Separation principle for linear quadratic control in statistically uncertain stochastic hybrid system*

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Abstract

We consider the minimax control problem for linear stochastic dynamic system with both continuous and discrete observations. It is assumed that the intensities of the continuous- and discrete-time noises are not known exactly. The only information available is that the noise intensities are constant and belong to a priori known compact uncertainty sets. The minimax control equations are provided in the explicit form.

1 Problem statement

Let a random process $y_t \in \mathbb{R}^p$ be described by the linear stochastic differential equation:

$$\mathrm{d}y_t = a_t y_t \mathrm{d}t + A_t u_t \mathrm{d}t + b_t \mathrm{d}w_t, \quad t \in [0,T], \quad y_0 = 0, \tag{1}$$

where a_t, A_t and b_t are some known piecewise continuous matrix functions; $u_t \in \mathbb{R}^l$ is a control process; $w_t \in \mathbb{R}^r$ is a zero-mean random process with orthogonal increments:

$$\mathbf{E}\left[w_t\right] = 0, \quad \mathbf{cov}(w_t, w_\tau) = \gamma_w \min(t,\tau). \tag{2}$$

The intensity matrix γ_w is assumed to belong to some known convex compact set Γ_w of positively semi-definite $r \times r$ matrices.

The continuous observation process $\zeta_t \in \mathbb{R}^m$ is governed by the linear stochastic differential equation:

$$\mathrm{d}\zeta_t = c_t y_t \mathrm{d}t + d_t \mathrm{d}w_t, \quad t \in [0,T], \tag{3}$$

where the matrix functions c_t and d_t are assumed to be known and piecewise continuous. It should be noted, that the same random process $\{w_t\}$ disturbing both the equations of system state (1) and continuous observations (3) allows to consider systems with correlated state disturbances and observation noises. At the same time this does not lead to loss of generality, because the classical case of uncorrelated noise processes can be obtained assuming $b_t = [B_t, O]$, $d_t = [O, D_t]$, and $\gamma_w = \mathrm{diag}(\gamma_y, \gamma_\zeta)$.

The discrete-time observation process $\{z_k\}$, $z_k \in \mathbb{R}^q$ is defined as follows:

$$z_k = f_k y_{t_k} + \nu_k, \quad k = 1, \ldots, N, \tag{4}$$

where $0 = t_0 \le t_1 < t_2 < \ldots < t_N \le t_{N+1} = T$ are given time instants, f_k are non-random known matrices, and ν_k is a zero-mean stationary white noise uncorrelated with the process $\{w_t\}$:

$$\mathbf{E}\left[\nu_k\right] = 0, \;\; \mathbf{cov}(\nu_k, \nu_l) = \gamma_\nu \delta_{k,l}, \;\; \mathbf{cov}(\nu_k, w_t) = 0, \tag{5}$$

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where $\delta_{k,l} = 1$ if $k = l$ and $\delta_{k,l} = 0$ otherwise.

The covariance matrix γ_ν is not known but assumed to belong to some given convex compact set Γ_ν of positively definite $q \times q$ matrices.

Introduce the *uncertainty set* of model (1)–(5):

$$\Gamma = \{\gamma \colon \gamma = \mathrm{diag}(\gamma_w, \gamma_\nu), \gamma_w \in \Gamma_w, \gamma_\nu \in \Gamma_\nu\}. \tag{6}$$

Let u_t be a linear non-anticipative control process:

$$u_t = U(Z_t) = \int_0^t g_c(t,\tau)\mathrm{d}z_\tau + \sum_{k\colon t_k \le t} g_d(t,t_k) z_k, \tag{7}$$

where $Z_t = \{z_k, \zeta_\tau \colon t_k \le t, \tau \in [0,t]\}$, U denotes a linear control operator, and its corresponding weight functions $g_c(t,\tau)$ and $g_d(t,t_k)$ are assumed to satisfy the following conditions:

$$\int_0^T \int_0^T \|g_c(t,\tau)\|^2 \mathrm{d}t\mathrm{d}\tau < \infty, \quad \int_0^T \|g_d(t,t_k)\|^2 \mathrm{d}t < \infty, \quad k = 1, \dots, N.$$

The whole class of non-anticipating controls that are linear with respect to the observations is denoted by $\mathcal{U}$.

