A QUICK LOOK AT ROW ECHELON FORM OF MULTI-INPUT CONTROL SYSTEMS

Miguel V. Carriegos
Department of Mathematics
University of León
Spain
miguel.carriegos@unileon.es

Montserrat M. Cb López
Department of Mathematics
University of León
Spain
mmlopc@unileon.es

Abstract

We give a first generalization of the invariants and canonical forms of single-input linear control systems over principal ideal domains to the multi-input case by means of quotient rings.

Key words

Multi-input, dynamical-systems, invariants, row-echelon, Bézout, pid.

1 Introduction

The feedback classification of linear dynamical systems over a commutative ring is an open problem on Control Theory, see [Brewer, 1986], [McDonald, 1984], [Sontag, 1998] and [Hermida, 2003] for a general lecture. In other words, we are about given a certain linear dynamical system $\Sigma = (A, B)$ over a particular commutative ring R, find its feedback invariants, that is to say, finding the canonical dynamical system $\hat{\Sigma} = (\hat{A}, \hat{B})$ over R feedback equivalent to Σ. Eventually, some cases have been studied and solved, see for example [Brewer and Klinger, 2001], [Brunovsky, 1970], [Carriegos and García, 2004] and [Carriegos and Sánchez, 2001].

In this paper, we focus our interest in applying for digital systems or coding case. So, we deal with linear dynamical systems over $R = \mathbb{Z}$ or finite ring, see [Carriegos and Hermida, 2003] for reading a canonical form for single-input n-dimensional linear systems. In this way, the main section of this study deal with rising from single-input to multi-input over a principal ideal domain R. Finally, under some conditions, we find row-echelon form $\hat{\Sigma} = (\hat{A}, \hat{B})$ corresponding to a given linear control system $\Sigma = (A, B)$ over R.

2 Feedback equivalence

Let R be a commutative ring with identity element. An m-input n-dimensional linear control system Σ over R is a pair (A, B), i.e. $A = (a_{ij})$ an $n \times n$ matrix and $B = (b_{ij})$ an $n \times m$ matrix with entries in R.

We say that two m-input n-dimensional systems $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ are (static) feedback equivalent, and write $\Sigma \sim_R \Sigma'$, if there exist invertible matrices P and Q, and a feedback matrix F such that $B' = PBQ$ and $PA - A'P = BF$. The objective of the feedback relation is to obtain a matrix F such that $A' = P(A - BF)P^{-1}$ has some desired property. Note that, one of the principal difficulty of this problem is to find change of basis P and Q in the respective sampling spaces. In this way, in some cases, the difficulty of the static feedback classification is tackled through enlargement systems, i.e. for playing a technique called dynamic feedback, see [Brewer and Klinger, 1988] for reading general case and [Hermida and Trobajo, 2003] for reading case R a principal ideal domain, and for playing a technique called weakly feedback, see [Hermida and López, 2006].

So, on the one hand, we say that two m-input n-dimensional systems $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ are dynamically feedback equivalent,and write $\Sigma \approx_R \Sigma'$, if $\Sigma(r)$ is feedback equivalent to $\Sigma'(r)$ for some positive integer r, where

$$\Sigma(r) = \left(\left(\begin{array}{c|c} 0_{r\times r} & 0 \\ 0 & A \end{array} \right), \left(\begin{array}{c|c} \mathrm{Id}_r & 0 \\ 0 & B \end{array} \right) \right).$$

On the other hand, we say that two m-input n-dimensional systems $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ are weakly feedback equivalent if $\Sigma[s]$ is feedback equivalent to $\Sigma'[s]$ for some positive integer s, where

$$\Sigma[s] = \left(A, \left(B \middle| 0_{n\times s} \right) \right).$$

3 Single-input case

Let R be a Bezout domain and let $\Sigma = (A, \underline{b})$ be an n-dimensional single-input linear dynamical system over

R. In the sense of feedback equivalence, Σ can be reduced to a row echelon form. That is

$$\Sigma \sim_R \left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ d_2 & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & \cdots & d_n & a_{nn} \end{pmatrix}, \begin{pmatrix} d_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \right),$$

and we say that $\{d_1, d_2, \ldots, d_n\}$ is the diagonal sequence of the system Σ. Moreover, the diagonal sequence of a reduced form is a feedback invariant up to units and it determines equivalence class of the control system Σ, see [Carriegos and Hermida, 2003].

