters and variables of markets while also annihilating financial stability conditions [Guegan, 2009], [Lorenz, 1993], [Chian, 2000], [Haas, 1998], [Holyst et al., 1996], [Holyst et al., 2000], [Serrano et al., 2012], [Fanti et al., 2007]. Systematic approaches for the control of financial systems exhibiting chaotic dynamics have been first presented in [Rigatos, 2017]. To harness chaotic dynamics and to develop methods that stabilize chaotic finance systems, much work has been done during the last years. One can note results on chaotic finance systems synchronization [Cai et al., 2013], [Cai et al., 2012], [Zhao et al, 2011], [Vargas et al., 2015]. Systems' theoretic results have been used in [Chen et al., 2014], [Chian et al., 2006], [Danca et al., 2013], [Xin et al., 2015], [Wang et al., 2011], [Ma and

Chen, 2001]. for analyzing the dynamics of chaos in finance. Moreover, methods for feedback control and stabilization of chaotic systems appearing in finance have been given in [Yu et al., 2012], [Andrievskii and Fradkov, 2004], [Zhao and Wang, 2014], [Chen, 2008], [Wang et al., 2012].

Elaborating on the developments of [Rigatos, 2017], this article presents an adaptive fuzzy control method for a chaotic finance system that shows interaction between variables such as the interest rate, the investment demand and the price exponent. The method is based on differential flatness theory and on diffeomorphisms (change of state variables) which allow transformation of the initial nonlinear description of the system, into an equivalent linear form. Moreover, the method is implemented only with output feedback thus requiring to monitor only a limited number of state variables in the financial system.

First, it is proven that the dynamic model of the chaotic dynamical system is a differentially flat one. This means that all its state variables and its control inputs can be expressed as differential functions of a primary state variable which is the system;s flat output. Moreover, the flat output and its derivatives are differentially independent which means that they are not connected between them with a relation of the type of a differential equation [Rigatos, 2011], [Rigatos, 2013], [Rigatos, 2015], [Rigatos, 2017]. Next, by applying a change of state variables (diffeomorphism) which is in accordance to differential flatness theory, one arrives at an input-output linearized system. This description is also written in the linear canonical (Brunovsky) state-space form [Rudolph, 2003], [Fliess and Mounier, 1999], [Sira-Ramirez and Agrawal, 2004]. For the latter description the design of a stabilizing state feedback controller becomes possible.

Since there is no knowledge about the financial system's dynamics and the control method is a model-free one, the unknown parts of the dynamics are identified in real-time with the use of neurofuzzy approxmators. The information obtained about the system's dynamics is used for the computation of the control input, and thus an indirect adaptive control scheme is established. The update of the approximators' weights is based on a gradient-type algorithm [Rigatos and Tzafestas, 2007], [Baseville and Nikiforov, 1993], [Rigatos and Zhang, 2009]. The learning rate of the neurofuzzy approximators is obtained from the requirement the first derivative of the system's Lyapunov function to be always negative. The computation of the control signal requires also the solution of two algebraic Riccati equations. Lyapunov stability analysis proves that the control loop satisfies the H-infinity tracking performance ciretion and this signifies elevated robustness agaisnt model uncertainty and external perturbations. Morever, under moderate conditions global asymptotic stability is proven.

2 Dynamic Model of the Chaotic Finance System

The considered macroeconomics model is derived after using accumulated knowledge about the interaction between parameters such as the interest rate, the investments demand and the price exponent (this indirectly expresses the inflation rate) [Ma and Chen, 2001], [Yu et al., 2012]. Thus one has:
(i) The change of the interest rate in time time is proportional to the difference between investments demand and savings. Moreover, it is proportional to the price exponent (interest rate) which implies an adjustment to consumption goods' prices. The above can written in the form of the differential equation:

$$\dot{x} = f_1(y - SV)x + f_2 z \tag{1}$$

where y is the investments demand, SV is the amount of savings and f_1, f_2 are constants.
(ii) The change of the investment demand is proportional to the benefit from the rate of investments, while (a) it is inhibited in a proportional manner by the investments demand itself, (b) it is inhibited in an exponential (square) manner by the value of the interest rate. The previous are expressed through the following relation:

$$\dot{y} = f_2(BEN - \alpha y - \beta x^2) \tag{2}$$

where BEN is the benefit rate of investments, f_2, α and β are constants.

(iii) The price exponent expresses a contradiction (discrepancy) between supply and demand in a commercial market. The price exponent is an indication of the inflation rate. The change of the price exponent is inhibited in a proportional manner by the value of the inflation rate itself. It is also inhibited in a proportional manner by the value of the interest rate. The previous are expressed through the following relation

$$\dot{z} = -f_4 z - f_5 x \tag{3}$$

where f_4 and f_5 are constants.

2.1 State-space Model of the Chaotic Financial System

The following state variables notation is used next: $x_1 = x$, $x_2 = y$ and $x_3 = z$. Moreover, the coefficient of the previous equations are denoted as a,b and

c. Thus, the dynamics of the chaotic finance system is now given by [Yu et al., 2012]

$$\begin{aligned} \dot{x}_1 &= x_3 + (x_2 - a)x_1 \\ \dot{x}_2 &= 1 - bx_2 - x_1^2 \\ \dot{x}_3 &= -x_1 - cx_3 \end{aligned} \tag{4}$$

As previously noted, in state vector $x = [x_1, x_2, x_3]^T$, x_1 is the interest rate, x_2 is the investment demand and x_3 is the price exponent. Moreover, a is the savings amount, b is the cost per investment, and c is the elasticity of demand. The dynamics of the financial system is complemented with the inclusion of control inputs [Yu et al., 2012]

$$\begin{aligned} \dot{x}_1 &= x_3 + (x_2 - a)x_1 \\ \dot{x}_2 &= 1 - bx_2 - x_1^2 + u \\ \dot{x}_3 &= -x_1 - cx_3 \end{aligned} \tag{5}$$

The financial system is also written in the state-space form:

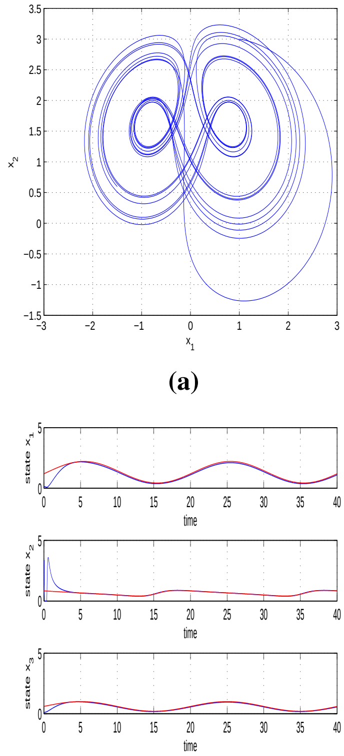
$$\dot{x} = f(x) + g(x)u \tag{6}$$

where

$$f(x) = \begin{pmatrix} x_3 + (x_2 - a)x_1 \\ 1 - bx_2 - x_1^2 \\ -x_1 - cx_3 \end{pmatrix} \quad g(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{7}$$

2.2 Chaotic Dynamics of the Finance System

The finance system exhibits chaotic dynamics. This means that in steady state it has a behavior that can be neither characterized as a stable equilibrium nor as a periodic or almost periodic oscillation. As time advances, the behavior of the system changes in a random-like manner and this depends on its initial conditions. Although the system is deterministic, it exhibits randomness in the way it evolves in time. By selecting the parameters' values to be $a = 0.9$, $b = 0.2$, $c = 1.2$ and the initial condition to be $x_0 = [1, 3, 2]$ one arrives at a chaotic behavior for the finance system as depicted in Fig. 1 and Fig. 2.



