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CONTROL OF A CART WITH VISCOELASTIC LINKS UNDER UNCERTAINTY

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Abstract
A control problem for a system, consisting of a rigid body with one or several viscoelastic links is considered. Such a system is modelled as a cart with linear dissipative oscillators attached to it. The cart moves along a horizontal line under the action of a control force and unknown disturbance, for example, a medium resistance, the parameters of which are unknown and impermanent. The phase states of the oscillators are assumed to be unavailable for measuring. A bounded feedback control which brings the cart to a prescribed terminal state in a finite time is proposed.

Key words
Linear controllable system, observability, disturbance, feedback control.

1 Introduction

We consider a control problem for a system representing a simplified model of a precision platform carrying one or several viscoelastic links or a vessel with a viscous liquid. The platform is driven by a tug or pusher, which generates a control force. Precise positioning of the platform is hampered by a medium resistance, for example, dry friction acting between the platform and the surface along which it moves, as well as disturbance from the viscoelastic links. The parameters of medium resistance are unknown beforehand and may change in the process of motion. The current states of the viscoelastic links is not available for measuring. A control algorithm which stops the platform in a prescribed terminal position in a finite time is proposed. The states of the viscoelastic links at the final moment is unimportant.

The difficulty in controlling such a mechanical system is related to the fact that it has many degrees of freedom and only a single scalar control input. More over, the current states of most its phase variables are unknown, while the system is subject to external perturbations. Nevertheless, our control, given by a smooth function, brings the cart into a given terminal state in a finite time.

The proposed algorithm consists of two stages. At the first stage, using the available information on the motion of the platform, we restore unknown phase variables that describe the dynamics of the viscoelastic links, and estimate the error in calculating these variables. For the control law applied at the first stage, this error is not significant, while the system is far from the terminal state.

At the second stage, when the energy of viscoelastic links, as well as the perturbations caused by the oscillations of the link are sufficiently small, a control law is applied, which depends only on the coordinate and velocity of the platform.

At both stages, for constructing the control we use the approach proposed in [Ovseevich, 2015] and developed in [Anan'evskii and Ishkhanyan, 2016].

2 Equations of the Motion and the Statement of the Problem

Consider a cart of mass m_0 with n linear dissipative oscillators attached to it. The cart moves along a straight horizontal line under the action of a control force u_0 and unknown disturbance v_0. Oscillators are assumed to be horizontally oscillating particles with masses m_i, connected to the cart via springs of stiffnesses κ_i, $i = 1, \dots, n$.

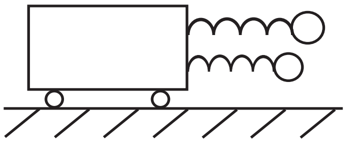

Figure 1. A cart with oscillators.

The system is governed by the equations

$$\begin{aligned} m_0\ddot{\xi} &= \sum_{i=1}^{n}(\kappa_i\varphi_i + \gamma_i\dot{\varphi}_i) + u_0 + v_0, \\ m_1(\ddot{\xi} + \ddot{\varphi}) &= -\sum_{i=1}^{n}(\kappa_i\varphi_i + \gamma_i\dot{\varphi}_i), \end{aligned} \tag{1}$$

where ξ describes the position of the cart on the horizontal line, φ_i is the elongation of the spring of the i-th oscillator, and $\gamma_i > 0$ is a coefficient of viscous friction, $i = 1, \ldots, n$.

The control force u_0 is bounded and exceeds the disturbance v_0, that is,

$$|u_0| \leq U_0,\ |v_0| \leq \rho U_0, \quad U_0 > 0,\ 0 < \rho < 1. \tag{2}$$

We assume that variables $\xi, \dot{\xi}$ that describe the phase state of the cart are known at every instant of time, while the vectors of the phase coordinates and velocities $\varphi, \dot{\varphi} \in R^n$ of the oscillators are not available for measuring.

The problem is to design a feedback control law $u(\xi, \dot{\xi})$ that brings the cart to the origin in a finite time. The states of the oscillators are not important at the instant when the cart reaches the origin.

