Bifurcation diagrams of families of regularizable singular systems under proportional and derivative feedback

M. Isabel García-Planas

Abstract— **In this work we consider differentiable families of triples of matrices $\varphi(\xi) = (E(\xi), A(\xi), B(\xi))$ with the parameter vector $\xi \in R^k$, representing families of regularizable singular linear time invariant systems in the form $E(\xi)\dot{x}(t) = A(\xi)x(t) + B(\xi)u(t)$, with $E(\xi), A(\xi) \in M_n(C)$, $B(\xi) \in M_{n\times m}(C)$ for each ξ, under proportional and derivative feedback.**

The knowledge of a complete system of invariants for regularizable systems permit us to obtain a canonical reduced form and describe generic families permitting to analyze the neighborhood of a given system showing bifurcation diagrams of a critical points.

I. Introduction

We denote by M the space of triples of matrices (E, A, B) representing families of singular linear time invariant systems in the form $E\dot{x}(t) = Ax(t) + Bu(t)$, with $E, A \in M_n(C)$ n-square matrices, and $B \in M_{n\times m}(C)$ a rectangular matrix with n-rows and m-columns.

It is well know, in the case where the pencil $\lambda E + \mu A$ is regular the system has a unique solution. So we are interested in systems with regular pencil or regularizable by proportional or derivative feedback, that is to say, if there exists a proportional F_A and/or a derivative F_E feedback such that the pencil $\alpha(E + BF_E) + \beta(A + BF_A)$ is regular. We will write M_R the open and dense subset of regularizable triples.

Different useful and interesting equivalence relations between singular systems have been defined. We deal with the equivalence relation $(E', A', B') = (QEP+QBF_E, QAP+QBF_A, QBR)$. with $Q, P \in Gl(n; C)$, $R \in Gl(m; C)$, $F_A, F_E \in M_{m\times n}(C)$, that is to say the equivalence relation accepting one or more, of the following standard transformations: basis change in the state space, input space, feedback and derivative feedback.

This equivalence relation preserves the standardizability, regularizability and controllability characters.

In this paper, we present a complete system of structural invariants in terms of ranks of certain matrices associated to the triple $(E, A, B) \in M_R$, that permit us to give the explicit form of reduced triple (E_1, A_1, B_1) without knowing the transformation matrices reducing the triple.

The knowledge of the complete system of invariants permit us to study bifurcation diagrams for differentiable families of triples of matrices $\varphi(\xi) = (E(\xi), A(\xi), B(\xi))$ with the parameter vector $\xi \in R^k$ in M_R.

M. Isabel García-Planas is with Universitat Politècnica de Catalunya, C. Mineria 1, Esc. C, 1-3, 08038 Barcelona, Spain; maria.isabel.garcia@upc.edu

Structurally stable elements are those whose behavior does not change when applying small perturbations. The concept of structural stability, in the qualitative theory of dynamical systems has been widely studied by several authors in control theory (see [3], [4], for example).

In the case where the system is not structurally stable, we are interested in the knowledge of the different kind of triples that we can find in a small neighborhood of the given system. It is well known that any algorithm for computation reduced forms computes the exact structure of a nearby element. In order to make the best decision during the computation of the reduced form it is important to know how the classes of triples of the different structures are related to each other.

The Arnold technique of constructing a local canonical form, called versal deformation, of a differentiable family of square matrices under conjugation [1] provide a special parametrization of matrix spaces, which can be effectively applied to perturbation analysis and investigation of complicated objects like singularities and bifurcations in multi-parameter dynamical systems [1], [2], [3], [6]. This technique has been generalized by several authors to matrix pencils under the strict equivalence [2], [3], pairs or triples of matrices under the action of the general linear group [8], pairs of matrices under the feedback similarity [6], among others. In this paper we generalize the study to the n-dimensional multi-input linear dynamical systems, we obtain explicit versal deformation and we apply it to analyze bifurcation diagrams in singular points of family of triples of matrices representing regularizable singular systems.

In the sequel we identify triples of matrices (E, A, B) with rectangular matrices $(E\ A\ B)$ in order to use matrix expressions.

II. Collection of invariants

First of all, we remember the equivalence relation considered over the space of triples of matrices.

Definition 1: Two triples (E', A', B') and (E, A, B) in M are called equivalent if, and only if, there exist matrices $Q, P \in Gl(n; C)$, $R \in Gl(m; C)$, $F_E, F_A \in M_{m\times n}(C)$, such that

$$(E', A', B') = (QEP + QBF_E, QAP + QBF_A, QBR),$$

or in a matrix form

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_E & F_A & R \end{pmatrix}$$

It is easy to check that this relation is an equivalence relation.