For any $\gamma \in \Gamma$ the quality of the control u_t is measured by the following mean-square criterion:

$$J(U,\gamma) = \mathrm{E}\left\{ \int_0^T \big(y_t^* Q_1(t) y_t + u_t^* Q_2(t) u_t\big)\mathrm{d}t + y_T^* Q_0 y_T \right\}, \tag{8}$$

where Q_0, $Q_1(t)$ and $Q_2(t)$ are known matrices and

$$Q_0 \succeq 0, \quad Q_1(t) \succeq 0, \quad Q_2(t) \succ 0 \quad \forall t \in [0,T].$$

It is also assumed that $Q_1(t)$ and $Q_2(t)$ are piece-wise continuous.

2 Main results

In order to obtain an optimal control process in the considered system with a priori uncertainty in parameters description we propose to use the game-theoretic approach, i.e. to find a control process $\hat{U}$, that is minimax w.r.t. criteria $J(U,\gamma)$ on $\mathcal{U} \times \Gamma$:

$$\hat{U} \in \operatorname*{argmin}_{U \in \mathcal{U}} \sup_{\gamma \in \Gamma} J(U,\gamma). \tag{9}$$

It is known [1] that that the problem (9) can not be solved without some serious additional restrictions on the set Γ, so we introduce the dual problem: uncertainty set

$$\hat{\gamma} \in \operatorname*{argmax}_{\gamma \in \Gamma} J^0(\gamma), \tag{10}$$

where $J^0(\gamma)$ denotes the dual criterion:

$$J^0(\gamma) = \inf_{U \in \mathcal{U}} J(U,\gamma). \tag{11}$$

This problem is a finite-dimensional convex program and its solution exists under a natural assumption on compactness of the uncertainty set.

The main results of our research are the following.

1. The properties of the dual criterion $J^0(\gamma)$ are studied in detail and for an arbitrary compact uncertainty set Γ the existence of the dual problem solution (10) is proved.

2. Using the results previously obtained for the minimax discrete-continuous filtering problem and minimax control problem for purely continuous systems [2–4] an analytical representation for the minimax control $\hat{U}$ is obtained under natural restrictions on non-degeneracy of the observation processes:

$$\begin{gathered} \hat{u}_t = -L_t \hat{y}_t, \quad L_t = Q_2^{-1}(t) A_t^* S_t, \\ -\dot{S}_t = a_t^* S_t + S_t a_t + Q_1(t) - S_t A_t Q_2^{-1}(t) A_t^* S_t, \quad S_T = Q_0, \end{gathered} \tag{12}$$

   where $\hat{y}_t$ is a recursive estimate of the system state. It should be noted, that this estimate depends on the solution to the dual problem (10), $\hat{y}_t = \hat{y}_t(\hat{\gamma})$.

3. It is proved that $(\hat{U}, \hat{\gamma})$ is a saddle point of the criterion $J(U, \gamma)$. The guaranteed value of the criterion $\hat{J} = J(\hat{U}, \hat{\gamma})$ is also provided.

4. A convergent iterative procedure for the dual problem solution computation is proposed. Some particular cases when the solution of (10) can be obtained analytically are also considered.

5. It is shown, that the *separation principle* [5] which holds for the classical linear stochastic control problem with mean-square criterion is violated for the minimax control problem for the general case of the uncertainty set Γ. In classical settings the optimal estimate $\hat{y}_t$ used for the optimal control design does not depend on the parameters of the control quality criterion. For the case of parameter uncertainty it turns out that this key feature does not take palace: the estimate depends on the solution to the dual problem $\hat{\gamma}$, which, in turn, depends on Q_0, $Q_1(t)$, $Q_2(t)$, so any change of the optimization criterion leads to the necessity of recomputation of the estimate $\hat{y}_t$. Hence, unlike the case of full a priori information, the problem of control process design and the observation processing problem are not separated in the minimax case. It should be noted that for a particular case of uncertainty set Γ containing a maximal element $\bar{\gamma}$ ($\bar{\gamma} \succeq \gamma$ for any $\gamma \in \Gamma$) the separation principle holds, since then $\hat{\gamma} = \bar{\gamma}$ and do not depend on the criterion (8).

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