4 Multi-input case

Through this section, let R be a principal ideal domain. Let $\Sigma = (A, B)$ be an n-dimensional m input linear dynamical system over R. Since R is a pid, without loss of generality we can assume that multi-input matrix B is rewrote, by some changes of basis P and Q, as

$$B = \left(\begin{array}{c|c} D & 0_{t\times(m-t)} \\ \hline 0_{(n-t)\times t} & 0_{(n-t)\times(m-t)} \end{array} \right) \in \mathcal{M}_{n\times m}(R),$$

where $D = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_k \end{pmatrix} \in \mathcal{M}_{t\times t}(R)$, $D_i = \begin{pmatrix} d_i & & \\ & \ddots & \\ & & d_i \end{pmatrix} \in \mathcal{M}_{t_i\times t_i}(R)$, $t = \sum_{i=1}^{k} t_i$ and $d_1/d_2/\ldots/d_k$.

Now, let $\Sigma' = (A', B')$ be another n-dimensional m input linear dynamical system over R feedback equivalent to Σ. By some changes of basis P' and Q', multi-input matrix B' is assumed as matrix B. So, at this moment we have two linear systems $\Sigma = (A, B)$ and $\Sigma' = (A', B)$ feedback equivalent over R with input matrix B in the above form. Furthermore, if we consider the systems

$$\Sigma_w = \left(A, \left(\frac{D}{0_{(n-t)\times t}} \right) \right)$$

and, analogously Σ'_w, then it is clear that

$$\Sigma \sim_R \Sigma' \Leftrightarrow \Sigma_w[m-t] \sim_R \Sigma'_w[m-t].$$

Remark 4.1. *it is known that feedback equivalence and weakly feedback equivalence are equivalent concepts over principal ideal domains, see [Hermida and López, 2006].*

Hence, following this idea

$$\Sigma \sim_R \Sigma' \Leftrightarrow \Sigma_w \sim_R \Sigma'_w.$$

Lemma 4.2. *Let R be a commutative ring with unit element. Let Σ be the $(t_i + t_{i+1})$-input $(t_i + t_{i+1})$-dimensional linear system given by*

$$\Sigma = \left(\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline B_1 & A_1 \end{array} \right), \left(\begin{array}{c|c} D_i & 0 \\ \hline 0 & D_{i+1} \end{array} \right) \right)$$

with $D_i = d_i \mathrm{Id}_{t_i}$, $D_{i+1} = \alpha_i d_i \mathrm{Id}_{t_{i+1}}$ and d_i a nonzero element of R. Suppose that the t_i-input $(n - t_i)$-dimensional system (A_1, B_1) is feedback equivalent to (A'_1, B'_1). Then there exist C_{11} and C_{12} matrices such that Σ is feedback equivalent to the system

$$\Sigma' = \left(\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline B'_1 & A'_1 \end{array} \right), \left(\begin{array}{c|c} D_i & 0 \\ \hline 0 & D_{i+1} \end{array} \right) \right).$$

Proof. Let (P_1, Q_1, F_1) the feedback action between (A_1, B_1) and (A'_1, B'_1). That is

$$P_1 A_1 - A'_1 P_1 = B'_1 F_1, \qquad P_1 B_1 = B'_1 Q_1.$$

Consider the invertible $n \times n$ block matrix P, the invertible $(t_i + t_{i+1}) \times (t_i + t_{i+1})$ matrix Q and the $(t_i + t_{i+1}) \times (t_i + t_{i+1})$ matrix F given by

$$P = \left(\begin{array}{c|c} Q_1 & F_1 \\ \hline 0 & P_1 \end{array} \right), \qquad Q = \left(\begin{array}{c|c} Q_1 & \alpha_1 F_1 \\ \hline 0 & P_1 \end{array} \right), \qquad F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

