NORMAL FORM REDUCTION FOR MULTIPLE-ZERO EIGENVALUE USING FRACTIONAL SCALES

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Abstract
In this paper, we present a new method for finding normal form equation and invariant manifold in the case of multiple zero eigenvalue with a single Jordan block. The method utilizes the concept of fractional scale. This allows using a single scale parameter in the normal form reduction for systems with multiple variables and parameters. The use of fractional scales substantially simplifies the procedure of system reduction. As an example, we perform the normal form reduction near the point of triple zero bifurcation for a double pendulum under a follower force.

Key words
Normal form, multiple eigenvalue, fractional scale.

1 Fraction scales in dynamical system reduction

We consider a nonlinear system of ordinary differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{p}), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a vector of dynamical variables, $\mathbf{p} \in \mathbb{R}^m$ is a constant parameter vector, and $\mathbf{F}(\mathbf{x}, \mathbf{p})$ is a smooth function. We assume that

$$\mathbf{F}(0, \mathbf{p}) = 0 \tag{2}$$

for all $\mathbf{p}$, i.e., $\mathbf{x} = 0$ is always an equilibrium. We will study equation (1) near the critical point $\mathbf{p} = 0$, at which the Jacobian matrix $\mathbf{F}_x = d\mathbf{F}/d\mathbf{x}$ evaluated at $(\mathbf{x}, \mathbf{p}) = (0, 0)$ possesses a multiple zero eigenvalue with a single Jordan block of order k. This point represents a degeneracy of codimension k for general systems, i.e., it can be found by tuning k parameters of the system [Arnold, 1983].

Let all the eigenvalues of the matrix $\mathbf{F}_x$, except for zero eigenvalue, have nonzero real part. In practical applications, we are usually interested in the case when all of them have negative real parts, so the point $\mathbf{p} = 0$ lies on the stability boundary of the trivial equilibrium. Then, in the neighborhood of $(\mathbf{x}, \mathbf{p}) = (0, 0)$ system (1) can be reduced to a k-dimensional invariant central manifold. The behavior on this manifold is governed by a k-dimensional reduced system. The reduced system plays the key role in the analysis of the system dynamics, since it describes important properties of the full original system like bifurcations of equilibria, limit cycles, stability etc.

Let $\mathbf{x} = \mathbf{x}(a, \dot{a}, \ddot{a}, \dots, a^{(k-1)}, \mathbf{p})$ be the equation for the stable manifold, where a new variable a and its derivatives $\dot{a}, \ddot{a}, \dots, a^{(k-1)}$ are considered as dynamical variables of the reduced system. We will look for the reduced system in the form

$$a^{(k)} = f(a, \dot{a}, \ddot{a}, \dots, a^{(k-1)}, \mathbf{p}), \tag{3}$$

where $f(a, \dot{a}, \dots, a^{(k-1)}, \mathbf{p})$ is a scalar smooth function. The unknown functions $\mathbf{x}(a, \dot{a}, \dots, a^{(k-1)}, \mathbf{p})$ and $f(a, \dot{a}, \dots, a^{(k-1)}, \mathbf{p})$ can be found in the form of Taylor series by substituting into equation (1) and comparing similar terms. This gives an infinite number of coupled equations.

The purpose of this paper is to give a constructive method for solving these equations by using the concept of fractional powers. In this way, we establish the relationship between the reduction problem considered here and the methods of perturbation theory in linear algebra [Mailybaev, 2001; Seyranian and Mailybaev, 2004] and multiple-scale methods in nonlinear equations [Luongo et al., 1999; Luongo et al., 2003]. In our case, the main idea is to introduce a small scale $\varepsilon > 0$ such that

$$a \sim \varepsilon, \quad \mathbf{p} \sim \varepsilon, \quad \frac{d}{dt} \sim \varepsilon^{1/k}. \tag{4}$$

These conditions determine the scale of each term in the Taylor expansion with respect to $a, \dot{a}, \dots, a^{(k-1)}$

and $\mathbf{p}$, e.g.