Now, we consider a list of ranks of a certain matrices associated to the matrices E, A, B in the triple $(E, A, B) \in M$.

Definition 2: We consider the following numbers

1) $r_0 = \operatorname{rank} B$
2) $r_1 = \operatorname{rank} \begin{pmatrix} \lambda E + \mu A & B \end{pmatrix}$, $\forall \lambda, \mu \in C$.
3) $r_2 = \operatorname{rank} \begin{pmatrix} E & A & B \end{pmatrix}$
4) $r_3 = (r_3^1, \dots, r_3^\ell, \dots)$, where
   1) $r_3^1 = \operatorname{rank} M_3^1$ with $M_3^1 = \begin{pmatrix} E & B & 0 \\ A & 0 & B \end{pmatrix} \in M_{2n \times (n+2m)}(C)$

   $\vdots$

   ℓ) $r_3^\ell = \operatorname{rank} M_3^\ell$ with $M_3^\ell = \begin{pmatrix} E & B & 0 & 0 & 0 & 0 & \\ A & 0 & B & E & B & 0 & \\ 0 & 0 & 0 & A & 0 & B & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & \ddots \end{pmatrix} \in M_{(\ell+1)n \times (\ell n + 2\ell m)}(C)$
5) $r_4 = (r_4^1, \dots, r_4^\ell, \dots)$, where
   j) $r_4^j = \operatorname{rank} M_4^j$ with $M_4^j = \begin{pmatrix} E & B & & & & C_n & 0 \\ -\lambda_0 E + A & 0 & & & & C_{n-1} & 0 \\ 0 & 0 & & & & C_{n-2} & 0 \\ & & \ddots & & & & \\ & & & E & B & C_2 & 0 \\ & & & -\lambda_0 E + A & 0 & C_1 & B \end{pmatrix}$
   where $C_i = \begin{cases} -\lambda_0 E + A & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$, for $j = 1, 2, \dots$.
6) $r_5 = (r_5^1, \dots, r_5^\ell, \dots)$, where
   p) $r_5^j = \operatorname{rank} M_5^j$ with $M_5^j = \begin{pmatrix} A & B & \dots & 0 & 0 & C_\ell & 0 \\ E & 0 & & & & & \\ & & \ddots & & & & \\ & & & A & B & C_1 & 0 \\ & & & E & 0 & C_0 & B \end{pmatrix}$
   where $C_i = \begin{cases} E & i = j, \\ 0 & i \neq j \end{cases}$, for $j = 1, 2, \dots$.
7) Each $\lambda_0 \in C$ such that $\operatorname{rank}(\lambda_0 E - A) < \operatorname{rank}(\lambda E - A\ B) \leq \operatorname{rank}(\lambda E + \mu A\ B)$.

We denote by $\sigma(E, A, B) = \{\lambda_0 \mid \operatorname{rank}(\lambda_0 E - A\ B) < \operatorname{rank}(\lambda E - A\ B)\}$ and we will call spectra of the triple.

Remark 1: We are interested in r_4 for each $\lambda_0 \in \sigma(E, A, B)$.

Proposition 1: In the set M of singular systems, the r_i numbers as well all $\lambda_0 \in \sigma(E, A, B)$, are invariant under the equivalence relation considered.

Proof: Let (E, A, B), (E', A', B') be two equivalent triples in M, then, there exist matrices $Q, P \in Gl(n; C)$, $R \in Gl(m; C)$, $F_E, F_A \in M_{m \times n}(C)$ such that

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_E & F_A & R \end{pmatrix}$$

So,

1) $r_1' = \operatorname{rank} \begin{pmatrix} \lambda E' + \mu A' & B' \end{pmatrix} = \operatorname{rank} Q \begin{pmatrix} \lambda E + \mu A & B \end{pmatrix} \begin{pmatrix} P & 0 \\ \lambda F_E + \mu F_A & R \end{pmatrix} = \operatorname{rank} \begin{pmatrix} E & B \end{pmatrix} = r_3$,
2) Obvious after equivalence relation definition.
3) ${r_3^1}' = \operatorname{rank} \begin{pmatrix} E' & B' & 0 \\ A' & 0 & B' \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} E & B & 0 \\ A & 0 & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ F_E & R & 0 \\ F_A & 0 & R \end{pmatrix} = \operatorname{rank} \begin{pmatrix} E & B & 0 \\ A & 0 & B \end{pmatrix} = r_5^1$.

