OPTIMAL CONTROL OF MULTILINK SYSTEMS IN A FLUID

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Abstract

Locomotion of a mechanical system consisting of a main body and one or two links attached to it by cylindrical joints is considered. The system moves in a resistive fluid and is controlled by periodic angular oscillations of the links relative to the main body. The resistance force acting upon each body is a quadratic function of its velocity. Under certain assumptions, a nonlinear equation of motion is derived and simplified. The optimal control of oscillations is found that corresponds to the maximal average locomotion speed.

Key words

Control, optimization, nonlinear dynamics, robotics, locomotion.

1 Introduction

It is well-known that a multilink mechanical system, whose links perform specific oscillations relative to each other, can move progressively in a resistive medium. This locomotion principle is used by fish, snakes, insects, and some animals [Gray, J. (1968)], [Lighthill, J. (1975)], [Blake, R.W. (1983)]. In robotics, the same principle is applied to locomotion of snake-like robots along a surface [Hirose, S.(1993)].

Dynamics and optimization of snake-like multilink mechanisms that move along a plane in the presence of Coulomb's dry friction forces acting between the mechanism and the plane, have been studied in [Chernousko, F.L.(2001)], [Chernousko, F.L.(2003)], [Chernousko, F.L.(2005)].

Various aspects of fish-like locomotion in a fluid are considered in many papers, and a number of swimming robotic systems have been developed [Terada, Y., Yamamoto, I.(1999)], [Mason, R., Burdick, J.(2000)], [Colgate, J.E., Lynch, K.M.(2004)], [wikipedia. org/wiki/RoboTuna].

In this paper, we consider a progressive motion of a multilink system in a fluid in the presence of resistance

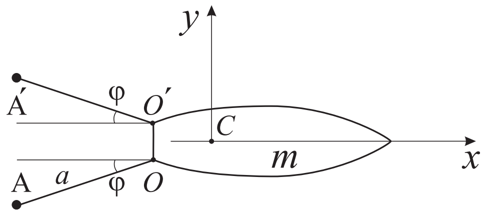

Figure 1. System with two links.

forces proportional to the squared velocity of the moving body. The mechanical model is described and simplified. The optimal control problem for the motion of the links is formulated, and its exact solution is presented. Similar problem for the case of small angles of deflection of links relative to the main body was considered in [Chernousko, F.L. (2010)]. Here, the angles of deflection are finite.

2 Mechanical Model

Consider a mechanical system consisting of the main body and two symmetric links OA and $O'A'$ attached to it by cylindrical joints (Fig. 1). The length of the links is denoted by a, and their mass is negligible compared to the mass m of the main body. The main body is symmetric with respect to the axis Cx.

Let us introduce the Cartesian coordinate frame Cxy connected with the main body and denote by $\mathbf{i}$ and $\mathbf{j}$ the unit vectors directed along the axes Cx and Cy, respectively.

The links perform symmetric periodic oscillations of period T about the joints O and O' so that the angle φ between the links and the axis Cx satisfies the condition

$$\varphi(t+T) = \varphi(t) \tag{1}$$

for each time instant t. During the period $[0, T]$, the angle φ first increases from 0 to φ_0 and then decreases from φ_0 to 0.

Denote by v the velocity of the progressive motion of the main body along the axis Cx. We will consider only forward motions, so that $v \geq 0$. The value of the resistance force acting upon the body is denoted by $c_0 v^2$, where c_0 is a positive constant.

Suppose for simplicity that the resistance forces acting upon each link can be reduced to a force applied at the end points A and A'. Denote by $\mathbf{V}$ the velocity of point A and by ω the angular velocity of the link OA. We have (see Fig. 1):

$$\mathbf{V} = v\mathbf{i} + a\omega \sin\varphi \mathbf{i} - a\omega\cos\varphi\mathbf{j}, \quad \omega = \dot{\varphi}. \qquad (2)$$

Here and below, dots denote derivatives with respect to time t.

The quadratic resistance force applied to the point A is given by

$$\mathbf{F} = -k_0 V \mathbf{V}, \qquad (3)$$

where $k_0 > 0$ is a constant coefficient.