$$\dot{a} \sim \varepsilon^{1+1/k}, \quad a\mathbf{p} \sim \varepsilon^2, \quad a\ddot{a}\mathbf{p} \sim \varepsilon^{3+2/k}. \tag{5}$$

Then we can group terms in Taylor expansions according to their fractional orders in ε:

$$\begin{aligned} \mathbf{x}(a, \dot{a}, \dots, a^{(k-1)}, \mathbf{p}) &= \varepsilon \mathbf{x}_1 + \varepsilon^{1+1/k}\mathbf{x}_{1+1/k} \\ &\quad + \varepsilon^{1+2/k}\mathbf{x}_{1+2/k} + \cdots \\ f(a, \dot{a}, \dots, a^{(k-1)}, \mathbf{p}) &= \varepsilon^2 f_2 + \varepsilon^{2+1/k} f_{2+1/k} \\ &\quad + \varepsilon^{2+2/k} f_{2+2/k} + \cdots \end{aligned} \tag{6}$$

Here the factors ε^α show the order of the corresponding terms $\mathbf{x}_\alpha$ and f_α (after the computations ε is set to 1).

As we will see below, the use of fractional scales substantially simplifies the procedure of system reduction. Another advantage of the method is that fractional scales explicitly determine the "importance" (scale) of each term when writing an approximate reduced equation.

2 Double zero eigenvalue

In this section, we consider the case of a double zero eigenvalue, which is essentially similar to the case of arbitrary multiplicity. In this case (6) becomes

$$\begin{aligned} \mathbf{x}(a, \dot{a}, \mathbf{p}) &= \varepsilon \mathbf{x}_1 + \varepsilon^{3/2}\mathbf{x}_{3/2} + \varepsilon^2 \mathbf{x}_2 + \cdots, \\ f(a, \dot{a}, \mathbf{p}) &= \varepsilon^2 f_2 + \varepsilon^{5/2} f_{5/2} + \varepsilon^3 f_3 + \cdots \end{aligned} \tag{7}$$

Substituting (7) into the right-hand side of (1), we obtain

$$\begin{aligned} &\mathbf{F}(\mathbf{x}, \mathbf{p}) = \mathbf{F}_x \mathbf{x} + \tfrac{1}{2}\mathbf{F}_{xx}\mathbf{x}^2 + \mathbf{F}_{xp}\mathbf{x}\mathbf{p} + \cdots \\ &= \varepsilon \mathbf{F}_x \mathbf{x}_1 + \varepsilon^{3/2}\mathbf{F}_x \mathbf{x}_{3/2} \\ &+ \varepsilon^2 \left(\mathbf{F}_x \mathbf{x}_2 + \tfrac{1}{2}\mathbf{F}_{xx}\mathbf{x}_1^2 + \mathbf{F}_{xp}\mathbf{x}_1 \mathbf{p}\right) \\ &+ \varepsilon^{5/2} \left(\mathbf{F}_x \mathbf{x}_{5/2} + \mathbf{F}_{xx}\mathbf{x}_1 \mathbf{x}_{3/2} + \mathbf{F}_{xp}\mathbf{x}_{3/2}\mathbf{p}\right) + \cdots \end{aligned} \tag{8}$$

Here we used condition (2) and the short notation for derivatives

$$\begin{aligned} \mathbf{F}_{xx}\mathbf{x}\mathbf{x}' &= \textstyle\sum_{i,j=1}^{n} \frac{\partial \mathbf{F}}{\partial x_i \partial x_j} x_i x_j', \\ \mathbf{F}_{xp}\mathbf{x}\mathbf{p} &= \textstyle\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial \mathbf{F}}{\partial x_i \partial p_j} x_i p_j, \ \dots \end{aligned} \tag{9}$$

taken at $(\mathbf{x}, \mathbf{p}) = (0, 0)$; similar notation can be used for higher order derivatives. Similarly, substituting (7) into the left-hand side of (1) and using (3), we obtain