   ${r_3^\ell}' = \operatorname{rank} \begin{pmatrix} E' & B' & 0 & 0 & 0 & 0 & \\ A' & 0 & B' & E' & B' & 0 & \\ 0 & 0 & 0 & A' & 0 & B' & \\ & & & & & & \ddots \end{pmatrix} = \begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix} M_3^\ell \begin{pmatrix} P & 0 & 0 & & & & \\ F_E & R & 0 & & & & \\ F_A & 0 & R & & & & \\ & & & P & 0 & 0 & \\ & & & F_E & R & 0 & \\ & & & F_A & 0 & R & \\ & & & & & & \ddots \end{pmatrix} =$

   $\operatorname{rank} M_5^j = r_5^j$.

Analogously, we can prove the invariance for 4, 5, 6. ■

This collection of invariants constitutes a complete system because they permit to deduce the canonical reduced form for regularizable systems.

The r_3-numbers permit us to obtain controllability indices in the following manner. Calling $r_3^0 = \operatorname{rank} B$, we define the ρ-numbers in the following manner

$$\begin{aligned} \rho_0 &= r_3^0 \\ \rho_1 &= r_3^1 - r_3^0 - n \\ \rho_2 &= r_3^2 - r_3^1 - n \\ &\vdots \\ \rho_s &= r_3^{s-1} - r_3^s - n \end{aligned}$$

Finally the *controllability indices* $k_1, \dots, k_p$ for singular systems as the integers $k_1 \geq \dots \geq k_p$ such that $[k_1, \dots, k_p]$ is the conjugate partition of $[\rho_0, \rho_1, \dots, \rho_s]$. Recall that if the system is controllable then $k_1 + \dots + k_p = n$.

For each $\lambda_1, \dots, \lambda_s$ in $\sigma(E, A, B)$, the r_4-numbers permit us to obtain the Segre characteristic in the following manner, for each eigenvalue λ_i the Segre characteristic of this eigenvalue is the conjugate partition of a $[n - \nu_1, \nu_1 - \nu_2, \dots.]$, where

$$\nu_l = \tau_l^{n-1} - (n-1)n$$

Analogously we prove that r_5-numbers describe the ∞-poles.

Now, we are going to describe the canonical reduced forms under this equivalence relation for multi-input n-dimensional standardizable generalized systems and the case where $n = 2$ and $m = 1$.

Proposition 2: Let (E, A, B) be a n-dimensional m-input regularizable system. Then, it can be reduced to (E_c, A_c, B_c)

with $E_c = \begin{pmatrix} I_1 & \\ & N_1 \end{pmatrix}$, $A_c = \begin{pmatrix} A_2 & \\ & I_2 \end{pmatrix}$, $B_c = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ where (A'_1, B'_1) is in its Kronecker canonical form and the Kronecker indices, eigenvalues and Segre characteristic of (A_1, B_1) are controllability indices, eigenvalues and Segre characteristic of (E, A, B), as well the r_5-numbers gives us the structure of nilpotent matrix N_1.

Remark 2: In general, let (E, A, B) be a singular not necessarily regularizable, it can be reduced to

$$\left(\begin{pmatrix} 0 \\ E'_1 \end{pmatrix}, \begin{pmatrix} 0 \\ A'_1 \end{pmatrix}, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \right)$$

and (E'_1, A'_1) in its Kronecker canonical form as a pencil $\lambda E'_1 + A'_1$, (see [5]).

III. Equivalence relation as a Lie group action

Let us consider the following Lie group $\mathcal{G} = Gl(n; C) \times Gl(n; C) \times Gl(m; C) \times M_{m\times n}(C) \times M_{m\times n}(C)$, acting on M. The product $\star$ in $\mathcal{G}$ is given by

$$(Q_1, P_1, R_1, F_{E_1}, F_{A_1}) \star (Q_2, P_2, R_2, F_{E_2}, F_{A_2}) = (Q_2Q_1, P_1P_2, R_1R_2, FE_1P_2 + R_1F_{E_2}, F_{A_1}P_2 + R_1F_{A_2})$$

being $e = (I_n, I_n, I_m, 0, 0)$ its unit element.

The action $\alpha : \mathcal{G} \times M \longrightarrow M$ of the Lie group $\mathcal{G}$ on M defined by

$$\alpha((Q, P, R, F_E, F_A), (E, A, B)) = (QEP + QBF_E, QAP + QBF_A, QBR)$$

give rise to the equivalence relations in M in §1 which we will call p-d-feedback-equivalence.