(a)

(b)

Figure 1. Chaotic dynamics of the finance system: (a) phase diagram of state variables x_1 and x_2, (b) phase diagram of state variables x_1 and x_3

3 Flatness-based Control of the Chaotic Finance Dynamics

3.1 Differential Flatness of the Chaotic Finance System

The state-space model of the chaotic finance system, complemented by the application of an external control input, is

$$\begin{aligned} \dot{x}_1 &= x_3 + (x_2 - a)x_1 \\ \dot{x}_2 &= 1 - bx_2 - x_1^2 + u \\ \dot{x}_3 &= -x_1 - cx_3 \end{aligned} \tag{8}$$

The flat output of the system is taken to be the state variable $y = x_3$. From the third row of Eq. (8) one has

$$x_1 = -\dot{x}_3 - cx_3 \Rightarrow x_1 = f_1(y, \dot{y}) \tag{9}$$

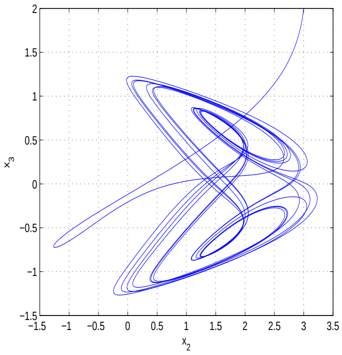
From the first row of Eq. (8) one has

$$x_2 = \frac{\dot{x}_1 - x_3}{x_1} + a \Rightarrow x_2 = f_2(y, \dot{y}, \ddot{y}) \tag{10}$$

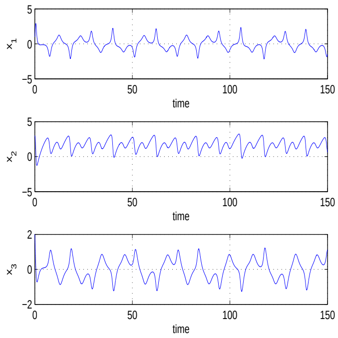
From the second row of Eq. (8) one has

$$u = \dot{x}_2 - 1 + bx_2 + x_1^2 \Rightarrow u = f_3(y, \dot{y}, \ddot{y}, y^{(3)}) \tag{11}$$

Since all state variables and the control input can be written as differential functions of the flat output, it is confirmed that the system is differentially flat.



(a)



(b)

Figure 2. Chaotic dynamics of the finance system: (a) phase diagram of state variables x_2 and x_3, (b) variation in time of state variables x_1,x_3 and x_3

3.2 Design of a Stabilizing Feedback Controller

By deriving twice the third row of Eq. (8) with respect to time, and by substituting the time derivatives $\dot{x}_i,\ i = 1,2,3$ again in accordance to the rows of Eq. (8) one has

$$\begin{array}{c} x_3^{(3)} = (x_1 + cx_3)(1 - bx_2 - x_1^2)x_1 + \\ +x_2[x_3 + (x_2 - a)x_1] + a[x_3 + (x_2 - a)x_1] + \\ +c[x_3 + (x_2 - a)x_1] + c^2[-x_1 - cx_3] - x_1 u \end{array} \tag{12}$$

or equivalently

$$x_3^{(3)} = f(x) + g(x)u \tag{13}$$

or in the form

$$y^{(3)} = f(y, \dot{y}, \ddot{y}) + g(y, \dot{y}, \ddot{y})u \tag{14}$$

where

$$\begin{array}{c} f(y, \dot{y}, \ddot{y}) = (x_1 + cx_3)(1 - bx_2 - x_1^2)x_1 + \\ +x_2[x_3 + (x_2 - a)x_1] + a[x_3 + (x_2 - a)x_1] + \\ +c[x_3 + (x_2 - a)x_1] + c^2[-x_1 - cx_3] \end{array} \tag{15}$$

$$g(y, \dot{y}, \ddot{y}) = -x_1 \tag{16}$$

By defining the transformed control input $v = f(y, \dot{y}, \ddot{y}) + g(y, \dot{y}, \ddot{y})u$ one has that

$$y^{(3)} = v \tag{17}$$

For the linearized description of the finance system given in Eq. (17), and using the notation $z_1 = y$, $z_2 = \dot{y}$ and $z_3 = \ddot{y}$, and $v = f(y, \dot{y}, \ddot{y}) + g(y, \dot{y}, \ddot{y})u$ one arrives also at the state-space description

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v \tag{18}$$

$$z^{meas} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{19}$$

and the stabilizing feedback control input is given by

$$v = y_d^{(3)} - k_1(\ddot{y} - \ddot{y}_d) - k_2(\dot{y} - \dot{y}_d) - k_3(y - y_d) \tag{20}$$

and the control input that is actually applied to the financial system is

$$u = g^{-1}(y, \dot{y}, \ddot{y})[v - f(y, \dot{y}, \ddot{y})] \tag{21}$$

The previous control signal results in the tracking error dynamics of the form

$$e^{(3)}(t) + k_1\ddot{e}(t) + k_2\dot{e}(t) + k_3 e(t) = 0 \tag{22}$$

By selecting the feedback gains $k_i,\ i = 1,2,3$ such that the characteristic polynomial of Eq. (22) to be a Hurwitz one, it assured that $lim_{t\to\infty} e(t) = 0$.

4 Adaptive Fuzzy Control of the Chaotic Finance System Using Output Feedback

4.1 Problem Statement

Adaptive fuzzy control aims at solving the control problem of the chaotic finance system in case that its dynamics are unknown and the state vector is not completely measurable. It has been shown that after applying the differential flatness theory-based transformation, the following non-linear SISO system is obtained:

$$x^{(n)} = f(x,t) + g(x,t)u + \tilde{d} \tag{23}$$

where $f(x,t)$, $g(x,t)$ are unknown nonlinear functions and $\tilde{d}$ is an unknown additive disturbance. The objective is to force the system's output $y = x$ to follow a given bounded reference signal x_d. In the presence of non-gaussian disturbances w, successful tracking of the reference signal is denoted by the H_∞ criterion [Rigatos, 2015]

$$\int_0^T e^T Q e dt \leq \rho^2 \int_0^T w^T w dt \tag{24}$$

where ρ is the attenuation level and corresponds to the maximum singular value of the transfer function $G(s)$ of the linearized equivalent of Eq. (23).