$$\begin{aligned} &\dot{\mathbf{x}} = \tfrac{\partial \mathbf{x}}{\partial a}\dot{a} + \tfrac{\partial \mathbf{x}}{\partial \dot{a}} f(a, \dot{a}, \mathbf{p}) \\ &= \varepsilon^{3/2}\tfrac{\partial \mathbf{x}_1}{\partial a}\dot{a} + \varepsilon^2 \left(\tfrac{\partial \mathbf{x}_{3/2}}{\partial a}\dot{a} + \tfrac{\partial \mathbf{x}_{3/2}}{\partial \dot{a}} f_2\right) \\ &+ \varepsilon^{5/2}\left(\tfrac{\partial \mathbf{x}_2}{\partial a}\dot{a} + \tfrac{\partial \mathbf{x}_2}{\partial \dot{a}} f_2 + \tfrac{\partial \mathbf{x}_{3/2}}{\partial \dot{a}} f_{5/2}\right) \\ &+ \varepsilon^3 \left(\tfrac{\partial \mathbf{x}_{5/2}}{\partial a}\dot{a} + \tfrac{\partial \mathbf{x}_{5/2}}{\partial \dot{a}} f_2 + \tfrac{\partial \mathbf{x}_2}{\partial \dot{a}} f_{5/2} + \tfrac{\partial \mathbf{x}_{3/2}}{\partial \dot{a}} f_3\right) + \cdots \end{aligned} \tag{10}$$

Let $\mathbf{u}_1, \mathbf{u}_2$ be the real generalized eigenvectors (eigenvector and associated vector) corresponding to zero eigenvalue and satisfying the equations

$$\mathbf{F}_x \mathbf{u}_1 = 0, \quad \mathbf{F}_x \mathbf{u}_2 = \mathbf{u}_1. \tag{11}$$

The left generalized eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (which are row-vectors) are given by

$$\mathbf{v}_1 \mathbf{F}_x = 0, \quad \mathbf{v}_2 \mathbf{F}_x = \mathbf{v}_1. \tag{12}$$

The vectors can be normalized such that

$$\mathbf{v}_1 \mathbf{u}_1 = \mathbf{v}_2 \mathbf{u}_2 = 0, \quad \mathbf{v}_1 \mathbf{u}_2 = \mathbf{v}_2 \mathbf{u}_1 = 1. \tag{13}$$

Let us introduce the nonsingular matrix $\mathbf{G} = (\mathbf{F}_x + \mathbf{u}_2\mathbf{v}_2)^{-1} - \mathbf{u}_1\mathbf{v}_1$, which gives a particular solution $\mathbf{G}\mathbf{y}$ to the equation $\mathbf{F}_x\mathbf{y} = 0$, assuming that the solution exists. It is easy to see that

$$\mathbf{G}\mathbf{u}_1 = \mathbf{u}_2, \ \mathbf{G}\mathbf{u}_2 = 0, \ \mathbf{v}_1\mathbf{G} = \mathbf{v}_2, \ \mathbf{v}_2\mathbf{G} = 0. \tag{14}$$

According to the central manifold theorem, we can take

$$\frac{\partial \mathbf{x}_1}{\partial a} = \mathbf{u}_1, \quad \frac{\partial \mathbf{x}_{3/2}}{\partial \dot{a}} = \mathbf{u}_2. \tag{15}$$

This implies that the central manifold is tangent to the central manifold for the linearized system.

Now we can substitute (8), (10) and (15) into (1) and compare the terms of equal order in ε. For the order ε we obtain:

$$\varepsilon: \quad 0 = \mathbf{F}_x \mathbf{x}_1. \tag{16}$$

With the use of (11) and (15), we find

$$\mathbf{x}_1 = a\mathbf{u}_1. \tag{17}$$

For $\varepsilon^{3/2}$, we have

$$\varepsilon^{3/2}: \quad \dot{a}\mathbf{u}_1 = \mathbf{F}_x \mathbf{x}_{3/2}. \tag{18}$$

Using (11) and (15), we find

$$\mathbf{x}_{3/2} = \dot{a}\mathbf{u}_2. \tag{19}$$

Terms of order ε^2 yield

$$\varepsilon^2: \quad f_2\mathbf{u}_2 = \mathbf{F}_x\mathbf{x}_2 + \frac{1}{2}\mathbf{F}_{xx}\mathbf{x}_1^2 + \mathbf{F}_{xp}\mathbf{x}_1\mathbf{p}. \tag{20}$$