3 Auxiliary Control Problem

In this section, we assume that $v_0 \equiv 0$. Using a linear transformation and adding an extra linear feedback, one can bring system (1) to a canonical form [Brunovsky, 1970], [Ovseevich, 2015]:

$$\dot{x} = \bar{A}x + \bar{B}u, \quad x \in R^N, \quad N = 2n + 2, \tag{3}$$

where $N \times N$-matrix $\bar{A}$ and N-vector $\bar{B}$ are given in the form

$$\bar{A} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & \ldots & 0 & 0 \\ 0 & -2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & \ldots & -(N-1) & 0 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Below we present an algorithm for constructing a bounded feedback control that brings system (3) to the origin in a finite time. In the following sections, this algorithm will be used to solve the previously formulated problem.

We introduce the $N \times N$-matrices q, Q

$$q_{ij} = [(i + j)(i + j - 1)]^{-1}, \quad Q = q^{-1},$$

and a matrix function of a positive parameter T

$$\delta(T) = \mathrm{diag}\{T^{-1}, T^{-2}, \ldots, T^{-N}\}.$$

In what follows the parameter T will be a function $T(x)$ of the state vector.

Now we define a feedback control by the formula

$$u(x) = -\frac{1}{2}\bar{B}^{\top}Q\delta(T(x))x, \tag{4}$$

where the function $T = T(x)$ is defined implicitly by the equation

$$(Q\delta(T)x, \delta(T)x) = 1 \tag{5}$$

(from now on, $(\cdot, \cdot)$ means a scalar product).

The basic result on steering canonical system (3) to the origin is as follows [Ovseevich, 2015]:
1) equation (5) defines $T = T(x)$ uniquely;
2) control (4) is bounded: $|u(x)| \leq \frac{1}{2}\sqrt{Q_{11}}$;
3) control (4) brings x to 0 in time $T(x)$.

4 The First Stage of Control

At the first stage, the control algorithm is aimed at transferring the cart to a neighborhood of the terminal state and reducing the elastic oscillations of viscoelastic links to an acceptable level. To this end, based on the available information about the motion of the cart, we estimate unknown phase variables that describe the dynamics of the viscoelastic links, and use these estimates to construct the desired feedback control. For simplicity, we shall carry out reasoning for a system with a single oscillator.

First, we introduce the dimensionless time and notations

$$\begin{gathered} \tau = \frac{\kappa t}{\gamma}, \quad \gamma \neq 0, \\ \bar{\xi} = \frac{\kappa^2 m_0}{\gamma^2 U_0}\xi, \quad \bar{\varphi} = \frac{\kappa}{U_0}\varphi, \quad u_1 = \frac{u_0}{U_0}, \\ v_1 = \frac{v_0}{U_0}, \quad a = \frac{\gamma^2}{\kappa m}, \quad b = \frac{\gamma^2}{\kappa m_0}. \end{gathered}$$

Then system (1) takes the form

$$\begin{aligned} &\ddot{\bar{\xi}} = \bar{\varphi} + \dot{\bar{\varphi}} + u_1 + v_1, \\ &\ddot{\bar{\varphi}} = -a(\bar{\varphi} + \dot{\bar{\varphi}}) - b(u_1 + v_1), \\ &|u_1| \leq 1, \quad |v_1| \leq \rho, \end{aligned}$$

where dots designate derivatives with respect to the dimensionless time. We introduce the vector $x \in R^4$ of new variables

$$x_1 = \bar{\xi}, \quad x_2 = \dot{\bar{\xi}}, \quad x_3 = \bar{\varphi}, \quad x_4 = \dot{\bar{\varphi}}$$

to obtain

$$\dot{x} = A_0 x + B_0(u_1 + v_1), \tag{6}$$
$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -a & -a \end{pmatrix}, \ B_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -b \end{pmatrix}.$$

The matrix $F = [B_0 \ A_0B_0 \ A_0^2B_0 \ A_0^3B_0]$ is nonsingular because $\det F = b^2(a-b)^2$ and $a-b, b > 0$. Therefore, system (6) is completely controllable when $v_1 \equiv 0$.