From now on, we will make use of the following notation: $g = (P, Q, R, U, V) \in \mathcal{G}$, and $x = (E, A, B) \in M$.

Given a triple $x_0 = (E_0, A_0, B_0) \in M$ we define the maps

$$\alpha_{x_0}(g) = \alpha(g, x_0). \tag{1}$$

The equivalence class of the triple x_0 with respect to the $\mathcal{G}$-action, called the $\mathcal{G}$-orbit of x_0, is the range of the function α_{x_0} and is denoted by

$$\mathcal{O}(x_0) = \operatorname{Im} \alpha_{x_0} = \{\alpha_{x_0}(g) \mid g \in \mathcal{G}\}. \tag{2}$$

The stabilizer of x_0 under the $\mathcal{G}$-action is the null-space of the function $\alpha_{x_0} - x_0$. We denote it by

$$Stab(x_0) = \{g \in \mathcal{G} \mid \alpha_{i x_0}(g) = x_0\}. \tag{3}$$

Remark 3: The maps α_{x_0} are clearly differentiable and $\mathcal{O}(x_0)$, $Stab(x_0)$ are smooth submanifolds of M and $\mathcal{G}$, respectively.

IV. Miniversal Deformations

First, we recall the definition of versal deformations. Let M be a smooth manifold.

Definition 3: Let $\mathcal{U}_0$ be a neighborhood of the origin of C^k. A *deformation* $\varphi(\mu)$ of x_0 is a smooth mapping

$$\varphi : \mathcal{U}_0 \longrightarrow M$$

such that $\varphi(0) = x_0$. The vector $\lambda = (\mu_1, \ldots, \mu_\ell) \in \mathcal{U}_0$ is called the parameter vector.
The deformation $\varphi(\mu)$ is also called *differentiable family* of elements of M.

Let $\mathcal{G}$ be a Lie group acting smoothly on M. We denote the action of $g \in \mathcal{G}$ on $x \in M$ by $g \circ x$.

Definition 4: The deformation $\varphi(\mu)$ of x_0 is called *versal* if any deformation $\varphi'(\xi)$ of x_0, where $\xi = (\xi_1, \ldots, \xi_k) \in \mathcal{U}'_0 \subset C^k$ is the parameter vector, can be represented in some neighborhood of the origin as

$$\varphi'(\xi) = g(\xi) \circ \varphi(\phi(\xi)), \quad \xi \in \mathcal{U}''_0 \subset \mathcal{U}'_0, \tag{4}$$

where $\phi : \mathcal{U}''_0 \longrightarrow C^\ell$ and $g : \mathcal{U}''_0 \longrightarrow \mathcal{G}$ are differentiable mappings such that $\phi(0) = 0$ and $g(0)$ is the identity element of $\mathcal{G}$. Expression means that any deformation $\varphi'(\xi)$ of x_0 can be obtained from the versal deformation $\varphi(\lambda)$ of x_0 by an appropriate smooth change of parameters $\lambda = \phi(\xi)$ and an equivalence transformation $g(\xi)$ smoothly depending on parameters.

A versal deformation having minimal number of parameters is called *miniversal*.

The following result was proved by Arnold [1], in the case where $\mathrm{Gl}(n; C)$ acts on $M_n(C)$. It provides the relationship between a versal deformation of x_0 and the local structure of the orbit.

Theorem 1 (8): 1. A deformation $\varphi(\lambda)$ of x_0 is versal if and only if it is transversal to the orbit $\mathcal{O}(x_0)$ at x_0.
2. Minimal number of parameters of a versal deformation is equal to the codimension of the orbit of x_0 in M, $\ell = \operatorname{codim} \mathcal{O}(x_0)$.

Let $\{v_1, \ldots, v_k\}$ be a basis of any arbitrary complementary subspace $(T_{x_0}\mathcal{O}(x_0))^c$ to $T_{x_0}\mathcal{O}(x_0)$ (for example, $(T_{x_0}\mathcal{O}(x_0))^\perp$).

Corollary 1: The deformation

$$x : \mathcal{U}_0 \subset C^k \longrightarrow M, \quad x(\mu) = x_0 + \sum_{i=1}^{k} \mu_i v_i \tag{5}$$

is a miniversal deformation.

The Lie group $\mathcal{G}$ act smoothly on M. Thus we can apply these results to deduce explicit miniversal deformations.