An easy calculation shows that

$$P \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline B_1 & A_1 \end{array} \right) - \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline B'_1 & A'_1 \end{array} \right) P = \left(\begin{array}{c|c} D_i & 0 \\ \hline 0 & D_{i+1} \end{array} \right) F,$$

where $C_{12} = (Q_1 A_{11} + F_1 B_1 - C_{11} F_1) P_1^{-1}$ and $C_{11} = (Q_1 A_{11} + F_1 B_1) Q_1^{-1}$ and

$$P \left(\begin{array}{c|c} D_i & 0 \\ \hline 0 & D_{i+1} \end{array} \right) = \left(\begin{array}{c|c} D_i & 0 \\ \hline 0 & D_{i+1} \end{array} \right) Q. \qquad \square$$

Corollary 4.3. *Let R be a principal ideal domain. Let Σ be the t_i-input n-dimensional linear system given by*

$$\Sigma = \left(\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline B_1 & A_1 \end{array} \right), \left(\frac{D_i}{0} \right) \right)$$

with $D_i = d_i \mathrm{Id}_{t_i}$ and d_i a nonzero element of R. Suppose that the t_i-input $(n - t_i)$-dimensional system (A_1, B_1) is feedback equivalent to (A'_1, B'_1). Then there exist C_{11} and C_{12} matrices such that Σ is feedback equivalent to the system

$$\Sigma' = \left(\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline B'_1 & A'_1 \end{array} \right), \left(\frac{D_i}{0} \right) \right)$$

Proof. The result is obtained by remark 4.1and by previous lemma 4.2 with $\alpha_i = 0$. $\square$

Note that in under result, we deal with $\pi : R \longrightarrow R(d)$ the canonical ring homomorphism of R onto the quotient ring $R/(d)$, where $d \neq 0$ is a non-unit of R. The extension of a system $\Sigma = (A, B)$ to $R/(d)$ is the linear system $\pi(\Sigma) = (\pi(A), \pi(B))$ where $\pi(A) = (\pi(a_{ij}))$ and $\pi(B) = (\pi(b_{ij}))$.

Theorem 4.4. *Let R be a principal ideal domain. Let $\Sigma = (A, B)$ and $\Sigma' = (A', B)$ be the $(t_i + t_{i+1})$-input $(t_i + t_{i+1})$-dimensional linear systems given by*

$$\Sigma = \left(\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline B_1 & A_1 \end{array} \right), \left(\begin{array}{c|c} D_i & 0 \\ \hline 0 & D_{i+1} \end{array} \right) \right)$$

and

$$\Sigma' = \left(\left(\begin{array}{c|c} A'_{11} & A'_{12} \\ \hline B'_1 & A'_1 \end{array} \right), \left(\begin{array}{c|c} D_i & 0 \\ \hline 0 & D_{i+1} \end{array} \right) \right)$$

with $D_i = d_i \mathrm{Id}_{t_i}$, $D_{i+1} = \alpha_i d_i \mathrm{Id}_{t_{i+1}}$, d_i a nonzero element and $d_{i+1} = \alpha_i d_i$ a non-unit of R. Assume that extended systems $\pi(\Sigma)$ and $\pi(\Sigma')$ are feedback equivalent over $R/(\alpha_i d_i)$. Then the linear systems Σ and Σ' are dynamically feedback equivalent over R.

Proof. From Theorem 2.6 of [Hermida and López, 2006], we have that system

$$\Sigma_{d_{i+1}} = \left(A, \left(\begin{array}{c|c|c|c} D_i & 0 & d_{i+1}\mathrm{Id}_{t_i} & 0 \\ \hline 0 & D_{i+1} & 0 & D_{i+1} \end{array} \right) \right)$$

and the analogous system $\Sigma'_{d_{i+1}}$ are dynamically feedback equivalent over R. We follow the proof by considering the invertible matrix

$$Q = \begin{pmatrix} \mathrm{Id} & 0 & -\alpha_i \mathrm{Id} & 0 \\ 0 & \mathrm{Id} & 0 & -\mathrm{Id} \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & \mathrm{Id} \end{pmatrix}$$

as a feedback action over each of the systems Σ and Σ'. So, we obtain that $\Sigma[(t_i + t_{i+1})]$ and $\Sigma'[(t_i + t_{i+1})]$ are dynamically equivalent. Finally, we conclude Σ is dynamically feedback equivalent to $\Sigma'[(t_i + t_{i+1})]$ by remark 4.1. $\square$