Since, by assumptions, only the coordinate and velocity of the cart are known, a measured output is

$$y = C_0 x, \quad C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad y \in R^2.$$

The pair (A_0, C_0) is observable because the observability matrix

$$[C_0^\top \ A_0^\top C_0^\top \ {A_0^\top}^2 C_0^\top \ {A_0^\top}^3 C_0^\top] =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -a & -a & -a+a^2 \\ 0 & 0 & 0 & 1 & 1 & 1-a & 1-a & -2a+a^2 \end{pmatrix}$$

has a full rank of 4.

To transform system (6) to a canonical form we denote $c = (a-b)^{-1}$ and introduce new variables $z = Sx, \ z \in R^4$, where

$$S = \begin{pmatrix} 0 & 0 & 0 & -1/b \\ 0 & 0 & 1/b & 0 \\ 0 & 2c & 2/b & 2c/b \\ -6c & 6c & 6(a-b-1)c/b & 6c/b \end{pmatrix}.$$

In the variables z system (6) takes the form

$$\begin{aligned} \dot{z} &= Az + B(u + v_1), \\ y &= Cz, \end{aligned} \tag{7}$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$
$$C = C_0 S^{-1} = \begin{pmatrix} 0 & -c & 1/2 & -1/6 \\ c & -1 & 1/2 & 0 \end{pmatrix}, \tag{8}$$

and

$$u(z) = u_1(z) - a(z_1 + z_2). \tag{9}$$

Now, using the measured output, we find approximately the vector of the current phase state of system (7). Let $Z(t)$ be the fundamental matrix of (7)

$$Z(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ t^2 & -2t & 1 & 0 \\ -t^3 & 3t^2 & -3t & 1 \end{pmatrix}. \tag{10}$$

Then, the solution of (7) with the initial state z^0 can be written as follows

$$z(t) = Z(t)\left(z^0 + \int_0^t Z^{-1}(\tau)B(u(\tau) + v_1(\tau))d\tau\right),$$

and the measured output $y(t), \ t \in [0, t_1]$, is

$$y(t) = H(t)\left(z^0 + \int_0^t Z^{-1}(\tau)B(u(\tau) + v_1(\tau))d\tau\right)$$

with the notation $H(t) = CZ(t)$. Let us find the expected initial state $z_{t_1}^0$ with the same output, but in case when there are no disturbance v. We have

$$H(t)z_{t_1}^0 = y(t) - CZ(t)\int_0^t Z^{-1}(\tau)Bu(\tau)d\tau. \tag{11}$$

Multiplying equation (11) by $H^\top(t)$ from the left and integrating it over the interval $[0, t_1]$ yield

$$\hat{H}(t_1)z_{t_1}^0 = \int_0^{t_1} H^\top(t)y(t)dt -$$
$$\int_0^{t_1} H^\top(t)H(t)\int_0^t Z^{-1}(\tau)Bu(\tau)d\tau dt,$$

where

$$\hat{H}(t_1) = \int_0^{t_1} H^\top(t)H(t)dt.$$

The pair (A, C) is observable and, consequently, the matrix $\hat{H}(t_1)$ is invertible. Hence, for the trajectory under consideration, the difference between the true initial state z^0 and the expected one $z_{t_1}^0$ equals

$$z^0 - z_{t_1}^0 =$$
$$\hat{H}^{-1}(t_1)\int_0^{t_1} H^\top(t)H(t)\int_0^t Z^{-1}(\tau)Bv_1(\tau)d\tau dt.$$

Taking into account expressions (8) and (10) for matrices C and $Z(t)$, we come to the following approximation which is valid for small t_1:

$$z^0_{t_1} - z^0 \approx \rho p_0, \quad p_0 = \begin{pmatrix} 1 \\ 1-c \\ 2-4c \\ 6-18c+6c^2 \end{pmatrix}.$$

Assuming no disturbance, we expect the following state of the system at the time instant t_1:

$$\hat{z}(t_1) = Z(t_1)\left(z^0_{t_1} + \int_0^{t_1} Z^{-1}(\tau)Bu(\tau)d\tau\right).$$