Proposition 3: Let $x_0 = (E, A, B)$ be a triple of matrices. Let $\{u_1, \ldots, u_k\}$, be a basis of the vector subspace

$$T_{x_0}\mathcal{O}(x_0)^\perp = \{(X, Y, Z) \in M \mid EX^* + AY^* + BZ^* = 0, X^*E + Y^*A = 0, X^*B = 0, Y^*B = 0, Z^*B = 0\}$$

Then the maps defined by

$$\varphi(\mu_1, \ldots, \mu_k) = x_0 + \mu_1 u_1 + \ldots + \mu_k u_k$$

is a miniversal deformation with respect to the $\mathcal{G}$-action.

Remark 4: If $E = I_n$, $T_{x_0}\mathcal{O}(x_0)^\perp = \{(X, Y, Z) \mid X^* = -Y^*, AY^* - Y^*A + BZ^* = 0, X^*B = 0, Y^*B = 0, Z^*B = 0\}$, that corresponds with the miniversal orthogonal deformation of (A, B) under block-similarity $T_{(A,B)}\mathcal{O}(A, B)^\perp = \{(Y, Z) \mid AY^* - Y^*A + BZ^* = 0, X^*B = 0, Y^*B = 0, Z^*B = 0\}$, (see [6]).

In order to describe the miniversal orthogonal deformation for regularizable triples, we observe that we can consider the triple in its canonical reduced form. So, partitioning the matrices $X^* = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, $Y^* = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$,

$Z^* = \begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$ following the blocks on the matrices E, A, B in its canonical reduced form, we obtain the following independent systems

$$\left.\begin{array}{rl} X_1 + A_2Y_1 + B_1Z_1 & = 0 \\ X_1 + Y_1A_2 & = 0 \\ X_1B_1 & = 0 \\ Y_1B_1 & = 0 \\ Z_1B_1 & = 0 \end{array}\right\}$$

according remark, this system corresponds to the miniversal orthogonal deformation to the standard system (I, A_2, B_1)

$$\left.\begin{array}{rl} N_1X_4 + Y_4 & = 0 \\ X_4N_1 + Y_4 & = 0 \end{array}\right\} \Leftrightarrow \quad \left.\begin{array}{rl} -N_1X_4 & = Y_4 \\ X_4N_1 - N_1X_4 & = 0 \end{array}\right\}$$

this system corresponds to the miniversal orthogonal deformation to the square matrix N_1 (see [1],

$$\left.\begin{array}{rl} N_1X_3 + Y_3 & = 0 \\ X_3 + Y_3A_2 & = 0 \\ X_3B_1 & = 0 \\ Y_3B_1 & = 0 \end{array}\right\}$$

having zero-solution, and

$$\left.\begin{array}{rl} X_2 + A_2Y_2 + B_1Z_2 & = 0 \\ X_2N_1 + Y_2 & = 0 \end{array}\right\}.$$

In order to solve the last system, we partition the systems into independent subsystems corresponding to the blocks in the matrix $A_2 = \begin{pmatrix} N_2 & \\ & J \end{pmatrix}$, so $B_1 = \begin{pmatrix} B_{11} \\ 0 \end{pmatrix}$, obtaining

$$X_1^2 - N_2X_1^2N_1 + B_1Z_2 = 0\}$$

and

$$X_2^2 - JX_2^2N_1 = 0\}$$

with solutions

$$X_2^2 = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & & \vdots \\ X_{s1} & \dots & X_{sr} \end{pmatrix},$$

$$X_{ij} = \begin{pmatrix} 0 & \dots & 0 & x_1 & \dots & x_m \\ 0 & \dots & x_1 & x_2 & \dots & x_{m+1} \\ & \iddots & & \iddots & \iddots & \\ x_1 & \dots & & & \dots & x_n \end{pmatrix}$$

$$X_{ij} = \begin{pmatrix} 0 & \dots & 0 & x_1 \\ 0 & \dots & x_1 & x_2 \\ & \iddots & & \\ x_1 & \dots & n_{-1} & x_n \end{pmatrix}$$

or

$$X_{ij} = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & x_1 \\ 0 & \dots & x_1 & x_2 \\ & \iddots & \iddots & \\ x_1 & \dots & x_{n-1} & x_n \end{pmatrix}$$

depending on the size of the nilpotent submatrices in N_1 and N_2.

And $X_2^2 = 0$.