Example 4.5. *Let $\Sigma = (A, B)$ and $\Sigma' = (A', B)$ be the 4-input 4-dimensional reduced forms over $R = \mathbb{Z}$ given by*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 5 & 2 & 4 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad A' = \begin{pmatrix} 3 & -4 & 1 & 5 \\ 3 & 2 & 7 & 6 \\ 0 & 5 & 6 & 9 \\ 0 & 0 & 2 & 7 \end{pmatrix},$$

$$B = \left(\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{array} \right).$$

We prove that Σ and Σ' are dynamically feedback equivalent an we give a procedure for finding (P, F) feedback equivalence action between Σ and Σ'.

- Firstly, we consider $\pi(\Sigma) = (\pi(A), \pi(B))$ and $\pi(\Sigma') = (\pi(A'), \pi(B))$ extended systems over $R/(d)$ with $d = 6$. Hence, we can write

$$\pi(B) = \left(\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

- Secondly, in [Carriegos and Hermida, 2003] is presented a numerical procedure in order to obtain (P_1, F_1) matrices pair of feedback action for proving that $\Sigma_1 = (A, \underline{b})$ and $\Sigma'_1 = (A', \underline{b})$ single-input systems are feedback equivalent, with $\underline{b} = (2\ 0\ 0\ 0)^t$.

$$P_1 = \begin{pmatrix} 1 & 4 & 9 & 17 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} -7 \\ 6 \\ -2 \\ -3 \end{pmatrix}.$$

- Thirdly, $\pi(\Sigma) = (\pi(A), \pi(B))$ and $\pi(\Sigma') = (\pi(A'), \pi(B))$ systems are feedback equivalent by

$$P_2 = P_1, \quad Q_2 = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathrm{Id}_3 \end{array} \right), \quad F_2 = \left(\begin{array}{c} F_1 \\ \hline 0 \end{array} \right).$$

- Fourthly, by Theorem 2.1 of [Hermida and López, 2006], we have that if $\pi(\Sigma) = (\pi(A), \pi(B))$ and $\pi(\Sigma') = (\pi(A'), \pi(B))$ systems are feedback equivalent over $R/(d)$, then $\Sigma_2 = (A, (B \mid d\mathrm{Id}_4 \mid 0_{4\times 1}))$ and $\Sigma'_2 = (A, (B \mid d\mathrm{Id}_4 \mid 0_{4\times 1}))$ are dynamically feedback equivalent over R, by

$$P_3 = \begin{pmatrix} P'_2 & -H \\ d\mathrm{Id}_4 & P_2 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} P'_2 & -H\underline{b} & -dH & 0 \\ 0 & Q_2 & 0 & -dS \\ \mathrm{Id}_4 & N & P_2 & \underline{b}S \\ 0 & \mathrm{Id}_1 & 0 & Q'_2 \end{pmatrix},$$

$$F_3 = \begin{pmatrix} 0 & -HA \\ 0 & F_2 \\ -A' & M \\ 0 & 0 \end{pmatrix},$$

where $P'_2 P_2 + dH = \mathrm{Id}_4$ and $Q_2 Q'_2 + dS = \mathrm{Id}_1$. Observe that there exist P'_2, H, Q'_2 and S matrices over R

because P_2 and Q_2 are invertible matrices over $R/(d)$. Moreover, these matrices P_2', H, Q_2' and S can be calculated by means of Cayley-Hamilton theorem.