Under the assumptions made, the equation of the progressive motion of the main body can be written as follows:

$$(m + m_0)\dot{v} = -c_0 v^2 + 2F_x \ , \qquad (4)$$

where m_0 is the added mass of the main body, and F_x is the projection of the vector $\mathbf{F}$ from (3) onto the axis Cx. Note that the projection of the resistance force acting upon the link $O'A'$ is also equal to F_x. Introducing the notation

$$c_0/(m + m_0) = c, \qquad 2k_0/(m + m_0) = k$$

and using equations (2) and (3) to determine F_x, we convert equation (4) to the form:

$$\begin{aligned} &\dot{v} = -cv^2 - \\ &-k(v + a\omega\sin\varphi)\sqrt{v^2 + a^2\omega^2 + 2va\omega\sin\varphi}\ . \end{aligned} \qquad (5)$$

Similarly, the system with one link attached to the main body can be considered [Chernousko, F.L.(2010)]. The system with one link (Fig.2) imitates a fish with a tail, whereas the system with two symmetric links is a model of a swimming animal with two extremities (e.g., a frog). Under certain conditions, the system with one link can be described by the same equation (5). These conditions are listed below.

1. The main body is symmetric with respect to the axis Cx.

2. The oscillations of the link OA are periodic and symmetric, i.e., conditions (1) and

$$\varphi(t + T/2) = -\varphi(t)$$

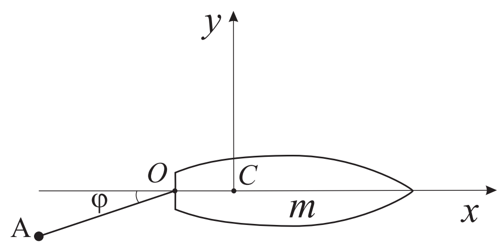

Figure 2. System with one link

are fulfilled.

3. The frequency of oscillations is sufficiently high, i.e.

$$T \ll a/v.$$

4. The moment of inertia of the main body is sufficiently high, so that the amplitude of the angular oscillations of the body is small.

5. The resistance force of the main body in the lateral direction (along the axis Cy) is much greater than the resistance force in the longitudinal direction (along the axis Cx).

Under these assumptions, the change in the orientation of the main body as well as its motion in the lateral direction (along the axis Cy) are insignificant, and the body will move mostly along the axis Cx. The oscillations of the tail (link OA in Fig.2) during two halves of the period T correspond to the oscillations of two links OA and $O'A'$ in Fig.1. The equation of motion of the model of Fig.2 is reduced to equation (5) with

$$k = k_0/(m + m_0).$$

In what follows, we consider equation (5) that describes the dynamics of the both models of Fig.1 and Fig.2.

Note that the coefficients c and k in equation (5) have the dimension of inverse length. To clarify the meaning of these coefficients, consider the motion of a unit mass point along a line in the presence of the quadratic resistance. The corresponding equations of the motion are

$$\dot{x} = v, \quad \dot{v} = -cv^2,$$

where x is the coordinate and v is velocity of the mass. Integrating these equations under the initial conditions

$$x(0) = 0, \quad v(0) = v_0 > 0,$$

we obtain

$$v = v_0 \exp(-cx).$$

Hence, the quantity c^{-1} is the distance where the initial velocity decreases by the factor e due to the quadratic resistance.

We assume that the following dimensionless parameter is small:

$$\varepsilon = a\sqrt{ck} \ll 1. \tag{6}$$

Let us introduce the new dimensionless variables $\tilde{t}$ and $\tilde{v}$ as follows:

$$t = \tilde{t}T, \quad v = (a/T)(k/c)^{1/2}\tilde{v}. \tag{7}$$

We substitute formulas (7) into equation (5) and simplify this equation by omitting terms of higher order of ε. After simplifications and replacing $\tilde{t}$ and $\tilde{v}$ by t and v, respectively, we obtain the equation

$$\frac{dv}{dt} = -\varepsilon\left(v^2 + \frac{d\varphi}{dt}\left|\frac{d\varphi}{dt}\right|\sin\varphi\right). \tag{8}$$

This equation contains a small parameter ε, and $\varphi(t)$ is a periodic function of t with a period equal to 1. Applying the asymptotic method of averaging [Bogoliubov, N.N., Mitropolsky, Y.A.,(1961)] to equation (8), we come to the following equation of the first approximation:

$$\frac{dv}{dt} = -\varepsilon(v^2 + I), \tag{9}$$

$$I = \int_0^1 \frac{d\varphi}{dt}\left|\frac{d\varphi}{dt}\right|\sin\varphi\, dt. \tag{10}$$

The solution $v(t)$ of the averaged equation (9) differs from the solution of the original equation (8), under the same initial conditions, by terms of an order of ε for the large time interval of an order of ε^{-1}.

If $I > 0$, then the right-hand side of equation (9) is positive for all $v > 0$. Hence, $dv/dt < -\varepsilon I < 0$, the velocity decreases and reaches zero in finite time. In this case, the forward motion of the system is impossible.

