

## OPTIMAL CONTROL OF NON-HOLONOMIC 3-STEP NILPOTENT SYSTEMS

**Alfonso Anzaldo-Meneses**

Basic Sciences Department  
UAM-Azcapotzalco  
México  
answald@ymail.com

**Felipe Monroy-Pérez**

Basic Sciences Department  
UAM-Azcapotzalco  
México  
fmp@correo.azc.uam.mx

### Abstract

This paper considers the optimal control problem for a control affine driftless system with quadratic cost, defined by means of a left invariant distribution on a Lie group satisfying the property that the Lie algebra generated by the distribution is 3-step nilpotent. The Pontryagin Maximum Principle provides necessary conditions for the extremal trajectories and in some low dimensions explicit integration for the adjoint system is carried out. The explicit integration of these cases yields the elements for completing the optimal synthesis. In dimension five these systems are related with the so called cross-chained form.

### Key words

Non-holonomic system, 3-step nilpotent Lie algebra, Pontryagin Maximum Principle.

### 1 Introduction

In the nineteenth century the German physicist H.R. Hertz (1857-1894) coined the term holonomic, (from the Greek roots *hólos* meaning *whole* and *nomos* meaning *law*), for describing some mechanical systems subject to velocity constraints. Generally speaking a system is said to be non-holonomic with respect to a given constrained motion, if the system can evolve between any two given configurations without violating the constraints, otherwise is said to be holonomic. A prototypical non-holonomic system is the one of a sphere rolling over the plane without slipping and twisting.

There is a large amount of literature regarding holonomic and non-holonomic constraints for mechanical systems, we refer the reader to the classical book in classical mechanics [Whittaker E.T., 1988] and also [Neimark J.I., 1972], both containing plenty of interesting examples.

To be precise, assume that  $q = (q_1, \dots, q_n)$  denotes the coordinates of the configuration space of the system, and that the evolution of the system obeys to an

ensemble of  $m$  linear constraints on the velocities written as  $\sum_{i=1}^n \alpha_{ij}(q) \dot{q}_i = 0$ ,  $j = 1, \dots, m$ . If it is possible to find constraints on the position only, say  $\beta_1(q) = \dots = \beta_m(q) = 0$ , in such a way that,  $\sum_{i=1}^n \frac{\partial \beta_j}{\partial q_i} \dot{q}_i = 0$ ,  $j = 1, \dots, m$ , determines the same ensemble of constraints for the system, then it is said that the constraints are *holonomic*, otherwise they are called *non-holonomic*.

In the control theory literature non-holonomic systems appear as models of mechanical systems with external forces, the constraints come up in two flavors: the ones that are obtained from the derivation of the equations of motion from the Euler-Lagrange equations (or from the Hamiltonian formalism), such constraints are not imposed on the system and it is better to take them as conservation laws rather than as genuine constraints; and those constraints that are direct consequence of the kinematics, such as the non slipping and twisting of the rolling, see for instance [Bloch A., 2003].

An important class of non-holonomic control systems is the one of driftless control-affine systems, such systems are defined either by a finite family of vector fields or by a Pfaffian system, that is, by the kernel of a finite family of differential 1-forms. A relevant example of this class is the so-called Goursat chained form, which provides a model for a robot towing a finite number of trailers, all of them satisfying the non slipping and twisting rolling conditions.

If  $Q$  denotes the configuration space of this mechanical system, the dynamical equation can be written as  $\dot{q} = u_1 X_1(q) + u_2 X_2(q)$ ,  $q \in Q$ , where  $u_1$  and  $u_2$  are the control parameters of the velocity of the center of mass of the leading robot, and the vector fields  $X_1$  and  $X_2$  satisfy the following commuting relations:  $[X_1, X_i] =: X_{i+1}$ ,  $i = 2, \dots, n$ , with  $n = \dim Q$ , (of course  $n$  depends on the number of trailers), and all other commutators are zero, for details see [Tilbury D. et al., 1995].

An important feature of this system is that the Lie algebra generated by the set of vector fields  $\Delta =$

$\{X_1, X_2\}$  is nilpotent. The property of a system of being nilpotent presents theoretical and computational advantages for tackling various problems in control: optimal synthesis, path planning, small time controllability, stability etc. However it is a very strong condition to impose, in part for this reason techniques of nilpotent approximations for control systems have been extensively developed.

In this paper we consider the optimal control problem of a driftless control-affine system with quadratic cost, for which the Lie algebra generated by the vector fields defining the system is 3-step nilpotent, that is, Lie brackets of length greater than three vanish.

Apart from this introduction the paper contains five sections, in section 2 we present a characterization of non-holonomic control systems and the basic definitions of nilpotent Lie algebras and nilpotent approximations. In section 3 we formulate the optimal control problem and apply the Pontryagin Maximum Principle that provides necessary conditions for the optimal controls. In section 4 we specialize the general results to some low dimensional cases, in particular we discuss the so-called cross-chained form. At the end in section 5 we derive some conclusions and perspectives of future work on the study of non-holonomic nilpotent control systems.