Remark: From the H_∞ control theory, the H_∞ norm of a linear system with transfer function $G(s)$, is denoted by $||G||_\infty$ and is defined as $||G||_\infty = sup_\omega \sigma_{max}[G(j\omega)]$ [Rigatos, 2015]. In this definition sup denotes the supremum or least upper bound of the function $\sigma_{max}[G(j(\omega)]$, and thus the H_∞ norm of $G(s)$ is the maximum value of $\sigma_{max}[G(j(\omega)]$ over all frequencies ω. H_∞ norm has a physically meaningful interpretation when considering the system $y(s) = G(s)u(s)$. When this system is driven with a unit sinusoidal input at a specific frequency, $\sigma_{max}|G(j\omega)|$ is the largest possible output for the corresponding sinusoidal output. Thus, the H_∞ norm is the largest possible amplification over all frequencies of a sinusoidal input.

4.2 Transformation of Tracking into a Regulation Problem

The H_∞ approach to nonlinear systems control consists of the following steps : i) linearization is applied: ii) the unknown system dynamics are approximated by neural of fuzzy estimators, iii) an H_∞ control term, is employed to compensate for estimation errors and external disturbances. If the state vector is not measurable, this can be reconstructed with the use of an observer.

For measurable state vector x, desirable state vector x_m and uncertain functions $f(x,t)$ and $g(x,t)$ an appropriate control law for (23) would be

$$u = \frac{1}{\hat{g}(x,t)}[x_m^{(n)} - \hat{f}(x,t) + K^T e + u_c] \tag{25}$$

where, $\hat{f}$ and $\hat{g}$ are the approximations of the unknown parts of the system dynamics f and g respectively, and which can be given by the outputs of suitably trained neuro-fuzzy networks. The term u_c denotes a supervisory controller which compensates for the approximation error $w = [f(x,t) - \hat{f}(x,t)] + [g(x,t) - \hat{g}(x,t)]u$, as well as for the additive disturbance $\tilde{d}$. Moreover the vectors $K^T = [k_n, k_{n-1}, \cdots, k_1]$, and $e^T = [e, \dot{e}, \ddot{e}, \cdots, e^{(n-1)}]^T$ are chosen such that the polynomial $e^{(n)} + k_1 e^{(n-1)} + k_2 e^{(n-2)} + \cdots + k_n e$ is Hurwitz. The substitution of the control law of Eq. (25) in Eq. (23) results into

$x^{(n)} = f(x,t) + g(x,t)\frac{1}{\hat{g}(x,t)}[x_m^{(n)} - \hat{f}(x,t) - K^T e + u_c] + \tilde{d} \Rightarrow x^{(n)} = f(x,t) + \{\hat{g}(x,t) + [g(x,t) - \hat{g}(x,t)]\}\frac{1}{\hat{g}(x,t)}[x_m^{(n)} - \hat{f}(x,t) - K^T e + u_c] + \tilde{d} \Rightarrow x^{(n)} = f(x,t) + \{\frac{\hat{g}(x,t)}{\hat{g}(x,t)}[x_m^{(n)} - \hat{f}(x,t) - K^T e + u_c] + [g(x,t) - \hat{g}(x,t)]u + \tilde{d} \Rightarrow x^{(n)} = f(x,t) + x_m^{(n)} - \hat{f}(x,t) - K^T e + u_c + [g(x,t) - \hat{g}(x,t)]u + u_c + \tilde{d} \Rightarrow x^{(n)} - x_m^{(n)} = -K^T e + [f(x,t) - \hat{f}(x,t)] + [g(x,t) - \hat{g}(x,t)]u + u_c + \tilde{d} \Rightarrow x^{(n)} = -K^T e + u_c + [f(x,t) - \hat{f}(x,t)] + [g(x,t) - \hat{g}(x,t)]u + \tilde{d}$

The above relation can be written in a state-equations form. The state vector is taken to be $e^T = [e, \dot{e}, \cdots, e^{(n-1)}]$, which yields

$$\begin{aligned}\dot{e} = Ae - BK^T e + Bu_c + B\{[f(x,t) - \hat{f}(x,t)] + \\ +[g(x,t) - \hat{g}(x,t)]u + \tilde{d}\}\end{aligned} \tag{26}$$

or equivalently

$$\begin{aligned}&\dot{e} = (A - BK^T)e + Bu_c + B\{[f(x,t) - \hat{f}(x,t)] + \\ &[g(x,t) - \hat{g}(x,t)]u + \tilde{d}\} \\ &e_1 = C^T e\end{aligned} \tag{27}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix}0, 0, \cdots, 0, 1\end{pmatrix}, \ C^T = \begin{pmatrix}1, 0, \cdots, 0, 0\end{pmatrix}$$

$$K^T = \begin{pmatrix}k_0, k_1, \cdots, k_{n-2}, k_{n-1}\end{pmatrix} \tag{28}$$

where e_1 denotes the output error $e_1 = x - x_m$. Eq. (27) describes a regulation problem.

4.3 Estimation of the State Vector

The control of the system described by Eq. (23) becomes more complicated when the state vector x is not directly measurable and has to be reconstructed through a state observer. The following definitions are used

error of the state vector $e = x - x_m$
error of the estimated state vector $\hat{e} = \hat{x} - x_m$
observation error $\tilde{e} = e - \hat{e} = (x - x_m) - (\hat{x} - x_m)$

When an observer is used to reconstruct the state vector, the control law of Eq. (25) is written as

$$u = \frac{1}{\hat{g}(\hat{x}, t)}[x_m^{(n)} - \hat{f}(\hat{x}, t) + K^T e + u_c] \tag{29}$$

Applying Eq. (29) to the nonlinear system described by Eq. (23), after some operations results into

$$x^{(n)} = x_m^{(n)} - K^T\hat{e} + u_c + [f(x,t) - \hat{f}(\hat{x}, t)] + [g(x,t) - \hat{g}(\hat{x}, t)]u + \tilde{d}$$

It holds $e = x - x_m \Rightarrow x^{(n)} = e^{(n)} + x_m^{(n)}$. Substituting $x^{(n)}$ in the above equation gives

$$e^{(n)} + x_m^{(n)} = x_m^{(n)} - K^T\hat{e} + u_c + [f(x,t) - \hat{f}(\hat{x}, t)] + +[g(x,t) - \hat{g}(\hat{x}, t)]u + \tilde{d} \Rightarrow \tag{30}$$

$$\dot{e} = Ae - BK^T\hat{e} + Bu_c + B\{[f(x,t) - \hat{f}(\hat{x}, t)] + +[g(x,t) - \hat{g}(\hat{x}, t)]u + \tilde{d}\} \tag{31}$$

$$e_1 = C^T e \tag{32}$$

where $e = [e, \dot{e}, \ddot{e}, \cdots, e^{(n-1)}]^T$, and $\hat{e} = [\hat{e}, \dot{\hat{e}}, \ddot{\hat{e}}, \cdots, \hat{e}^{(n-1)}]^T$.