Thus, due to the disturbance v_1, the accumulated error in the determination of the state of system (7) at the instant t_1 equals

$$z(t_1, z^0_{t_1}) - z(t_1, z^0) = \\ Z(t_1)\left(z^0_{t_1} - z^0 - \int_0^{t_1} Z^{-1}(\tau)Bv_1(\tau)d\tau\right),$$

and the following asymptotic estimate holds

$$\lim_{t_1 \to o} \|z(t_1, z^0_{t_1}) - z(t_1, z^0)\| = \rho\|p_0\|.$$

Let $z(t)$ be the true current state vector of the system, $\hat{z}(t)$ be its estimate, and

$$\triangle z(t) = \hat{z}(t) - z(t).$$

Substituting $\hat{z}$ into control function (9) gives

$$u_1(z + \triangle z) = u(z + \triangle z) + \\ a(z_1 + z_2) + a(\triangle z_1 + \triangle z_2).$$

Now, system (7) becomes

$$\dot{z} = Az + B(u(z) + v), \qquad (12)$$

where

$$v = v_1 + v_2 + v_3, \\ v_2 = u(z + \triangle z) - u(z), \\ v_3 = a(\triangle z_1 + \triangle z_2).$$

Based on the approach presented above, we construct a bounded feedback control that we apply at first stage.

We introduce the scalar function $T(\hat{z})$, the diagonal matrices $\delta(T)$ and M, the vector f, and the positive definite matrix Q:

$$\delta(T) = \mathrm{diag}(T^{-1}, T^{-2}, T^{-3}, T^{-4}), \\ M = \mathrm{diag}(-1, -2, -3, -4), \\ f^\top = (-10, 90, -210, 140), \\ Q = \begin{pmatrix} 20 & -180 & 420 & -280 \\ -180 & 2220 & -5880 & 4200 \\ 420 & -5880 & 16800 & -12600 \\ -280 & 4200 & -12600 & 9800 \end{pmatrix}.$$

We compose the matrix A_3 by filling the top row of the matrix A with the components of the vector f

$$A_3 = \begin{pmatrix} -10 & 90 & -210 & 140 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}.$$

The following property holds:

$$QA_3 + A_3^\top Q = QM + MQ = P < 0,$$

where P is the negative definite constant matrix

$$P = \begin{pmatrix} -40 & 540 & -1680 & 1400 \\ 540 & -8880 & 29400 & -25200 \\ -1680 & 29400 & -100800 & 88200 \\ 14000 & -25200 & 88200 & -78400 \end{pmatrix}.$$

We define the function $T(\hat{z})$ implicitly by the equation

$$(Q\delta(T)\hat{z}, \delta(T)\hat{z}) = 1/5, \quad \hat{z} \neq 0. \qquad (13)$$

As was noted earlier, equation (13) has only one positive solution for T in the whole phase space $\hat{z} \in R^4$, except zero. This solution is given by an analytic function. Moreover, the function $T(\hat{z})$ can be defined at zero as $T(0) = 0$, which preserves the continuity of it.

We specify the feedback control law

$$u(\hat{z}) = (f, \delta(T)\hat{z}), \quad \hat{z} \neq 0. \qquad (14)$$

The coefficients of the feedback control function (14) at the estimates of the phase variables $\hat{z}$ increase infinitely as $\hat{z}$ tends to zero. Nevertheless, control (14) meets the constraint $|u(\hat{z})| \leq 1$.

Denote $r = \delta(T(z)z), \quad r \in R^4$. Then system (12) becomes

$$\dot{r} = T^{-1}\left(A_3 r + Bv + M\dot{T}r\right).$$

Differentiating the function T by virtue of (12) gives

$$\dot{T} = -\frac{(Pr, r) + 2v(QB, r)}{(Pr, r)}.$$

The following theorem is valid.

Theorem 1. *There exists $\rho_1 > 0$ such that if $|v| \leq \rho_1$, then the derivative of the function T by virtue of (12) meets the inequality $\dot{T} < -\lambda,\ \lambda > 0$.*

Let the function $T_1(\xi, \dot{\xi})$ be implicitly defined by the equation

$$dT_1^4 - \dot{\xi}^2 T_1^2 - 4\xi\dot{\xi}T_1 - 6\xi^2 = 0, \quad d = \frac{U_0^2}{9m_0^2}.$$

Similarly to T, the function T_1 is analytic and positive in R^2, except zero, and can be defined at zero as $T(0, 0) = 0$, which preserves its continuity.