## 2 Non-holonomic nilpotent systems

We present in this section a characterization of non-holonomic control systems and the basic definitions of nilpotent Lie algebras and nilpotent approximations.

### 2.1 Non-holonomic control systems

Let  $G$  be a smooth manifold, and let  $\Delta = \{X_1, \dots, X_n\}$  with  $n < \dim(G)$  be a distribution of smooth vector fields on  $G$ , the Lie algebra generated by the distribution is denoted as  $\mathcal{G}(\Delta)$  and it consists of the Lie algebra spanned by iterations of all the Lie brackets of elements of  $\Delta$ . It is said that the distribution  $\Delta$  is of *full rank*,<sup>1</sup> if for all  $g \in G$  it holds that  $\mathcal{G}(\Delta)_g = T_g G$ .

For  $k = 1, 2, \dots$ , the modules of vector fields  $\Delta^k$  are defined inductively as follows:  $\Delta^1 := \Delta$  and  $\Delta^{k+1} := \Delta^k + [\Delta, \Delta^k]$ . For each  $g \in G$ , the full rank condition implies the existence of an integer  $\nu(g)$  such that  $\Delta_g^{\nu(g)} = T_g G$ , together with a flag of modules of vector fields naturally defined as

$$\Delta_g^1 \subset \Delta_g^2 \subset \dots \subset \Delta_g^{\nu(g)} = T_g G.$$

Furthermore if  $n_i(g) = \dim \Delta_g^i$ ,  $i = 1, \dots, \nu$ , then the vector  $(n_1(g), n_2(g), \dots, n_\nu(g))$  is called the *growth*

<sup>1</sup>This condition goes also in the literature under the names of bracket generating distribution, Lie algebra rank condition or Hörmander condition.

vector of the distribution  $\Delta$  at  $g$ , and  $\nu(g)$  the *non-holonomy degree* of the distribution at  $g$ . The distribution is said to be regular at  $g$  if the growth vector is constant on a neighborhood of  $g$ , it is said to be regular in  $G$  if it is regular for all  $g$  with the same degree of nonholonomy.

Assume that  $\Delta$  is a full rank, regular distribution of vector fields in  $G$ , an absolutely continuous curve  $g : [0, T_g] \rightarrow G$  is said to be *admissible* for the distribution  $\Delta$  if satisfies  $\dot{g}(t) \in \Delta(g(t))$ , *a.e.*, which is tantamount of saying that, there is a measurable and bounded function  $t \mapsto \mathbf{u} = (u_1, \dots, u_n)$  such that  $g(t)$  is an admissible solution of the following control-affine system:

$$\dot{g}(t) = \sum_{i=1}^n u_i X_i(g(t)), \quad (1)$$

that we shall call it a *non-holonomic control system* on  $G$ ; the family of all admissible curves is denoted as  $\mathcal{A}$ , whereas the one of admissible control parameters is denoted as  $\mathcal{U}$ . It is known that the full rank condition guarantees the controllability of the system, see for instance [Jurdjevic V., 1997]. A smooth varying inner product  $g \mapsto \langle \cdot, \cdot \rangle_g$  on the vector spaces  $\Delta_g$  can be defined by means of  $\langle X_i, X_j \rangle_g = \delta_{ij}$ , in such a way that the energy functional  $\Lambda : \mathcal{A} \rightarrow \mathbb{R}^+$  for admissible curves is written as follows:

$$\Lambda(g, \mathbf{u}) = \int_0^{T_g} \langle \dot{g}(t), \dot{g}(t) \rangle = \int_0^{T_g} u_1^2 + \dots + u_n^2. \quad (2)$$

In this paper we study the optimal control problem defined by (1) and (2), for the particular case when  $\mathcal{G}(\Delta)$  is a 3-nilpotent and 2-solvable Lie algebra.

### 2.2 Nilpotent Lie algebras and control systems

In order to set the notation for presenting our results, we shall digress on some aspects of structure theory of  $n$ -step nilpotent Lie algebras, for more details we refer the reader to [Jacobson N., 1962] and [Corwin L., and Greenleaf F.P., 1990].

### 2.3 Nilpotent and solvable Lie algebras

Let  $\mathfrak{g}$  be a Lie algebra over  $\mathbb{R}$ , the *lower central series* is defined follows  $\mathfrak{g} := \mathfrak{g}^1 \supseteq [\mathfrak{g}^2, \mathfrak{g}] \supseteq [\mathfrak{g}^3, \mathfrak{g}] \dots$ , where  $\mathfrak{g}^j := [\mathfrak{g}^{j-1}, \mathfrak{g}]$  for  $j = 2, 3, \dots$ . The Lie algebra is said to be *nilpotent* if there is an integer  $n$  such that  $\mathfrak{g}^{n+1} = 0$ , if such a  $n$  is minimal in the sense that  $\mathfrak{g}^n \neq 0$  then the Lie algebra is said to be  *$n$ -step nilpotent*. The Jacobi identity together with an elementary induction argument clearly imply

$$[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j} \cup \text{for all } i \text{ and } j. \quad (3)$$