Remark 5: Given a triple $(E, A, B) \in M_R$ in its canonical reduced form, we can consider the minimal miniversal deformation $(E+X, A+Y, B+Z)$ with $X = \begin{pmatrix} 0 & 0 \\ X_3 & X_4 \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} Z_1 & 0 \end{pmatrix}$, $(A_2 + Y_1, B_1 + Z_1)$ a minimal deformation of the pair (A_1, B_1), $N_1 + X_4$ a miniversal deformation of the square matrix N and $X_3 = (X_{ij})$ with

$$X_{ij} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ & & \\ 0 & \dots & 0 \\ x_1 & \dots & x_n \end{pmatrix}$$

$$X_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ & & & & \\ 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & x_1 & \dots & x_n \end{pmatrix}$$

corresponding to size in the nilpotent submatrices N_1 and N_2.

V. The strata

The space M_R of all regularizable triples of matrices is formed by the disjoint union of all orbits of the triples and the frontier of each orbit is formed by orbits of strictly lower dimension. Given two triples (E_i, A_i, B_i) in M, we can ask when the closure of $\mathcal{O}(E_1, A_1, B_1)$ includes the closure of $\mathcal{O}(E_2, A_2, B_2)$.

There are infinitely orbits having the same discrete invariants varying only in the values of the continuous ones.

In order to obtain a finite partition preserving the orbit structure, we group the orbits with the same type, we call this set stratum in M_R and we will write $\mathcal{S}t(E, A, B)$. There are only finitely many strata, each an uncountable union of orbits or a unique orbit partitioning M_R.

Proposition 4: Any stratum is a constructible and connected subset of M_R.

Proof: Let (E, A, B) in its canonical reduced form with parameters $\lambda_1, \dots, \lambda_s \in C$, and $C' = \{(\lambda_1, \dots, \lambda_s\} \subset C \mid \lambda_i \neq \lambda_j, \forall 1 \leq i, j \leq s\}$. We consider the map $\rho : \mathcal{G} \times C' \longrightarrow M$ such that $\rho((P, Q, R, F_E, F_A), (\lambda_1, \dots, \lambda_s)) = \alpha((P, Q, R, F_E, F_A), (E, A, B))$. The domain $\mathcal{G} \times C'$ is constructible in $C^{2n^2+m^2+2nm} \times C'$, and the mapping is regular rational, so by Chevalley's theorem its image (stratum) is constructible.

Note also that this stratum is connected since the set $\mathcal{G} \times C'$ is connected and ρ is continuous.

∎

Lemma 1: Let $\varphi_1 : \Lambda \longrightarrow M_R$ be a deformation of (E, A, B) minitransversal to the orbit

$\mathcal{O}(E,A,B)$. Let $V \subset \mathcal{G}$ a subvariety minitransversal to $Stab(E,A,B) = \{(Q,P,R,F_E,F_A) \in \mathcal{G} \mid \alpha((Q,P,R,F_E,F_A),(E,A,B)) = (E,A,B)\}$. Then:

$$\beta : \Lambda \times V \longrightarrow M_R$$
$$(\mu,(Q,P,R,F_E,F_A)) \longrightarrow Q(E(\mu)\ A(\mu)\ B(\mu)) \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_E & F_A & R \end{pmatrix}$$

with $\mu = (\mu_1,...,\mu_s)$ and $(E(\mu),A(\mu),B(\mu)) = \varphi_1(\mu)$, is a diffeomorphism at $(0,I)$.

Proof: The inverse function theorem ensures that β is a local diffeomorphism at $(0,I)$, if and only if $d\beta_{(0,I)}$ is a diffeomorphism.

Taking into account that $\dim(V \times \Lambda) = 2n^2 + mn = \dim M$, it suffices to observe that $d\beta$ is surjective. ■

Lemma 2: Let (E,A,B) be a triple in M, $\mathcal{O}(E,A,B)$ its orbit, $\mathcal{S}t(E,A,B)$ its stratum, and Γ the variety transversal to the orbit considered in remark 4. Then, in a neighborhood of (E,A,B), $\mathcal{S}t(E,A,B)$ is a subvariety regular en (E,A,B) if and only if $\mathcal{S}t(E,A,B) \cap \Gamma$ is.

Proof: Suppose $\mathcal{S}t(E,A,B)$ regular at (E,A,B). Taking into account that Γ is transversal to $\mathcal{O}(E,A,B)$, it also is transversal to $\mathcal{S}t(E,A,B)$. Then, $\mathcal{S}t(E,A,B) \cap \Gamma$ is regular at (E,A,B).

Conversely, suppose $\mathcal{S}t(E,A,B) \cap \Gamma$ regular at (E,A,B). The local triviality given in lemma 2, we have

$$\mathcal{S}t(E,A,B) = \beta((\mathcal{S}t(E,A,B) \cap \Gamma) \times V)$$

locally in (E,A,B). Then $\mathcal{S}t(E,A,B)$ is regular at (E,A,B). ■

Now we analyze $\mathcal{S}t(E,A,B) \cap \Gamma$.