Consider Lyapunov function

$$V(\xi, \dot{\xi}, \varphi, \dot{\varphi}) = T_1(\xi, \dot{\xi}) + E(\dot{\xi}, \varphi, \dot{\varphi}),$$
$$E(\dot{\xi}, \varphi, \dot{\varphi}) = \frac{1}{2}\sum_{i=1}^{n}\left(\kappa_i\varphi_i^2 + m_i(\dot{\xi} + \dot{\varphi}_i)^2\right).$$

Theorem 2. *For a given $V_0 > 0$ there exists ρ, introduced in (3), such that outside the neighborhood*

$$G = \{(\xi, \dot{\xi}, \varphi, \dot{\varphi}) \in R^N : \|(\xi, \dot{\xi}, \varphi, \dot{\varphi})\| < V_0\} \quad (15)$$

inequality $|v| \leq \rho_1$ holds.

It follows from Theorems 1 and 2, that every trajectory of system (12) reaches the neighborhood G in a finite time.

5 Control Algorithm at the Second Stage

At the second stage, when the system moves within the neighborhood G, we consider the first equation of system (1) separately:

$$m_0\ddot{\xi} = u_0 + g, \quad g = \sum_{i=1}^{n}(\kappa_i\varphi_i + \gamma_i\dot{\varphi}_i) + v_0. \quad (16)$$

Now g is treated as an uncertain disturbance. We denote

$$W(\dot{\xi}, \varphi, \dot{\varphi}) = \dot{\xi}\sum_{i=1}^{n}(\kappa_i\varphi_i + \gamma_i\dot{\varphi}_i).$$

Theorem 3. *For a given $\lambda_1 > 0$ there exists $V_0 > 0$ such that in the neighborhood G the estimate*

$$|W(\dot{x}, y, \dot{y})| \leq \frac{\lambda_1}{2} \quad (17)$$

holds.

Theorem 4. *For sufficiently small ρ the number V_0 in (15) can be chosen so that in the neighborhood G the following inequality holds:*

$$|g| \leq \frac{3 - \sqrt{3}}{6}U_0. \quad (18)$$

In the neighborhood G we use the control function

$$u_0(\xi, \dot{\xi}) = -\frac{6m_0\xi}{T_1^2(\xi, \dot{\xi})} - \frac{3m_0\dot{\xi}}{T_1(\xi, \dot{\xi})}. \quad (19)$$

As it is shown in [Anan'evskii and Ishkhanyan, 2016], control function (19) meets constraint (2), and there exists $\lambda_1 > 0$ such that, under condition (18), the derivative of T_1 by virtue of equation (16) satisfies the inequality

$$\dot{T}_1 < -\lambda_1,\ \lambda_1 > 0. \quad (20)$$

Let us note that the trajectories of system (1) controlled by (19) do not leave the neighborhood G. Indeed, the function V is positive-definite and the derivative of it by virtue of system (1) has the form

$$\dot{V} = \dot{T}_1 - \sum_{i=1}^{n}\gamma_i y_i^2 - W(\dot{\xi}, \varphi, \dot{\varphi}).$$

This relation and inequalities (17),(20) imply $\dot{V} < 0$. Consequently, at the second stage of motion, the trajectories of system (1) stay within G.

Thus, the function T_1 vanishes to zero in a finite time, i.e. every trajectory of equation (16) reaches the origin of the phase space $\xi, \dot{\xi}$ in a finite time. This means that the cart will be stopped in the origin by control (19).

6 Conclusion

The proposed approach is effective due to the fact that, at the first stage, far from the terminal state, the feedback control used is insensitive to inaccurate knowledge of current phase variables. At the second stage, in a neighborhood of the terminal state, where the vibrations of the viscoelastic links are small, the control copes with perturbations caused by these vibrations.

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