As a consequence, any product of  $k$  elements of  $\mathfrak{g}$  is an element of  $\mathfrak{g}^k$ , independently of the order. Furthermore,  $\mathfrak{g}$  is  $n$ -step nilpotent if and only if all brackets of order greater than  $n$  vanish. A typical element  $x \in \mathfrak{g}^k$  is the monomial  $[X_{i_1}[X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}] \dots]]$ , with  $\{X_{i_1}, X_{i_2}, \dots, X_{i_k}\} \subset \mathfrak{g}$ , the degree of an such element is naturally defined as  $\deg(x) = k$ .

In a similar manner, the *derived series* of  $\mathfrak{g}$  is defined inductively as follows  $\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \dots$ , with  $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$  and  $\mathfrak{g}^{(j)} = [\mathfrak{g}^{(j-1)}, \mathfrak{g}^{(j-1)}]$  for  $j = 2, 3, \dots$ . The Lie algebra is said to be *solvable* if there is an integer  $m$  such that  $\mathfrak{g}^{(m)} = 0$ . For then, relation (3) implies  $\mathfrak{g}^{(k)} \supseteq \mathfrak{g}^{2^k}$ , for all  $k$ , therefore nilpotent Lie algebras are also solvable.

There is a collecting process for organizing the commutators of free Lie algebras generated by a finite number of elements, that was originally presented by Philip Hall in [Hall Ph., 1934] and has recently been utilized in applications to path planning problem and constructive controllability, see for instance [Laferriere G. and Sussmann H., 1992].

Let  $\mathfrak{g}$  be the free Lie algebra generated by the symbols  $\{X_1, \dots, X_p\}$  which are considered of being of degree one. For two given monomials  $m_1$  and  $m_2$  the relation  $\deg[m_1, m_2] = \deg(m_1) + \deg(m_2)$  readily follows. A linear combination of monomials of degree  $k$  is said to be homogeneous of degree  $k$ , and any element of  $\mathfrak{g}$  is written as linear combination of monomials.

Since there are only a finite number of monomials of a given degree, then for each  $n$ , a number of monomials, say  $m_{n_1}, \dots, m_{n_s}$ , denominated *standard monomials*, which are linearly independent and have the property that each homogeneous expression of degree  $n$  is written as linear combination of  $m_{n_1}, \dots, m_{n_s}$ , such a collection is defined recursively as follows:

**Definition 2.1.** *The standard monomials of degree one are  $X_1, \dots, X_p$ . Assume that the standard monomials of degree  $n - 1$ , are defined, and that they are  $\prec$ -ordered in such a way that  $u \prec v$  provided  $\deg(u) < \deg(v)$ . If  $\deg(x) = i, \deg(v) = j$  and  $\deg[x, v] = i + j$ , then  $[x, v]$  is a standard monomial if and only if satisfies:*

1.  $x$  and  $v$  are standard monomials with  $x \prec v$ .
2. If  $v = [y, z]$  then  $y \preceq x$  and  $y \prec z$ .

An element of the free Lie algebra  $\mathfrak{g}$  is said to be in *standard form* if it is written as linear combination of *standard monomials*.

**Theorem 2.1.** (M. Hall, 1950) *The standard monomials form a basis for the free Lie algebra  $\mathfrak{g}$  generated by  $X_1, \dots, X_p$ .*

Applying this process to a 3-step nilpotent free Lie algebra generated by  $\{X_1, X_2, \dots\}$  one has that a basis is given by

$$\begin{aligned} [X_{i_1}, X_{i_2}] &= X_{i_1 i_2}, \quad i_1 < i_2, \\ [X_{i_{12}}, X_{i_1 i_2}] &= X_{i_{12}, i_1 i_2}, \quad i_1 < i_2 \leq i_{12}; \\ [X_{i_2}, X_{i_1 i_{12}}] &= X_{i_2, i_1 i_{12}}, \quad i_1 \leq i_2 < i_{12}, \end{aligned}$$

and the remaining elements are again  $X_{i_2, i_1} = -X_{i_1, i_2}$  and  $[X_{i_1}, X_{i_2 i_{12}}] = X_{i_{12}, i_1 i_2} - X_{i_2, i_1 i_{12}}$ ,  $i_1 < i_2 < i_{12}$  with  $i_1, i_2, i_{12} = 1, \dots, n$ .