The state observer is designed according to Eq. (31) and (32) and is given by [Rigatos, 2015]:

$$\dot{\hat{e}} = A\hat{e} - BK^T\hat{e} + K_o[e_1 - C^T\hat{e}] \tag{33}$$

$$\hat{e}_1 = C^T\hat{e} \tag{34}$$

The observation gain $K_o = [k_{o_0}, k_{o_1}, \cdots, k_{o_{n-2}}, k_{o_{n-1}}]^T$ is selected so as to assure the convergence of the observer.

4.4 The Additional Control Term u_c

The additional term u_c which appeared in Eq. (25) is also introduced in the observer-based control to compensate for:

The external disturbances $\tilde{d}$
The state vector estimation error $\tilde{e} = e - \hat{e} = x - \hat{x}$
The approximation error of the nonlinear functions $f(x,t)$ and $g(x,t)$, denoted as $w = [f(x,t) - \hat{f}(\hat{x}, t)] + [g(x,t) - \hat{g}(\hat{x}, t)]u$

The control signal u_c consists of 2 terms, namely:

the H_∞ control term, $u_a = -\frac{1}{r}B^T P\tilde{e}$ for the compensation of d and w
the control term u_b for the compensation of the observation error $\tilde{e}$

4.5 Dynamics of the Observation Error

The observation error is defined as $\tilde{e} = e - \hat{e} = x - \hat{x}$. Subtracting Eq. (33) from Eq. (31) as well as Eq. (34) from Eq. (32) one gets

$$\dot{e} - \dot{\hat{e}} = A(e - \hat{e}) + Bu_c + B\{[f(x,t) - \hat{f}(\hat{x}, t)] + +[g(x,t) - \hat{g}(\hat{x}, t)]u + \tilde{d}\} - K_o C^T(e - \hat{e})$$

$$e_1 - \hat{e}_1 = C^T(e - \hat{e})$$

i.e.

$$\dot{\tilde{e}} = A\tilde{e} + Bu_c + B\{[f(x,t) - \hat{f}(\hat{x}, t)] + +[g(x,t) - \hat{g}(\hat{x}, t)]u + \tilde{d}\} - K_o C^T\tilde{e}$$

$$\tilde{e}_1 = C^T\tilde{e}$$

which can be written as

$$\dot{\tilde{e}} = (A - K_o C^T)\tilde{e} + Bu_c + B\{[f(x,t) - \hat{f}(\hat{x},t)] + \\ +[g(x,t) - \hat{g}(\hat{x},t)]u + \tilde{d}\} \tag{35}$$

$$\tilde{e}_1 = C\tilde{e} \tag{36}$$

4.6 Approximation of the Functions $f(x,t)$ and $g(x,t)$

Neurofuzzy networks can been trained on-line to approximate parts of the dynamic equation of non-linear systems, or to compensate for external disturbances. The approximation of functions $f(x,t)$ and $g(x,t)$ of Eq. (23) can be carried out with Takagi-Sugeno neuro-fuzzy networks of zero or first order (Fig. 3). These consist of rules of the form:

R^l : IF $\hat{x}$ is A_1^l AND $\dot{\hat{x}}$ is A_2^l AND $\cdots$ AND $\hat{x}^{(n-1)}$ is A_n^l THEN $\bar{y}^l = \sum_{i=1}^n w_i^l \hat{x}_i + b^l$, $l = 1, 2, \cdots, L$

The output of the neuro-fuzzy model is calculated by taking the average of the consequent part of the rules

$$\hat{y} = \frac{\sum_{l=1}^L \bar{y}^l \prod_{i=1}^n \mu_{A_i^l}(\hat{x}_i)}{\sum_{l=1}^L \prod_{i=1}^n \mu_{A_i^l}(\hat{x}_i)} \tag{37}$$

where $\mu_{A_i^l}$ is the membership function of x_i in the fuzzy set A_i^l. The training of the neuro-fuzzy networks is carried out with 1^{st} order gradient algorithms, in pattern mode, i.e. by processing only one data pair (x_i, y_i) at every time step i. The estimation of $f(x,t)$ and $g(x,t)$ can be written as

$$\hat{f}(\hat{x}|\theta_f) = \theta_f^T \phi(\hat{x}) \ \hat{g}(\hat{x}|\theta_g) = \theta_g^T \phi(\hat{x}) \tag{38}$$

where $\phi(\hat{x})$ are kernel functions with elements $\phi^l(\hat{x}) = \frac{\prod_{i=1}^n \mu_{A_i^l}(\hat{x}_i)}{\sum_{l=1}^L \prod_{i=1}^n \mu_{A_i^l}(\hat{x}_i)}$ $l = 1, 2, \cdots, L$. It is assumed that that the weights θ_f and θ_g vary in the bounded areas M_{θ_f} and M_{θ_g} which are defined as: $M_{\theta_f} = \{\theta_f \in R^h : ||\theta_f|| \leq m_{\theta_f}\}$ and $M_{\theta_g} = \{\theta_g \in R^h : ||\theta_g|| \leq m_{\theta_g}\}$, with m_{θ_f} and m_{θ_g} positive constants. The values of θ_f and θ_g for which optimal approximation is succeeded are:

$$\theta_f^* = arg\ min_{\theta_f \in M_{\theta_f}}[sup_{x \in U_x, \hat{x} \in U_{\hat{x}}}|f(x) - \hat{f}(\hat{x}|\theta_f)|] \\ \theta_g^* = arg\ min_{\theta_g \in M_{\theta_g}}[sup_{x \in U_x, \hat{x} \in U_{\hat{x}}}|g(x) - \hat{g}(\hat{x}|\theta_g)|]$$

The variation ranges of x and $\hat{x}$ are the compact sets

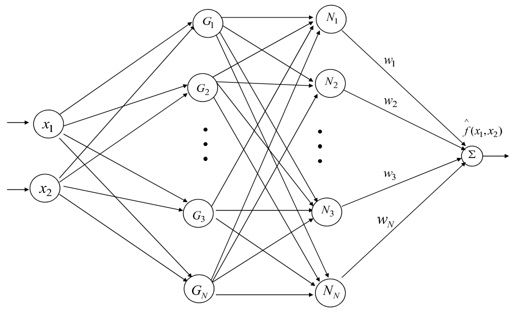


Figure 3. Neuro-fuzzy approximator: G_i Gaussian basis function, N_i: normalization unit

$$U_x = \{x \in R^n : ||x|| \leq m_x < \infty\}, \\ U_{\hat{x}} = \{\hat{x} \in R^n : ||\hat{x}|| \leq m_{\hat{x}} < \infty\} \tag{39}$$