Multiplying this equation by $\mathbf{v}_1$ and using (12), (13) and (17), we find

$$f_2 = \frac{1}{2}\mathbf{v}_1\mathbf{F}_{xx}\mathbf{u}_1^2 a^2 + \mathbf{v}_1\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p}a. \tag{21}$$

Under this condition, (20) can be solved with respect to $\mathbf{x}_2$ as

$$\mathbf{x}_2 = -\frac{1}{2}\mathbf{G}\mathbf{F}_{xx}\mathbf{u}_1^2 a^2 - \mathbf{G}\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p}a, \tag{22}$$

where we used (14).

Similarly, solving equation for the terms of order $\varepsilon^{5/3}$, we find

$$\begin{aligned} f_{5/2} &= (\mathbf{v}_1\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_2 + \mathbf{v}_2\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_1)a\dot{a} \\ &\quad + (\mathbf{v}_1\mathbf{F}_{xp}\mathbf{u}_2\mathbf{p} + \mathbf{v}_2\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p})\dot{a}, \\ \mathbf{x}_{5/2} &= -(\mathbf{G}\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_2 + \mathbf{G}^2\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_1)a\dot{a} \\ &\quad - (\mathbf{G}\mathbf{F}_{xp}\mathbf{u}_2\mathbf{p} + \mathbf{G}^2\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p})\dot{a}. \end{aligned} \tag{23}$$

Equations for higher orders $\varepsilon^{j/2}$ are solved similarly. At each step, we obtain the expression for $f_{j/2}$ and $\mathbf{x}_{j/2}$. In order to obtain approximate reduced equation, one must retain all the terms up to a certain order in ε. Note that the final formulae of this section are essentially the same as the reduction formulae found by the multiple time scale method [Luongo et al., 1999].

Note that $\mathbf{x}_{j/2}$ are particular solutions of equations (18), (20) etc. The general solution is obtained by adding the eigenvector $\mathbf{u}_1$ with an arbitrary constant factor. This means that we can add $c_{3/2}\dot{a}\mathbf{u}_1$ to $\mathbf{x}_{3/2}$, $(c_2'a^2 + \mathbf{c}''a\mathbf{p})\mathbf{u}_1$ to $\mathbf{x}_2$ with arbitrary constants $c_{3/2}$, c_2', $\mathbf{c}_2''$ (and similarly for terms of higher orders). These constants can be chosen in such a way that some of the terms in the reduced equation (3) vanish. One can show that this method gives the minimal number of nonzero terms prescribed by the general normal form theory [Guckenheimer and Holmes, 1983]. However, in the case of multiple zero eigenvalue, this procedure does not provide a significant simplification of reduced equation.

3 Multiple zero eigenvalue

There are k generalized eigenvectors corresponding to zero eigenvalue and satisfying

$$\mathbf{F}_x\mathbf{u}_1 = 0,\ \mathbf{F}_x\mathbf{u}_2 = \mathbf{u}_1,\ \ldots,\ \mathbf{F}_x\mathbf{u}_k = \mathbf{u}_{k-1}. \tag{24}$$

The left generalized eigenvectors are given by

$$\mathbf{v}_1\mathbf{F}_x = 0,\ \mathbf{v}_2\mathbf{F}_x = \mathbf{v}_1,\ \ldots,\ \mathbf{v}_k\mathbf{F}_x = \mathbf{v}_{k-1}. \tag{25}$$

The vectors can be normalized such that

$$\mathbf{v}_i\mathbf{u}_j = \begin{cases} 1,\ i = k-j+1, \\ 0,\ i \neq k-j+1 \end{cases}. \tag{26}$$