**Example 2.1.** *The basis for a 3-step nilpotent free Lie algebra generated by seven symbols  $\Delta = \{X_1, X_2, X_{12}, X_{112}, X_{212}, X_6, X_7\}$ .*

*The standard monomials of degree 2, are denoted as  $X_{ij} = [X_i, X_j], i < j$ . We have then the seven elements of  $\Delta$ , and the 21 elements of length two,*

$$\begin{aligned} \Delta_{12} &= \{X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}\}, \\ \Delta_{23} &= \{X_{23}, X_{24}, X_{25}, X_{26}, X_{27}\}, \\ \Delta_{34} &= \{X_{34}, X_{35}, X_{36}, X_{37}\}, \\ \Delta_{45} &= \{X_{45}, X_{46}, X_{47}\}, \\ \Delta_{56} &= \{X_{56}, X_{57}\}, \\ \Delta_{67} &= \{X_{67}\}, \end{aligned}$$

and if we denote as  $\Delta_{ijk} = \{[X_i, X_{jk} \mid X_{jk} \in \Delta_{jk}]\}$ , we further have the following 112 elements of length three

$$\begin{aligned} \Delta_{112} \cup \Delta_{212} \cup \Delta_{312} \cup \Delta_{412} \cup \Delta_{512} \cup \Delta_{612} \cup \Delta_{712} \\ \cup \Delta_{223} \cup \Delta_{323} \cup \Delta_{423} \cup \Delta_{523} \cup \Delta_{623} \cup \Delta_{723} \\ \cup \Delta_{334} \cup \Delta_{434} \cup \Delta_{534} \cup \Delta_{634} \cup \Delta_{734} \\ \cup \Delta_{445} \cup \Delta_{545} \cup \Delta_{645} \cup \Delta_{645} \\ \cup \Delta_{556} \cup \Delta_{656} \cup \Delta_{756} \\ \cup \Delta_{667} \cup \Delta_{767} \end{aligned}$$

## 2.4 Nilpotent approximations

A nilpotent approximation of a distribution of vector fields is another family of vector fields with the same generic properties that further has the property of generating a nilpotent Lie algebra. The definition of nilpotent approximations is based on the notion of *order of smooth functions and vector fields*, see for instance [Vendittelli M. et al., 2004].

Let  $G$  be a smooth manifold, and let  $\Delta = \{X_1, \dots, X_n\} \subset TG$  be a regular and full rank distribution of vector fields. A smooth function  $f : G \rightarrow \mathbb{R}$  is said to be of *order*  $\geq k$  at a point  $g \in G$ , if all its Lie derivatives, (with respect to vectors  $X_i$ 's) of order  $\leq k - 1$  vanish at  $g$ , if is of order  $\geq k$  but is not of order  $\geq k + 1$  at  $g$ , then it is said to be of order  $k$  at  $g$ .

A vector field  $X$  is said to be of *order*  $\geq k$  at a point  $g \in G$  if for every  $j$  and every function  $f$  of order  $j$  at

$g$ , the function  $X(f)$  has order  $\geq k+j$  at  $g$ , again if  $X$  is of order  $\geq k$  but is not of order  $\geq k+1$  at  $g$ , then it is said to be of order  $k$  at  $g$ . From this definition of order it is clear that the  $X_i$ s are of order  $-1$ , the brackets  $[X_i, X_j]$  are of order  $-2$ , etc.

A family of vector fields  $\tilde{\Delta} = \{\tilde{X}_1, \dots, \tilde{X}_n\}$  is said to be a *nilpotent approximation* of  $\Delta$  at  $g$  if

1. The vector fields  $X_i - \tilde{X}_i$  are of positive order at  $g$
2. The Lie algebra  $\mathcal{G}(\tilde{\Delta})$  is  $\kappa$ -step nilpotent, with  $\kappa > \nu(g)$ , that is Lie brackets of length greater than  $\kappa$  vanish.

The explicit computation of nilpotent approximations for a given distribution is rather technical and is based on the existence of the so-called privileged coordinates. There are in the literature various algorithmic processes for finding nilpotent approximations, see for instance [Vendittelli M. et al., 2004], however, those processes are far from the purposes of this paper. From now on we shall consider regular full rank distributions that generate 3-step nilpotent Lie algebras, or distributions that are 3-step nilpotent approximations of non-nilpotent ones

### 3 The optimal control problem

Following the technique of completing the Philip Hall basis for finitely generated Lie algebras, it has been shown in [Monroy-Pérez F. and Anzaldo-Meneses, A., 2011] that a 3-step nilpotent, 2-solvable Lie algebra  $\mathfrak{g}$  generated by a set  $\Delta$  of  $n$  symbols has dimension at most

$$\eta := \underbrace{n}_{\Delta^1} + \underbrace{\frac{(n-1)n}{2}}_{\Delta^2} + \underbrace{\frac{(n-1)n(n+1)}{3}}_{\Delta^3}. \quad (4)$$

The associated Lie group and corresponding group law can be obtained by standard BCH techniques. It has also been shown in the aforementioned reference that good models for this situation are the Lie the subgroups of  $\mathbb{R}^n \times \mathfrak{so}_n \times \mathbb{R}^D$  where  $D = (n-1)n(n+1)/3$ ; in this case the group law can be written as follows:

$$g \odot h = (\alpha + \beta, a + b + \alpha \wedge \beta, \check{a} + \check{b} + \Gamma), \quad (5)$$

where

$$\alpha \wedge \beta = \frac{1}{2}(\alpha \otimes \beta^T - \beta \otimes \alpha^T), \quad \text{and}$$

$$\Gamma = -\frac{1}{2}\varphi(a, \alpha, b, \beta) - \frac{1}{12}\zeta(\alpha \wedge \beta, \alpha, \beta),$$

for certain smooth functions  $\varphi$  and  $\zeta$ .