Proposition 5: Let $(E_c,A_c,B_c) \in \mathcal{S}t(E,A,B)$ for some stratum.

i) If $(X,Y,Z) \neq (0,0,0)$, then $(E_c,A_c,B_c)+(X,Y,Z) \notin \mathcal{O}(E,A,B)$.

ii) $(E_c,A_c,B_c)+(X,Y,Z) \in \mathcal{E}(E,A,B)$ if and only if $X = 0$, $Z = 0$, $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_1 + Y_1 = \begin{pmatrix} N_2 & \\ & J_2 \end{pmatrix} + \begin{pmatrix} 0 & \\ & Y_{14} \end{pmatrix}$ and $J + Y_{14}$ have the same Segre symbol than J (that is to say having the same Jordan form varying at the most in the values of eigenvalues).

Proof: It suffices to compute the collection of invariants for a minimal deformation of the triple. ■

Theorem 2: The strata are submanifolds of M_R.

Proof: Let $(E,A,B) \in M_R$.

It is obvious for strata $\mathcal{S}t(E,A,B)$ that they are orbits

We are going to proof for the other strata. By the homogeneity of the orbits we can suppose the triples (E,A,B) in its canonical reduced form.

By lemma 2 it suffices to prove that $\mathcal{S}t(E,A,B) \cap \Gamma$, is regular at (E,A,B). We denote by r the size of the matrix J in A_c, notice that if the stratum is not an orbit $r \neq 0$.

For each $\mathcal{S}t$-stratum we define the map:

$$\begin{aligned} \phi : M_r(C) &\longrightarrow M_R \\ C &\longrightarrow (E,A,B) \end{aligned}$$

with $E = E_c$, $A = \begin{pmatrix} N_2 & & \\ & J+C & \\ & & I_2 \end{pmatrix}$, $B = B_c$.

This map is an embbebding such that $\phi(\mathcal{S}t(J) \cap \Gamma(J) = \mathcal{S}t(E,A,B)$, where $\mathcal{S}t(J)$ corresponds to the Segre stratum in the space of square matrices and $\Gamma(J)$ the variety transversal to the orbit of J.

Taking into account that $\mathcal{S}t(J) \cap \Gamma(J)$ is regular (see [7]), we have that $\mathcal{S}t(E,A,B) \cap \Gamma$ is regular hence $\mathcal{S}t(E,A,B)$ is regular. ■

VI. Bifurcation diagrams

The knowledge of the versal deformation of a triple (E,A,B) gives us a method for investigating the possible canonical reduced form of a perturbation of (E,A,B).

A local perturbation of a triple of matrices (E,A,B), is a family of triples of matrices depending differentiable on parameters defined in a neighborhood of (E,A,B), and it is induced by a versal deformation. Taking into account that the miniversal deformation of a triple is transversal to its orbit so transversal to its stratum, the miniversal families are generic in the sense that in the space of differentiable families of triples of matrices, the transverse families constitutes a dense set. So we are going to analyze generic families with few parameters.

Given a triple of matrices the homogeneity of the orbit permit us to consider it in its canonical reduced form.

We can analyze for example, the case of 2-dimensional 1-input generalized systems, we have.

1) If the triple (E,A,B) is in the stratum $\mathcal{S}t(1) = \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right\}$, the miniversal deformation is (E,A,B), then any triple in a neighborhood is equivalent to it. So the triple is structurally stable.

2) Let (E,A,B) be a triple in $\mathcal{S}t(2) = \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right\}$, in a neighborhood of this triple, a generic family is the one-parametric:

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y_1 & a \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

Computing the controllability indices, eigenvalues and Segre characteristic for all different values of parameters we have that the bifurcation diagram is:

$$\begin{cases} \text{if } y_1 \neq 0 \text{ the triple is in } \mathcal{S}t(1) \\ \text{if } y_1 = 0 \text{ the triple is in } \mathcal{S}t(2) \end{cases}$$

3) Let (E,A,B) in $\mathcal{S}t(3) = \left\{\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right\}$, in this case the triple is not regularizable but it is easy to compute the minimal miniversal deformation Γ as well $\mathcal{S}t(3) \cap \Gamma$. In a neighborhood of this triple, a generic family is the two-parametric:

$$\left(\begin{pmatrix} 0 & 0 \\ x_2 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