The approximation error of $f(x,t)$ and $g(x,t)$ is given by

$$w = [\hat{f}(\hat{x}|\theta_f^*) - f(x,t)] + [\hat{g}(\hat{x}|\theta_g^*) - g(x,t)]u \Rightarrow \\ w = \{[\hat{f}(\hat{x}|\theta_f^*) - f(x|\theta_f^*)] + [f(x|\theta_f^*) - f(x,t)]\} + \\ \{[\hat{g}(\hat{x}|\theta_g^*) - g(\hat{x}|\theta_g^*)] + [g(\hat{x}|\theta_g^*)g(x,t)]\}u \tag{40}$$

where

$\hat{f}(\hat{x}|\theta_f^*)$ is the approximation of f for the best estimation θ_f^* of the weights' vector θ_f.
$\hat{g}(\hat{x}|\theta_g^*)$ is the approximation of g for the best estimation θ_g^* of the weights' vector θ_g.

The approximation error w can be decomposed into w_a and w_b, where

$$w_a = [\hat{f}(\hat{x}|\theta_f) - \hat{f}(\hat{x}|\theta_f^*)] + [\hat{g}(\hat{x}|\theta_g) - \hat{g}(\hat{x}|\theta_g^*)]u \\ w_b = [\hat{f}(\hat{x}|\theta_f^*) - f(x,t)] + [\hat{g}(\hat{x}|\theta_g^*) - g(x,t)]u$$

Finally, the following two parameters are defined:

$$\tilde{\theta}_f = \theta_f - \theta_f^*, \quad \tilde{\theta}_g = \theta_g - \theta_g^* \tag{41}$$

5 Lyapunov Stability Analysis

5.1 Design of the Lyapunov Function

The adaptation law of the neurofuzzy approximators' weights θ_f and θ_g as well as of the supervisory control term u_c are derived from the requirement for negative definiteness of the Lyapunov function

$$V=\frac{1}{2}\hat{e}^TP_1\hat{e}+\frac{1}{2}\tilde{e}^TP_2\tilde{e}+\frac{1}{2\gamma_1}\tilde{\theta}_f^T\tilde{\theta}_f+\frac{1}{2\gamma_2}\tilde{\theta}_g^T\tilde{\theta}_g \quad (42)$$

The selection of the Lyapunov function is based on the following principle of indirect adaptive control $\hat{e}$: $\lim_{t\to\infty}\hat{x}(t)=x_d(t)$ and $\tilde{e}$: $\lim_{t\to\infty}\hat{x}(t)=x(t)$. This yields $\lim_{t\to\infty}x(t)=x_d(t)$. Substituting Eq. (31), (32) and Eq. (35), (36) into Eq. (42) and differentiating results into

$$\begin{array}{c}\dot{V}=\frac{1}{2}\dot{\hat{e}}^TP_1\hat{e}+\frac{1}{2}\hat{e}^TP_1\dot{\hat{e}}++\frac{1}{2}\dot{\tilde{e}}^TP_2\tilde{e}+\frac{1}{2}\tilde{e}^TP_2\dot{\tilde{e}}+\\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f+\frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{array} \quad (43)$$

which in turn gives

$$\begin{array}{c}\dot{V}=\frac{1}{2}\{(A-BK^T)\hat{e}+K_oC^T\tilde{e}\}^TP_1\hat{e}+\\ +\frac{1}{2}\hat{e}^TP_1\{(A-BK^T)\hat{e}+K_oC^T\tilde{e}\}+\\ +\frac{1}{2}\{(A-K_oC^T)\tilde{e}+Bu_c+Bd+Bw\}^TP_2\tilde{e}+\\ +\frac{1}{2}\tilde{e}^TP_2\{(A-K_oC^T)\tilde{e}+Bu_c+Bd+Bw\}+\\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f+\frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{array} \quad (44)$$

or, equivalently

$$\begin{array}{c}\dot{V}=\frac{1}{2}\{\hat{e}^T(A-BK^T)^T+\tilde{e}^TCK_o^T\}P_1\hat{e}+\\ +\frac{1}{2}\hat{e}^TP_1\{(A-BK^T)\hat{e}+K_oC^T\tilde{e}\}+\\ +\frac{1}{2}\{\tilde{e}^T(A-K_oC^T)^T+B^Tu_c+B^Tw+B^Td\}P_2\tilde{e}+\\ +\frac{1}{2}\tilde{e}^TP_2\{(A-K_oC^T)\tilde{e}+Bu_c+Bw+Bd\}+\\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f+\frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{array} \quad (45)$$

which is also written as

$$\begin{array}{c}\dot{V}=\frac{1}{2}\hat{e}^T(A-BK^T)^TP_1\hat{e}+\frac{1}{2}\tilde{e}^TCK_o^TP_1\hat{e}+\\ +\frac{1}{2}\hat{e}^TP_1(A-BK^T)\hat{e}+\frac{1}{2}\hat{e}^TP_1K_oC^T\tilde{e}+\\ +\frac{1}{2}\tilde{e}^T(A-K_oC^T)^TP_2\tilde{e}+\frac{1}{2}B^TP_2\tilde{e}(u_c+w+d)+\\ +\frac{1}{2}\tilde{e}^TP_2(A-K_oC^T)\tilde{e}+\frac{1}{2}\tilde{e}^TP_2B(u_c+w+d)+\\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f+\frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{array} \quad (46)$$

Assumption 1: For given positive definite matrices Q_1 and Q_2 there exist positive definite matrices P_1 and P_2, which are the solution of the following Riccati equations [Rigatos, 2015]

$$(A-BK^T)^TP_1+P_1(A-BK^T)+Q_1=0 \quad (47)$$

$$\begin{array}{c}(A-K_oC^T)^TP_2+P_2(A-K_oC^T)-\\ -P_2B(\frac{2}{r}-\frac{1}{\rho^2})B^TP_2+Q_2=0\end{array} \quad (48)$$