We will consider below a more interesting case, where $I < 0$. Then equation (9) has a unique positive stationary solution

$$v_* = \sqrt{-I} \tag{11}$$

which is globally asymptotically stable. Thus, for any initial condition $v(t_0) = v_0 \geq 0$, we have $v(t) \to v_*$ as $t \to \infty$. To check the inequality $I < 0$ and evaluate the velocity v_*, we are to specify the periodic function $\varphi(t)$ and calculate the integral I from (10).

3 Piecewise constant angular velocity

Let us first consider the case where the angular velocity of links $\omega(t) = d\varphi/dt$ is a piecewise constant periodic function of time. Suppose that the angle $\varphi(t)$ first grows linearly from 0 to φ_0 on the time interval $(0, \theta)$ and then decreases linearly from φ_0 to 0 on the interval $(\theta, 1)$. Here, θ is a given time instant, $\theta \in (0, 1)$, and $\varphi_0 \in (0, \pi/2)$. We set

$$\varphi(t) = \begin{cases} \omega_+ t, & t \in (0, \theta) \\ \omega_-(1-t), & t \in (\theta, 1)\,, \end{cases} \tag{12}$$

where ω_+ and ω_- are fixed constant angular velocities of deflection and retrieval of the links, respectively. Since $\varphi(t)$ is a continuous function of t, it follows from (12):

$$\varphi_0 = \omega_+\theta = \omega_-(1-\theta). \tag{13}$$

We obtain from equation (13):

$$\theta = \frac{\varphi_0}{\omega_+}, \quad 1-\theta = \frac{\varphi_0}{\omega_-}, \quad \varphi_0 = \frac{\omega_+\omega_-}{\omega_+ + \omega_-}. \tag{14}$$

We substitute formulas (12) into equation (10) and calculate the integral I. Taking into account equations (14), we obtain

$$I = (\omega_+ - \omega_-)(1 - \cos\varphi_0). \tag{15}$$

It is evident from (15) that $I > 0$ for $\omega_+ > \omega_-$ and $I < 0$ for $\omega_+ < \omega_-$, if $\varphi_0 \in (0, \pi/2]$. Therefore, the average velocity v_* of the system is positive if and only if $\omega_+ < \omega_-$, i.e., the angular velocity ω_+ of deflection of the links from the axis of the system is smaller than the angular velocity of retrieval ω_-. The velocity v_* in this case is given by equations (11) and (15) as follows:

$$v_* = \left[(\omega_- - \omega_+)(1 - \cos\varphi_0)\right]^{1/2}. \tag{16}$$

4 Optimal Control

Let us consider the optimal control problem for the angular motion of the links. We will regard the dimensionless angular velocity ω as the control subject to the constraints

$$-\omega_- \leq \omega = d\varphi/d\tau \leq \omega_+, \tag{17}$$

where ω_- and ω_+ are given positive constants.

Suppose that the angle φ changes over the interval $t \in (0, 1)$ as follows: it grows from $\varphi(0) = 0$ to $\varphi(\theta) = \varphi_0 > 0$ and then decreases from φ_0 to $\varphi(1) = 0$. Here, $\theta \in (0, 1)$ and $\varphi_0 \in (0, \pi/2]$ are constant parameters.

The problem is to find functions $\omega(t)$ and $\varphi(t)$ that satisfy (17) and the boundary conditions imposed above and maximize the average velocity v_* defined by (16).

The solution of this problem is obtained by means of Pontryagin's maximum principle [Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E.F.(1986)]. After that, parameter $\theta \in (0,1)$ is chosen that maximizes v_*. Omitting this rather lengthy analysis, we present below the final results.

The optimal control $\omega(t)$ and the corresponding optimal time history of the normalized angle $\varphi(t)$ are given by equations

$$\begin{aligned}
&\omega = \omega_+, \quad \varphi = \omega_+ t \quad \text{for} \quad t \in (0, t_*),\ t_* = z/\omega_+,\\
&\omega^2 \sin\varphi = \omega_+^2 \sin z,\\
&\int_z^{\varphi} (\sin x)^{1/2} dx = \omega_+ (\sin z)^{1/2} (t - t_*)\\
&\text{for} \quad t \in (t_*, \theta),\\
&\omega = -\omega_-, \quad \varphi = \omega_-(1-t) \quad \text{for} \quad t \in (\theta, 1).
\end{aligned} \tag{18}$$

Here, parameters z and θ are defined by equations

$$\begin{aligned}
&\int_z^{\varphi_0} (\sin x)^{1/2} dx = (\omega_+ \theta - z)(\sin z)^{1/2},\\
&\theta = 1 - \varphi_0/\omega_-.
\end{aligned} \tag{19}$$

Note that equation (19) for z has a unique root in the interval $(0, \varphi_0)$.