For the remaining of the paper  $G$  shall be taken as the simply connected Lie group  $\mathbb{R}^n \times \mathfrak{so}_n \times \mathbb{R}^D$  of dimension  $\eta$  with group law (5), whose Lie algebra  $\mathfrak{g}$  is the 3-step nilpotent, 2-solvable Lie algebra generated by a given distribution  $\Delta = \{X_1, \dots, X_n\} \subsetneq TG$  of left invariant vector fields on  $G$ .

As explained in section 2, the left invariant distribution  $\Delta$  determines on the Lie group  $G$  an optimal control problem, namely: for certain given initial conditions find, among the solutions of (1), the one that minimizes the functional (2).

#### 3.1 The Hamiltonian formalism.

We approach the aforementioned optimal control problem by means of the Hamiltonian formalism on the cotangent bundle  $T^*G$  and the necessary condition for optimality given by the Pontryagin Maximum Principle of optimal control theory, see for instance [Agrachev A.A. and Sachkov Y.L., 2004] and [Jurjevic V., 1997]. We summarize in this paragraph the basic notation for the symplectic structure of the cotangent bundle and the Hamiltonian formalism.

The differential of the left translation  $g \mapsto L_g$  on  $G$ , yields the trivialization of the cotangent bundle  $T^*G \simeq G \times \mathfrak{g}^*$ , in such a way that Hamiltonian functions corresponding to left invariant vector fields on  $G$  are linear functions on  $\mathfrak{g}^*$ . Each left invariant vector field  $X$  on  $G$  defines a Hamiltonian function  $H_X$  on  $T^*G$  as  $H_X(g, p) = p(X(e))$ .

The double bundle  $T(T^*G)$  is identified with  $(G \times \mathfrak{g}) \times (\mathfrak{g}^* \times \mathfrak{g}^*)$ , and with this realization, any tangent vector  $((g, X), (p, Y^*)) \in T(T^*G)$ , is simply represented by means of the pair  $(X, Y^*) \in \mathfrak{g} \times \mathfrak{g}^*$ .

The canonical symplectic form on  $T^*G \simeq G \times \mathfrak{g}^*$ , allows to write, for each Hamiltonian function  $H$  on  $T^*G$ , the corresponding Hamiltonian vector field  $\vec{H}(g, p) = (X(g, p), Y^*(g, p))$  as follows

$$X(g, p) = \frac{\partial H}{\partial p}(g, p),$$

$$Y^*(g, p) = -dL_g^* \left( \frac{\partial H}{\partial g}(g, p) \right) - (\text{ad})^* X(p),$$

or equivalently, integral curves  $t \mapsto (g(t), p(t))$  of the Hamiltonian vector field  $\vec{H}$ , satisfy the Hamilton equations,

$$\frac{dg}{dt} = dL_g \left( \frac{\partial H}{\partial p} \right), \quad \text{and}$$

$$\frac{dp}{dt} = -dL_g^* \left( \frac{\partial H}{\partial g} \right) - \left( (\text{ad})^* \frac{\partial H}{\partial p} \right).$$

For details on the representation of tangent and cotangent bundles of Lie groups, and the integral curves of the Hamiltonian lifting of left invariant vector fields,

we refer the reader to V. Jurdjevic's book [Jurdjevic V., 1997].

### 3.2 The Pontryagin Maximum Principle.

Following the notation explained in example 2.1,  $X_{ij}$  denotes the Lie bracket  $[X_i, X_j]$  whereas  $X_{ijk}$  denotes  $[X_i, [X_j, X_k]]$ . The Hamiltonian functions associated to the left invariant vector fields  $X_i, X_{ij}$  and  $X_{ijk}$  shall be denoted as  $H_i, H_{ij}$  and  $H_{ijk}$ , respectively. These Hamiltonians span the dual Lie algebra  $\mathfrak{g}^*$ , endowed with the Poisson brackets that clearly satisfy the following commuting relations,

$$\{H_i, H_j\} = H_{ij} \text{ and } \{H_i, H_{jk}\} = H_{ijk},$$

evidently the Lie algebra  $\mathfrak{g}^*$  is also 3-step nilpotent and 2-solvable.