The conditions given in Eq. (47) to (48) are related to the requirement that the systems described by Eq. (33), (34) and Eq. (35), (36) are strictly positive real. Substituting Eq. (47) to (48) into $\dot{V}$ yields

$$\begin{array}{c}\dot{V}=\frac{1}{2}\hat{e}^T\{(A-BK^T)^TP_1+P_1(A-BK^T)\}\hat{e}+\\ +\tilde{e}^TCK_o^TP_1\hat{e}+\frac{1}{2}\tilde{e}^T\{(A-K_oC^T)^TP_2+\\ +P_2(A-K_oC^T)\}\tilde{e}+B^TP_2\tilde{e}(u_c+w+d)+\\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f+\frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{array} \quad (49)$$

which is also written as

$$\begin{array}{c}\dot{V}=-\frac{1}{2}\hat{e}^TQ_1\hat{e}+\tilde{e}^TCK_o^TP_1\hat{e}-\\ \frac{1}{2}\tilde{e}^T\{Q_2-P_2B(\frac{2}{r}-\frac{1}{\rho^2})B^TP_2\}\tilde{e}+\\ +B^TP_2\tilde{e}(u_c+w+d)+\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f+\frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{array} \quad (50)$$

The supervisory control u_c is decomposed in two terms, u_a and u_b

$$\begin{array}{c}u_a=-\frac{1}{r}p_{1n}\tilde{e}_1=-\frac{1}{r}\tilde{e}^TP_2B+\\ +\frac{1}{r}(p_{2n}\tilde{e}_2+\cdots+p_{nn}\tilde{e}_n)=-\frac{1}{r}\tilde{e}^TP_2B+\Delta u_a\end{array} \quad (51)$$

where p_{1n} stands for the last (n-th) element of the first row of matrix P_2, and

$$u_b=-[(P_2B)^T(P_2B)]^{-1}(P_2B)^TCK_o^TP_1\hat{e} \quad (52)$$

u_a is an H_∞ control used for the compensation of the approximation error w and the additive disturbance $\tilde{d}$. Its first component $-\frac{1}{r}\tilde{e}^TP_2B$ has been chosen so as to compensate for the term $\frac{1}{r}\tilde{e}^TP_2BB^TP_2\tilde{e}$, which appears in Eq. (50). By subtracting the second component $-\frac{1}{r}(p_{2n}\tilde{e}_2+\cdots+p_{nn}\tilde{e}_n)$ one has that $u_a=-\frac{1}{r}p_{1n}\tilde{e}_1$, which means that u_a is computed based on the feedback the measurable variable $\tilde{e}_1$. Eq. (51) is finally rewritten as $u_a=-\frac{1}{r}\tilde{e}^TP_2B+\Delta u_a$.

u_b is a control used for the compensation of the observation error (the control term u_b has been chosen so as to satisfy the condition $\tilde{e}^TP_2Bu_b=-\tilde{e}^TCK_o^TP_1\hat{e}$.

Substituting Eq. (51) and (52) in $\dot{V}$, one gets

$$\begin{array}{c}\dot{V}=-\frac{1}{2}\hat{e}^TQ_1\hat{e}+\tilde{e}^TCK_o^TP_1\hat{e}-\frac{1}{2}\tilde{e}^TQ_2\tilde{e}+\\ +\frac{1}{r}\tilde{e}^TP_2BB^TP_2\tilde{e}-\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}+\\ +\tilde{e}^TP_2Bu_b-\frac{1}{r}\tilde{e}^TP_2BB^TP_2\tilde{e}+B^TP_2\tilde{e}(w+d+\Delta u_a)+\\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f+\frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{array} \quad (53)$$

or equivalently,

$$\begin{array}{c}\dot{V}=-\frac{1}{2}\hat{e}^TQ_1\hat{e}-\frac{1}{2}\tilde{e}^TQ_2\tilde{e}-\\ -\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}+B^TP_2\tilde{e}(w+d+\Delta u_a)+\\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f+\frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{array} \quad (54)$$

It holds that $\dot{\tilde{\theta}}_f=\dot{\theta}_f-\dot{\theta}_f^*=\dot{\theta}_f$ and $\dot{\tilde{\theta}}_g=\dot{\theta}_g-\dot{\theta}_g^*=\dot{\theta}_g$. The following weight adaptation laws are considered:

$$\dot{\theta}_f=\{\begin{array}{c}-\gamma_1\tilde{e}^TP_2B\phi(\hat{x})\ \ if\ ||\theta_f||<m_{\theta_f}\\ 0\ \ ||\theta_f||\geq m_{\theta_f}\end{array} \quad (55)$$

$$\dot{\theta}_g=\{\begin{array}{c}-\gamma_2\tilde{e}^TP_2B\phi(\hat{x})u_c\ \ if\ ||\theta_g||<m_{\theta_g}\\ 0\ \ ||\theta_g||\geq m_{\theta_g}\end{array} \quad (56)$$

To set $\dot{\theta}_f$ and $\dot{\theta}_g$ equal to 0, when $||\theta_f \geq m_{\theta_f}||$, and $||\theta_g\geq m_{\theta_g}||$ the projection operator is employed [Rigatos, 2011]:

$$\begin{array}{c}P\{\gamma_1\tilde{e}^TP_2B\phi(\hat{x})\}=-\gamma_1\tilde{e}^TP_2B\phi(\hat{x})+\\ +\gamma_1\tilde{e}^TP_2B\frac{\theta_f\theta_f^T}{||\theta_f||^2}\phi(\hat{x})\end{array}$$

$$\begin{array}{c}P\{\gamma_1\tilde{e}^TP_2B\phi(\hat{x})u_c\}=-\gamma_1\tilde{e}^TP_2B\phi(\hat{x})u_c+\\ +\gamma_1\tilde{e}^TP_2B\frac{\theta_f\theta_f^T}{||\theta_f||^2}\phi(\hat{x})u_c\end{array}$$

The update of θ_f stems from a gradient algorithm on the cost function $\frac{1}{2}(f-\hat{f})^2$ [Baseville and Nikiforov, 1993], [Rigatos and Tzafestas, 2007],[Rigatos and Zhang, 2009]. The update of θ_g is also of the gradient type, while u_c implicitly tunes the adaptation gain γ_2. Substituting Eq. (55) and (56) in $\dot{V}$ gives

$$\begin{array}{c}\dot{V}=-\frac{1}{2}\hat{e}^TQ_1\hat{e}-\frac{1}{2}\tilde{e}^TQ_2\tilde{e}-\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}+\\ +B^TP_2\tilde{e}(w+d+\Delta u_a)+\\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T(-\gamma_1\tilde{e}^TP_2B\phi(\hat{x}))+\frac{1}{\gamma_2}\tilde{\theta}_g^T(-\gamma_2\tilde{e}^TP_2B\phi(\hat{x})u)\end{array} \quad (57)$$

which is also written as

$$\begin{array}{c}\dot{V}=-\frac{1}{2}\hat{e}^TQ_1\hat{e}-\frac{1}{2}\tilde{e}^TQ_2\tilde{e}-\\ -\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}+\tilde{e}^TP_2B(w+d+\Delta u_a)-\\ -\tilde{e}^TP_2B\tilde{\theta}_f^T\phi(\hat{x})-\tilde{e}^TP_2B\tilde{\theta}_g^T\phi(\hat{x})u\end{array} \quad (58)$$