- Fifthly, as $2 = d_1/d_2 = 6$ we have that $\Sigma_2 = (A, (B \mid d\mathrm{Id}_4 \mid 0_{4\times 1}))$ is feedback equivalent to $\Sigma[2]$ and $\Sigma_2' = (A', (B \mid d\mathrm{Id}_4 \mid 0_{4\times 1}))$ is feedback equivalent to $\Sigma'[2]$ by

$$P_4 = \mathrm{Id}_4, \quad Q_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad F = 0_{6\times 4}.$$

Note that, on the one hand the input matrix of new systems $\Sigma[2]$ and $\Sigma'[2]$ is

$$B[2] = \left(\begin{array}{c|ccc|cc} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \end{array}\right)$$

and that, on the other hand, we have the chain of equivalences

$$\Sigma[2] \sim_R \Sigma_2 \approx_R \Sigma_2' \sim_R \Sigma'[2].$$

Hence, $\Sigma[2]$ and $\Sigma'[2]$ systems are dynamically equivalent by

$$P_5 = P_4 P_3 P_4^{-1}, \quad Q_5 = Q_4 Q_3 Q_4^{-1},$$

$$F_5 = F_4 P_4 P_3 P_4^{-1} + Q_4(F_3 P_4^{-1} + Q_3(-Q_4^{-1} F_4 P_4^{-1})).$$

- Sixthly and finally, by remark 4.1, we have that $\Sigma = (A, B)$ and $\Sigma' = (A', B)$ linear systems are dynamically feedback equivalent. Furthermore, if we write

$$Q_5 = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad F_5 = \begin{pmatrix} F_{11} \\ F_{21} \end{pmatrix}$$

with Q_{11} a 4×4 matrix and F_{11} a 4×4 matrix, then the (P, Q, F) feedback action of the dynamic equivalence over R between $\Sigma = (A, B)$ and $\Sigma' = (A', B)$, is given by

$$P = P_5, \quad Q = Q_{11}, \quad F = F_{11}.$$

Note that, in Proposition 2.4 of [Hermida and López, 2006], it is proved that in above conditions Q_{11} matrix is invertible over R. $\square$

5 Conclusion

Since row echelon form of single-input case and throughout lifting from quotient rings, it is in our aim to determinate feedback invariants and canonical form of a multi-input linear dynamical system over principal ideal domains.

Acknowledgements

This research has been partially supported by INTECO-MInETur (Spain) and MinECo (Spain).

References

Brewer, J. W.,Bunce, J. W. and Van Vleck, F. S. (1986). *Linear Systems over Commutative Rings*. Marcel-Dekker. New York.

Brewer, J. W. and Klinger, L. (1988) Dynamic feedback over commutative rings. *Linear Algebra Appl.*, **98**, pp. 137–168.

Brewer, J. W. and Klinger, L. (2001) On feedback invariants for linear dynamical systems. *Linear Algebra Appl.*, **325**, pp. 209–220.

Brunovsky, P. A. (1970) A classification of linear controllable systems. *Kybernetika*, **3**, pp. 173–187.

Carriegos, M. and García Planas, I. (2004) On matrix inverses modulo a subspace. *Linear Algebra Appl.*, **379**, pp. 229–237.

Carriegos, M. and Hermida-Alonso, J. A. (2003) Canonical forms for single input linear systems. *Syst. Control Lett.*, **49**, pp. 99–110.

Carriegos, M. and Sánchez-Giralda, T. (2001) Canonical forms for linear dynamical systems over commutative rings: the local case. *Lecture Notes in Pure and Appl. Math.* , **221**, pp. 113–133.

Estes, D. and Ohm, J. (1967) Stable range in commutative rings. *J. Algebra*, **7(3)**, pp. 343–362.

Hermida-Alonso, J. A. (2003). *Handbook of Algebra* **vol.3)**, pp. 3–61. Elsevier Science.

Hermida, J. A. and López-Cabeceira, M. M. (2006) Dynamic feedback over principal ideal domains and quotient rings. *Linear Algebra Appl.*, **413**, pp. 235–244.

Hermida, J. A., López-Cabeceira, M. M. Trobajo, M. T. (2005) When are dynamic and static feedback equivalent? *Linear Algebra Appl.*, **405**, pp. 74–82.

Hermida, J. A. and Trobajo, M. T. (2003) The dynamic feedback equivalence over principal ideal domains. *Linear Algebra Appl.*, **368**, pp. 197–208.

McDonald, B. R. (1984). *Linear Algebra over Commutative Rings*. Marcel-Dekker. New York.

Sontag, E. D. (1998). *Mathematical Control Theory - 2nd ed*. Springer-Verlag. New York.