Each admissible control  $\mathbf{u} = (u_1, \dots, u_n) \in \mathcal{U}$ , determines a control dependent Hamiltonian function

$$\mathcal{H}_{\mathbf{u}}^{\lambda_0} = -\frac{\lambda_0}{2} [u_1^2 + \dots + u_n^2] + u_1 H_1 + \dots + u_n H_n.$$

Integral curves  $t \mapsto \xi(t) = (g(t), p(t))$  of the corresponding Hamiltonian vector field  $\mathcal{H}_{\mathbf{u}}^{\lambda_0}$  are called extremal curves, the ones for  $\lambda_0 \neq 0$ , are called normal, whereas the ones for  $\lambda_0 = 0$  are called abnormal. The necessary conditions for  $\Lambda$ -optimal trajectories, i.e., solutions of (1) that minimize (2), read as follows:

**Theorem 3.1.** (Pontryagin Maximum Principle) *If a trajectory  $t \mapsto (g, \hat{\mathbf{u}})$  is  $\Lambda$ -optimal then it is the projection of an extremal curve  $t \mapsto \xi = (g, p)$ , satisfying:*

- i  $\mathcal{H}_{\hat{\mathbf{u}}}^{\lambda_0}(\xi) \geq \mathcal{H}_{\mathbf{v}}^{\lambda_0}(\xi)$ , for all  $\mathbf{v} \in \mathcal{U}$
- ii  $\mathcal{H}_{\hat{\mathbf{u}}}^0(\xi)$  is not identically zero.

**Remark 3.1.** *Abnormal extremals, are trajectories independent of the cost functional, they play a very important role in the geometric analysis of the so-called optimal synthesis, these extremals deserve a careful treatment and shall not be discussed here. For the remaining of the paper we shall consider that all  $\Lambda$ -optimal trajectories are projections of normal extremals only.*

A direct application of the Pontryagin Maximum Principle yields the necessary condition for optimality. Observe that the dual variable can be expressed in terms of the dual basis as  $(h, \omega, \check{\mathbf{H}}) \in \mathbb{R}^n \times \mathfrak{so}_n \times \mathbb{R}^D$ , where  $h = (H_1, \dots, H_n)^T$ ,  $\omega = (H_{ij})_{i < j}$  and  $\check{\mathbf{H}} = (\Upsilon_1, \dots, \Upsilon_n)$  with  $\Upsilon_i$  the skew-symmetric matrix dual to the matrix  $A_k = (X_{ij k})_{ij}$  which result from the length three brackets organized according to the Philip Hall process, for details see [Monroy-Pérez F. and Anzaldo-Meneses, A., 2011].

**Theorem 3.2.** *If  $(g, \hat{\mathbf{u}})$  is a  $\Lambda$ -optimal then it is the projection of an extremal curve  $(h, \omega, \Upsilon_1, \dots, \Upsilon_n)$  along which  $\hat{\mathbf{u}} = (H_1, \dots, H_n)$  and*

$$\dot{h} = \omega h, \quad (6)$$

$$\dot{\omega} = \sum_{i=1}^n H_i \Upsilon_i \quad (7)$$

$$\dot{\Upsilon}_i = 0, \quad i = 1 \dots, n. \quad (8)$$

*Proof.* The maximality condition implies that along extremals we have  $\hat{\mathbf{u}} = (H_1, \dots, H_n)$ , for then the system Hamiltonian becomes quadratic

$$\mathcal{H} = \frac{1}{2}(H_1^2 + \dots + H_n^2). \quad (9)$$

A straightforward differentiation (Poisson bracketing with  $\mathcal{H}$ ), yields

$$\dot{H}_i = \{H_i, \mathcal{H}\} = \sum_{j \neq i} H_i H_{ij}, \quad (10)$$

similarly

$$\dot{H}_{ij} = \{H_{ij}, \mathcal{H}\} = - \sum_{k=1}^n H_k H_{ijk}, \quad (11)$$

and finally

$$\dot{H}_{ijk} = \{H_{ijk}, \mathcal{H}\} = 0, \quad (12)$$

as required.

**Remark 3.2.** *Equation (12) implies that all the length three Poisson brackets are constant along extremals, that is, we have already D integrals of motion.*

The coordinates  $g = (\alpha, a, \check{a})$  in  $G$ , can be chosen in such a way that the left invariant vector fields are written as  $X_i = \partial \alpha_i + \dots$ . In such a case, we have  $\dot{\alpha}_i = u_i = H_i$ , and therefore

$$\frac{d}{dt} \left( H_{ij} + \sum_{k=1}^n \alpha_k H_{kij} \right) = 0.$$

We introduce the skew-symmetric constant matrix  $c$  with elements

$$c_{ij} = H_{ij} + \sum_{k=1}^n \alpha_k H_{kij}.$$

therefore, from (10)

$$\ddot{\alpha}_i - \sum_{j=1}^n c_{ij} \dot{\alpha}_j = - \sum_{j,k=1}^n \alpha_k H_{kij} \dot{\alpha}_j,$$

that is,

$$\ddot{\alpha}_i + \sum_{j,k=1}^n \alpha_k H_{kij} \dot{\alpha}_j = 0 \quad (13)$$

Since the  $H_{kij}$  are constant, these equations are given in terms of the  $\alpha_i$ 's only. And together with the non-holonomic constraints (1), after plugging the optimal controls given by theorem (3.2), determine completely the optimal curves.