and using Eq. (38) and (41) results into

$$\begin{array}{c}\dot{V}=-\frac{1}{2}\hat{e}^TQ_1\hat{e}-\frac{1}{2}\tilde{e}^TQ_2\tilde{e}-\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}+\\ +\tilde{e}^TP_2B(w+d+\Delta u_a)-\tilde{e}^TP_2B\{[\hat{f}(\hat{x}|\theta_f)+\\ +\hat{g}(\hat{x}|\theta_f)u]-[\hat{f}(\hat{x}|\theta_f^*)+\hat{g}(\hat{x}|\theta_g^*)u]\}\end{array} \quad (59)$$

where $[\hat{f}(\hat{x}|\theta_f)+\hat{g}(\hat{x}|\theta_f)u]-[\hat{f}(\hat{x}|\theta_f^*)+\hat{g}(\hat{x}|\theta_g^*)u]=w_a$. Thus setting $w_1=w+w_a+d+\Delta u_a$ one gets

$$\begin{array}{c}\dot{V}=-\frac{1}{2}\hat{e}^TQ_1\hat{e}\frac{1}{2}\tilde{e}^TQ_2\tilde{e}-\\ -\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}+B^TP_2\tilde{e}w_1\Rightarrow\\ \dot{V}=-\frac{1}{2}\hat{e}^TQ_1\hat{e}\frac{1}{2}\tilde{e}^TQ_2\tilde{e}-\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}+\\ +\frac{1}{2}w_1^TB^TP_2\tilde{e}+\frac{1}{2}\tilde{e}^TP_2Bw_1\end{array} \quad (60)$$

Lemma: The following inequality holds

$$\begin{array}{c}\frac{1}{2}\tilde{e}^TP_2Bw_1+\frac{1}{2}w_1^TB^TP_2\tilde{e}-\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}\\ \leq\frac{1}{2}\rho^2w_1^Tw_1\end{array} \quad (61)$$

Proof: The binomial $(\rho a-\frac{1}{\rho}b)^2\geq 0$ is considered. Expanding the left part of the above inequality one gets

$$\begin{array}{c}\rho^2a^2+\frac{1}{\rho^2}b^2-2ab\geq 0\Rightarrow\\ \frac{1}{2}\rho^2a^2+\frac{1}{2\rho^2}b^2-ab\geq 0\Rightarrow\\ ab-\frac{1}{2\rho^2}b^2\leq\frac{1}{2}\rho^2a^2\Rightarrow\\ \frac{1}{2}ab+\frac{1}{2}ab-\frac{1}{2\rho^2}b^2\leq\frac{1}{2}\rho^2a^2\end{array} \quad (62)$$

The following substitutions are carried out: $a=w_1$ and $b=\tilde{e}^TP_2B$ and the previous relation becomes

$$\begin{array}{c}\frac{1}{2}w_1^TB^TP_2\tilde{e}+\frac{1}{2}\tilde{e}^TP_2Bw_1-\frac{1}{2\rho^2}\tilde{e}^TP_2BB^TP_2\tilde{e}\\ \leq\frac{1}{2}\rho^2w_1^Tw_1\end{array} \quad (63)$$

The above inequality is used in $\dot{V}$, and the right part of the associated inequality is enforced

$$\dot{V}\leq-\frac{1}{2}\hat{e}^TQ_1\hat{e}-\frac{1}{2}\tilde{e}^TQ_2\tilde{e}+\frac{1}{2}\rho^2w_1^Tw_1 \quad (64)$$

Thus, Eq. (64) can be written as

$$\dot{V}\leq-\frac{1}{2}E^TQE+\frac{1}{2}\rho^2w_1^Tw_1 \quad (65)$$

where

$$E=\begin{pmatrix}\hat{e}\\ \tilde{e}\end{pmatrix},\ \ Q=\begin{pmatrix}Q_1 & 0\\ 0 & Q_2\end{pmatrix}=diag[Q_1,Q_2] \quad (66)$$

Hence, the H_∞ performance criterion is derived. For ρ sufficiently small Eq. (64) will be true and the H_∞ tracking criterion will be satisfied. In that case, the integration of $\dot{V}$ from 0 to T gives

$$\int_0^T \dot{V}(t)dt \leq -\frac{1}{2}\int_0^T ||E||^2 dt + \frac{1}{2}\rho^2 \int_0^T ||w_1||^2 dt \Rightarrow$$
$$2V(T) - 2V(0) \leq -\int_0^T ||E||_Q^2 dt + \rho^2 \int_0^T ||w_1||^2 dt \Rightarrow$$
$$2V(T) + \int_0^T ||E||_Q^2 dt \leq 2V(0) + \rho^2 \int_0^T ||w_1||^2 dt$$

It is assumed that there exists a positive constant $M_w > 0$ such that $\int_0^\infty ||w_1||^2 dt \leq M_w$. Therefore for the integral $\int_0^T ||E||_Q^2 dt$ one gets $\int_0^\infty ||E||_Q^2 dt \leq 2V(0) + \rho^2 M_w$. Thus, the integral $\int_0^\infty ||E||_Q^2 dt$ is bounded and according to Barbalat's Lemma

$$\lim_{t\to\infty} E(t) = 0 \Rightarrow \begin{matrix} \lim_{t\to\infty} \hat{e}(t) = 0 \\ \lim_{t\to\infty} \tilde{e}(t) = 0 \end{matrix}$$

Therefore $\lim_{t\to\infty} e(t) = 0$.

6 Simulation Tests

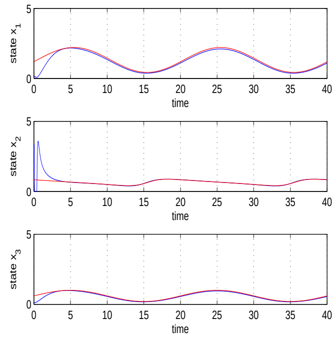
The proposed adaptive fuzzy control method has been applied to the problem of stabilization of the dynamics of the chaotic finance system defined in Eq. (5) and its performance has been checked through simulation experiments in the case of tracking of several reference trajectories. The presented results are depicted in Fig. 4 to Fig. 8. It has been confirmed that all state variables converged fast to the reference trajectories and that the tracking error was minimized. Moreover,the control inputs computed by the nonlinear H-infinity controller varied smoothly.

The estimation of the unknown dynamics of the system with the use of neuro-fuzzy approximators has been explained in subsection 4.6. The approximators' inputs were the system's state variables x_1: interest rate, x_2: investment demand and x_3: price exponent. Knowing that there are $i = 3$ state variables for the chaotic finance system and that each such variable comprises $n = 3$ fuzzy sets, the total number of rules in the fuzzy rule base should be $n^m = 3^3 = 27$. The aggregate output of the neuro-fuzzy approximator (rule-base) for function $f(x)$ is given by Eq. (37). Similar is the structure of the neuro-fuzzy approximator for function $g(x)$.