The maximal value of the velocity v_* of the system for the optimal solution (18) is expressed as follows:

$$\begin{aligned}
&v_* = \{\omega_-(1 - \cos\varphi_0) -\\
&\omega_+ [1 - \cos z + (\omega_+ \theta - z)\sin z]\}^{1/2}.
\end{aligned} \tag{20}$$

Let us consider two particular limit cases.

Suppose the constraint imposed upon the angular velocity of the deflection of links is absent,i.e., $\omega_+ \to \infty$ in (17). Then the first interval $(0, t_*)$ in (18) tends to zero $(t \to \infty)$, and the solution of our optimal control problem becomes

$$\begin{aligned}
&\omega^2 \sin\varphi = c,\\
&\int_0^{\varphi} (\sin x)^{1/2} dx = c^{1/2} t, \quad \text{for} \quad t \in (0, \theta),\\
&\omega = -\omega_-,\ \varphi = -\omega_-(1-t) \quad \text{for} \quad t \in (\theta, 1).
\end{aligned}$$

Here, the constant c is defined as follows:

$$c = \left[\theta^{-1} \int_0^{\varphi_0} (\sin x)^{1/2} dx\right]^2.$$

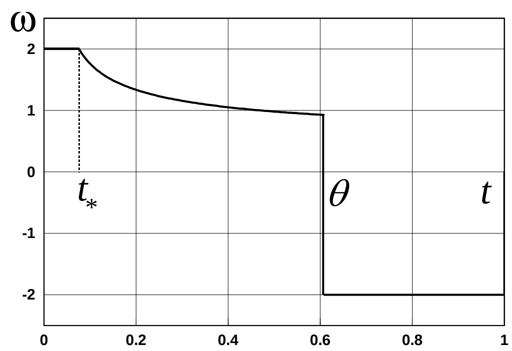

Figure 3. Optimal control $\omega(t)$.

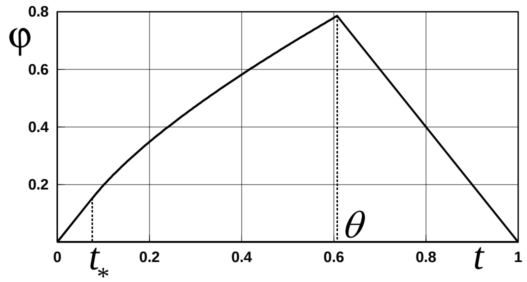

Figure 4. Optimal trajectory $\varphi(t)$.

Equation (20) for the velocity v_* takes the form:

$$v_* = [\omega_-(1 - \cos\varphi_0) - c\theta]^{1/2}.$$

If the constraint imposed upon the angular velocity of the retrieval of links is absent, i.e., $\omega_- = \infty$ in (17), then the third interval in the solution (18) tends to zero $(\theta \to 1)$. In this case, $I \to -\infty$, and $v_* \to \infty$.

Thus, the optimal control is completely determined in terms of normalized variables. To return to the original dimensional ones, one is to use equations (6) and (7).

5 Example

Let us consider a numerical example. We assume that

$$\varphi_0 = \pi/4, \quad \omega_+ = \omega_- = 2$$

and obtain from the optimal solution (18)–(20):

$$\begin{aligned}
&\theta = 0.6073,\ z = 0.1563,\\
&t_* = 0.0763,\ v_* = 0.4894.
\end{aligned} \tag{21}$$

The time histories of functions $\omega(t)$ and $\varphi(t)$obtained by means of equations (18) are shown in Figs. 3 and 4, respectively.

This optimal solution is close to the case of a piecewise constant angular velocity $\omega(t)$. If we choose the piecewise linear function $\varphi(t)$ so that it coincides with the optimal one at $t = 0,\ t = \theta,$ and $t = 1$, we obtain from equations (13)-(16):

$$\omega_+ = 1.293, \quad \omega_- = 2, \quad v_* = 0.4551.$$

Comparing these data with the optimal solution given by equations (21), we see that the difference in the average speed v_* does not exceed 7%.

6 Conclusions

A mechanical system consisting of a main body and one or two links attached to it by cylindrical joints can move progressively in a medium that acts upon moving bodies with forces proportional to the squared velocities of the bodies. Under the assumptions made, the equation of motion is simplified, and the average velocity of the progressive motion is evaluated.

The optimal time history of the angular oscillations of the links is obtained that corresponds to the maximal, under the conditions imposed, average speed of the progressive motion.

The obtained results correlate well with observations of the process of swimming.

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