#### 4 Some low dimensional cases

We discuss here two examples for illustrating the general results of the above sections.

##### 4.1 The Cartan Lie algebra n

This case corresponds to  $n = 2$  and  $\eta = 5$ , and is provided by the rank 2 distribution  $\Delta = \{X_1, X_2\}$  for which the only non-zero Lie brackets are  $X_{12}, X_{112}, X_{212}$ . The Lie algebra  $\mathfrak{n} = \text{span}\{X_1, X_2, X_{12}, X_{112}, X_{212}\}$ , is known as the Cartan Lie algebra.

If  $H_i$  denotes the left invariant Hamiltonian associated with the vector field  $X_i$ , then we have the non-trivial Poisson brackets  $H_{12}, H_{112}, H_{212}$ .

The corresponding system Hamiltonian writes as follows

$$\mathcal{H} = \frac{\lambda_0}{2}(u_1^2 + u_2^2) + u_1 H_1 + u_2 H_2 \quad (14)$$

For the normal case ( $\lambda_0 = 1$ ), the maximality condition of the Maximum Principle, readily yield  $u_1 = H_1$  and  $u_2 = H_2$ , therefore the system Hamiltonian is quadratic  $\mathcal{H} = H_1^2 + H_2^2$ , and the adjoint system can be directly written as follows

$$\dot{H}_1 = \frac{1}{2}\{H_1, \mathcal{H}\} = H_2 H_{12} \quad (15)$$

$$\dot{H}_2 = \frac{1}{2}\{H_2, \mathcal{H}\} = -H_1 H_{12} \quad (16)$$

$$\dot{H}_{12} = \frac{1}{2}\{H_{12}, \mathcal{H}\} = -H_1 H_{112} - H_2 H_{212} \quad (17)$$

$$\dot{H}_{112} = \dot{H}_{212} = 0. \quad (18)$$

$H_{112}$  and  $H_{212}$  are central elements, and multiplying third equation by  $H_{12}$ , we get

$$\frac{1}{2} \frac{d}{dt} (H_{12}^2) = \frac{d}{dt} (H_2 H_{112} - H_1 H_{212}),$$

therefore we obtain the constant of integration

$$c_2 := \frac{1}{2} H_{12}^2 - H_2 H_{112} + H_1 H_{212}. \quad (19)$$

Further derivation of (17) yields

$$\ddot{H}_{12} = c_2 H_{12} - \frac{1}{2} H_{12}^3,$$

in consequence

$$\begin{aligned} \dot{H}_{12} \ddot{H}_{12} &= \left[ \frac{1}{2} \frac{d}{dt} (\dot{H}_{12})^2 \right] = c_2 H_{12} \dot{H}_{12} - \frac{1}{2} H_{12}^3 \dot{H}_{12} \\ &= c_2 \left[ \frac{1}{2} \frac{d}{dt} (H_{12})^2 \right] - \frac{1}{2} \left[ \frac{1}{4} \frac{d}{dt} (H_{12})^4 \right] \end{aligned}$$

we obtain then another constant of integration

$$c_3 := \frac{1}{4} H_{12}^4 - c_2 H_{12}^2 + (H_1 H_{112} + H_2 H_{212})^2$$

**Lemma 4.1.** *The elements of set  $\{\mathcal{H}, H_{112}, H_{212}, c_2\}$  are independent first integrals in involution, whereas  $\mathcal{K} := H_{112}^2 + H_{212}^2$  and  $c_3$  are neither independent nor in involution.*

*Proof.* A straightforward calculation shows that  $\{c_2, H_1\} = \{c_2, H_2\} = 0$ , therefore  $\{c_2, \mathcal{H}\} = 0$ . On the contrary we have,  $\{c_3, H_1\} = -2H_2 H_{12} \mathcal{K}$  and  $\{c_3, H_2\} = 2H_1 H_{12} \mathcal{K}$ . Consequently  $\{c_3, \mathcal{H}\} = 0$ , but  $c_3 = 2\mathcal{H}\mathcal{K} - c^2$  as can be easily shown.  $\square$

Thus the trajectories in cotangent space are given by the intersection of the cylinder  $H_1^2 + H_2^2 = 1$ , with the parabolic cylinder  $\frac{1}{2} H_{12} - H_2 H_{112} + H_1 H_{212} = c_2$ . A simple way to represent this curves is to note that they can be visualized as curves on the sphere  $(H_1 + H_{212})^2 + (H_2 - H_{112})^2 + H_{12}^2 = \mathcal{H} + 2c_2 + H_{112}^2 + H_{212}^2$ .