Computing the invariants we have that the bifurcation diagram is:

$$\begin{cases} \text{if } x_2 \neq 0 \text{ the triple is in } \mathcal{S}t(1) \\ \text{if } x_4 \neq 0, x_2 = 0 \text{ the triple is in } \mathcal{S}t(2) \\ \text{if } x_4 = 0, x_2 = 0 \text{ the triple is in } \mathcal{S}t(3). \end{cases}$$

4) There are not three-parametric generic families

In the case of 3-dimensional 1-input generalized systems, we present the following example

Let (E, A, B) be a triple with $E = \begin{pmatrix} 5 & 8 & 3 \\ 1 & 4 & 2 \\ 1 & -1 & 2 \end{pmatrix}$ $A = \begin{pmatrix} 5 & 8\lambda & 8+3\lambda \\ 1 & 4\lambda & 4+2\lambda \\ -1 & -\lambda & -1+2\lambda \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$ computing the invariants given in proposition 1, we obtain that the continuous invariants are λ and the discrete ones $k_1 = 1$, $\nu_1 = 2$, $\nu_2 = 1$, so its canonical reduced form is $E_1 = I_3$ $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in M_{3\times 1}(C)$.

A generic family in a neighborhood of the triple is the equivalent family of the following three-parametric $(E_1, A_1, B_1) + \{(0, Y, 0)\}$ with $Y = \begin{pmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & y_{32} & 0 \end{pmatrix}$.

The family contains the same type of triples than $(E_1, A_1, B_1 + \{(0, Y, 0)\}$, so we analyze this one, and that it contain the following types of triples

a) if $y_{32}y_{21}^2 + y_{21}y_{31}y_{33} - y_{31}^2 \neq 0$ the triple is equivalent to (E', A', B') with $E' = I_3$, $A' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in M_{3\times 1}(C)$.

b) if $y_{32}y_{21}^2 + y_{21}y_{31}y_{33} - y_{31}^2 = 0$ If we make a change of coordinates defined by $y_{21} = x$, $y_{32} = z - \frac{t^2}{4}$, $y_{31} = y + x\frac{t}{2}$, $y_{33} = t$, the above equation become $x^2z - y^2 = 0$. This is the Whitney umbrella surface along the t axis. In this case we can find the following triples

   a) $x = y = z = 0$ for all t, the triple is equivalent to (E', A', B'), with $E' = I_3$ $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda' & 1 \\ 0 & 0 & \lambda' \end{pmatrix}$, $B' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in M_{3\times 1}(C)$.

   b) $x = y = 0$, $z \neq 0$ for all t, the triple is equivalent to (E', A', B'), with $E' = I_3$ $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \lambda'' \end{pmatrix}$, $B' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in M_{3\times 1}(C)$.

   c) $x^2z - y^2 = 0$, $x \neq 0$ or $y \neq 0$ for all t, the triple is equivalent to (E', A', B'), with $E' = I_n$ $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda' \end{pmatrix}$, $B' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in M_{3\times 1}(C)$.

VII. Conclusions

Knowing a complete system of invariants for triples of matrices representing regularizable singular linear systems under proportional and derivative feedback, we can explicit a miniversal deformation that permit us to analyze the bifurcation diagrams of singular points of differentiable family of triples.

VIII. References

[1] V. I. Arnold, *On matrices depending on parameters*, Russian Math. Surveys, 26: 2 (1971), pp. 29–43.

[2] A. Edelman, E. Elmroth, and B. Kågström, *A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 653-692.

[3] J. Ferrer, Mª I. García, F. Puerta, *Regularity of the Brunovsky-Kronecker Stratification*. SIAM Journal on Matrix Analysis **21** (2), pp. 724-742 (2000).

[4] J. Ferrer, Mª I. García-Planas, *Structural Stability of Quadruples of Matrices*. Linear Algebra and its Applications **241/243**, pp. 279–290, (1996).

[5] Mª I. García-Planas, M. D. Magret, *Polynomial Matrices and generalized linear multivariable dynamical systems*. IRBS of Wseas Press, pp. 17-22 (2000).

[6] Mª I. García-Planas, A. Mailybaev, *Reduction to versal deformations of matrix pencils and matrix pairs with application to control theory*. SIAM Journal on Matrix Analysis. vol. 24, **4**, pp. 943-962, (2003).

[7] C. G. Gibson, *Regularity of the Segre Stratification*, Math. Proc. Camb. Phil. Soc. 80 (1976), pp. 91-97.

[8] A. Tannenbaum , *Invariance and System Theory: Alge braic and geometric Aspects*, Lecture Notes in Math. 845, Springer-Verlag (1981).