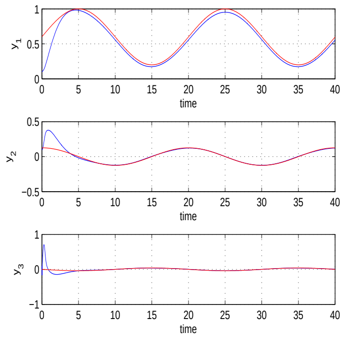
The control loop was based on simultaneous estimation of the unknown chaotic finance system's dynamics (this was performed with the use of neuro-fuzzy approximators) and of the nonmeasurable elements of the chaotic finance system's state vector, that is of the interest rate x_1 and of the investment demand x_2 (this was performed with the use of the state observer).

7 Conclusions

The article has presented an adaptive fuzzy control method for chaotic finance systems, that is is based on differential flatness theory and which uses feedback of



(a)



(b)

Figure 4. Tracking of reference setpoint 1 (red line): (a) state variables x_1, x_2, and x_3 (blue line), (b) flat output $y_1 = x_3$ and its time derivatives $y_2 = \dot{y}_1$, $y_3 = \ddot{y}_1$ (blue lines)

only the system's output. By proving the differential flatness properties of the chaotic finance system, and by applying a change of state variables (diffeomorphism) the nonlinear state-space description of the system was transformed into an equivalent linear one. For the latter description of the system's dynamics the design of a stabilizing state feedback controller became possible. Next, a an adaptive control scheme was implemented without any prior knowledge of the system's dynamics. The unknown dynamics of the chaotic finance system was identified with the use of neurofuzzy approximators. The learning rate of these approximators was defined by the requirement the first derivative of the system's Lyapunov function to be always a negative one. The information provided by the approximators was used for computing the control input of the system, thus establishing an indirect adaptive control loop. Moreover, for the computation of the control signal, at each iteration of the control algorithm, two algebraic Riccati equations had to be solved. Through Lyapunov stability analysis the global asymptotic stability of the control scheme was proven.

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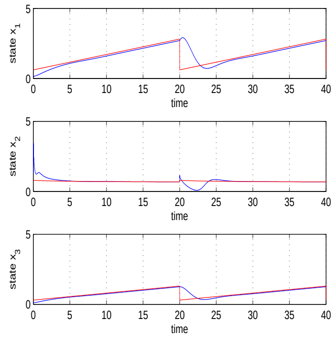

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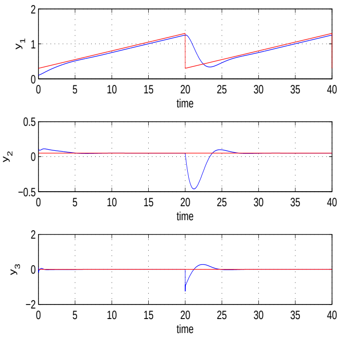
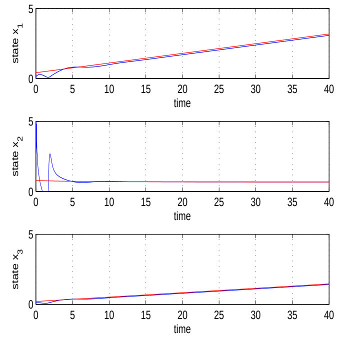
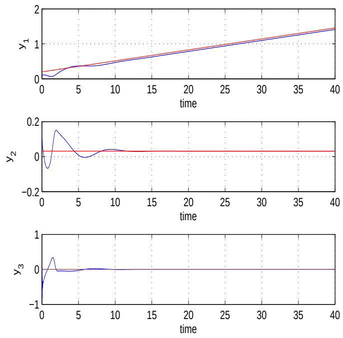


(b)

Figure 5. Tracking of reference setpoint 2 (red line): (a) state variables x_1, x_2, and x_3 (blue line), (b) flat output $y_1 = x_3$ and its time derivatives $y_2 = \dot{y}_1, y_3 = \ddot{y}_1$ (blue lines)



(a)



(b)

Figure 6. Tracking of reference setpoint 3 (red line): (a) state variables x_1, x_2, and x_3 (blue line), (b) flat output $y_1 = x_3$ and its time derivatives $y_2 = \dot{y}_1, y_3 = \ddot{y}_1$ (blue lines)

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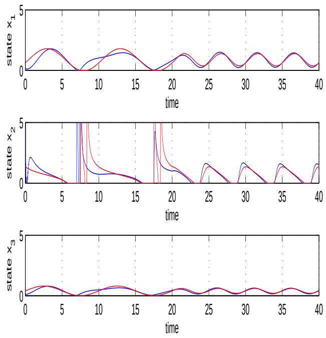
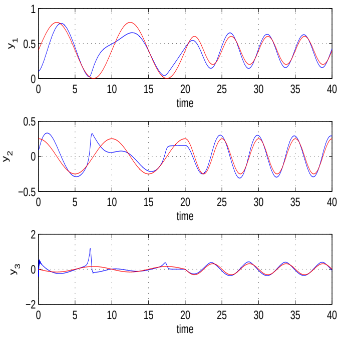
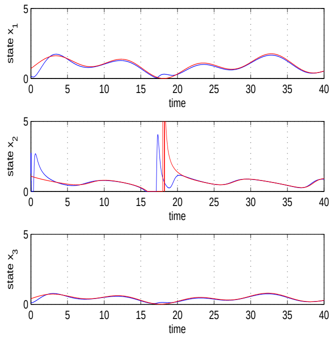


(a)

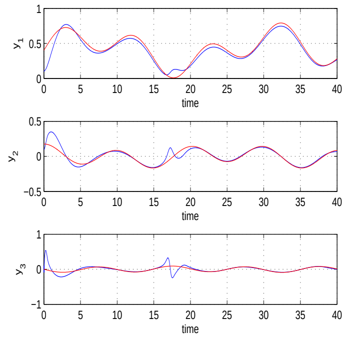


(b)

Figure 7. Tracking of reference setpoint 4 (red line): (a) state variables x_1, x_2, and x_3 (blue line), (b) flat output $y_1 = x_3$ and its time derivatives $y_2 = \dot{y}_1$, $y_3 = \ddot{y}_1$ (blue lines)



(a)



(b)

Figure 8. Tracking of reference setpoint 5 (red line): (a) state variables x_1, x_2, and x_3 (blue line), (b) flat output $y_1 = x_3$ and its time derivatives $y_2 = \dot{y}_1$, $y_3 = \ddot{y}_1$ (blue lines)

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