## 4.2 Cross-chained form

An alternative normal form to the Goursat [Tilbury D. et al., 1995] is the so-called cross-chained introduced by [Ishikawa M. and Kuroiwa T., 2009]. It turns out that such normal form yields the same 3-step nilpotent Lie algebra as the Cartan discussed above. The normal form is defined in  $\mathbb{R}^5$  by the kernel of the Pfaffian system

$$dx_5 - x_3 dx_2, dx_4 - x_3 dx_1, dx_3 + x_2 dx_1 - x_1 dx_2 \quad (20)$$

The non-holonomic control system consisting of a ball rolling on a plane without twisting and slipping and driven by the another plane can be modelled in the Lie group  $G = \mathbb{R}^2 \times \text{SO}_3$ : the first coordinate of a state  $(p, M) \in G$  yields the center of the ball, whereas the second provides the evolution of a moving frame attached to its center, for details see [Jurđjevic, V., 1993]. It can also be shown that the non-holonomic control system in this case can be reduced to the kernel of the following Pfaffian system in  $\mathbb{R}^5$

$$dx_3 - x_1 dx_2, dx_4 - x_1^2 dx_2, dx_5 - x_1 x_2 dx_2. \quad (21)$$

After a lengthy calculation including a change of variables and exterior differentiation it can be shown that system (21) can be taken into the crossed-chained normal form (20). Furthermore the crossed-chained normal encodes a driftless control system with two inputs given by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}$$

$$X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_5},$$

The Lie algebra generated by the distribution  $\Delta = \{X_1, X_2\}$  has the same Lie structure of  $\mathfrak{n}$ , as can be easily verified.

## 5 Conclusions

We have studied general properties for optimal trajectories of a problem defined by a driftless non-holonomic control system and a quadratic cost. We have considered the case when the Lie algebra generated by the distribution of vector fields is 3-step and 2-solvable. We have derived the geometric properties of the trajectories using the Pontryagin Maximum Principle and the associated Hamiltonian formalism. We have discussed a five dimensional case that models and interesting non-holonomic mechanical system.

## Acknowledgements

The final version of this paper was prepared during the sabbatical leave of F. Monroy-Pérez at the Laboratoire des Sciences de l'information et des Systèmes (LSIS, UMR 7296) in the Université du Sud Toulon-Var, France. The author was financially supported by the CONACYT under the program of sabbatical leaves abroad for the reinforcement of the research groups, project number 204051.

## References

- Whittaker E. T. (1988). *A treatise on the Analytical Dynamics of Particles and Rigid Bodies*. Cambridge University Press, 1988.
- Neimark J. I. and Fufaev N. A. (1972). *Dynamics of Nonholonomic systems*. Translations of Mathematical Monographs, AMS, **33**.
- Bloch A. M. (2003), *Nonholonomic Mechanics and Control*. Interdisciplinary applied mathematics, Springer, 2003.
- Tilbury D., Murray R. and Sastry S. (1995). Trajectory Generation for the N-Trailer Problem using Goursat Normal Form. *IEEE Transactions on Automatic Control*, **40**(5), pp. 802–819.
- Jurđjevic V. (1997). *Geometric control theory*, Cambridge studies in advanced mathematics 51, Cambridge University Press, 1997.
- Jacobson N. (1962). *Lie Algebras*. Intersciences tracts in pure and applied mathematics, Number 10, John Wiley & Sons, New York, 1962.
- Corwin L. and Greenleaf F.P. (1990). *Representations of nilpotent Lie groups and their applications. Part I: Basic theory and examples*, Cambridge University Press, 1990.
- Hall Ph. (1934). A contribution to the theory of groups of prime power order, *Proc. London Math. Soc.*, **36**, pp. 29–95.
- Laferrriere G. and Sussmann H.J. (1992). A differential geometric approach to motion planning in *Nonholonomic Motion Planning*, Eds.: Zexiang Li, Canny J.F. The Kluwer International Series in Engineering and Computer Science. V. 192, 1992.
- Vendittelli M., Oriolo G., Jean F. and Laumond J. P. (2004). Nonhomogeneous nilpotent approximations for nonholonomic systems with singularities. (2004). *IEEE Transactions on Automatic Control*, **49**(6), pp. 261–266.
- Monroy-Pérez F. and Anzaldo-Meneses A. (2011). A survey on 3-step nilpotent sub-Riemannian geometry *International journal of pure and applied mathematics*, **70**(4), pp. 521–548.
- Agrachev A. A. and Sachkov Y. L. (2004). *Control theory from the geometric viewpoint*, Encyclopædia of Mathematical Sciences, Vol. 87, Springer-Verlag, 2004.
- Ishikawa M. and Kuroiwa T. (2009). Nilpotent normal form for non-chained driftless systems: characterization of the cross-chained form. In *Joint 48th IEEE*

*Conference on Decision and Control and the 28th Chinese Control Conference* Shanghai, P.R. China, Dec. 16-18. pp. 4390–4395.

Jurdjevic V. (1993). The Geometry of the plate-ball problem, *Arch. Rational Mech. Anal.* **124**(4) , pp. 305–328.