The nonsingular matrix $\mathbf{G} = (\mathbf{F}_x + \mathbf{u}_k\mathbf{v}_k)^{-1} - \mathbf{u}_1\mathbf{v}_1$ gives a particular solution $\mathbf{G}\mathbf{y}$ to the equation $\mathbf{F}_x\mathbf{y} = 0$, assuming that the solution exists. One can show that

$$\begin{aligned} &\mathbf{G}\mathbf{u}_1 = \mathbf{u}_2,\ \ldots,\ \mathbf{G}\mathbf{u}_{k-1} = \mathbf{u}_k,\ \mathbf{G}\mathbf{u}_k = 0, \\ &\mathbf{v}_1\mathbf{G} = \mathbf{v}_2,\ \ldots,\ \mathbf{v}_{k-1}\mathbf{G} = \mathbf{v}_k,\ \mathbf{v}_k\mathbf{G} = 0. \end{aligned} \tag{27}$$

The procedure for finding the functions $f(a, \dot{a}, \ldots, a^{(k-1)}\mathbf{p})$ $\mathbf{x}(a, \dot{a}, \ldots, a^{(k-1)}\mathbf{p})$ describing the reduced system (3) and the central manifold is absolutely analogous to the case $k = 2$.

For the triple zero eigenvalue, we have

$$\begin{aligned} \mathbf{x}(a, \dot{a}, \ddot{a}, \mathbf{p}) &= \varepsilon\mathbf{x}_1 + \varepsilon^{4/3}\mathbf{x}_{4/3} + \varepsilon^{5/3}\mathbf{x}_{5/3} + \cdots, \\ f(a, \dot{a}, \ddot{a}, \mathbf{p}) &= \varepsilon^2 f_2 + \varepsilon^{7/3} f_{7/3} + \varepsilon^{8/3} f_{8/3} + \cdots \end{aligned} \tag{28}$$

and

$$\begin{aligned} &\mathbf{x}_1 = a\mathbf{u}_1,\ \ \mathbf{x}_{4/3} = \dot{a}\mathbf{u}_2,\ \ \mathbf{x}_{5/3} = \ddot{a}\mathbf{u}_3, \\ &f_2 = \tfrac{1}{2}\mathbf{v}_1\mathbf{F}_{xx}\mathbf{u}_1^2 a^2 + \mathbf{v}_1\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p}a, \\ &\mathbf{x}_2 = -\tfrac{1}{2}\mathbf{G}\mathbf{F}_{xx}\mathbf{u}_1^2 a^2 - \mathbf{G}\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p}a, \\ &f_{7/3} = (\mathbf{v}_1\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_2 + \mathbf{v}_2\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_1)a\dot{a} \\ &\qquad + (\mathbf{v}_1\mathbf{F}_{xp}\mathbf{u}_2\mathbf{p} + \mathbf{v}_2\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p})\dot{a}, \\ &\mathbf{x}_{7/3} = -(\mathbf{G}\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_2 + \mathbf{G}^2\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_1)a\dot{a} \\ &\qquad - (\mathbf{G}\mathbf{F}_{xp}\mathbf{u}_2\mathbf{p} + \mathbf{G}^2\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p})\dot{a}. \\ &f_{8/3} = \\ &\quad (\mathbf{v}_1\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_3 + \mathbf{v}_2\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_2 + \mathbf{v}_3\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_1)a\ddot{a} \\ &\quad + (\mathbf{v}_1\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_2 + \mathbf{v}_2\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_1)\dot{a}^2 \\ &\quad + (\mathbf{v}_1\mathbf{F}_{xp}\mathbf{u}_3\mathbf{p} + \mathbf{v}_2\mathbf{F}_{xp}\mathbf{u}_2\mathbf{p} + \mathbf{v}_3\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p})\ddot{a}, \\ &\mathbf{x}_{8/3} = \\ &\quad -(\mathbf{G}\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_3 + \mathbf{G}^2\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_2 + \mathbf{G}^3\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_1)a\ddot{a} \\ &\quad -(\mathbf{G}\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_2 + \mathbf{G}^2\mathbf{F}_{xx}\mathbf{u}_1\mathbf{u}_1)\dot{a}^2 \\ &\quad -(\mathbf{G}\mathbf{F}_{xp}\mathbf{u}_3\mathbf{p} + \mathbf{G}^2\mathbf{F}_{xp}\mathbf{u}_2\mathbf{p} + \mathbf{G}^3\mathbf{F}_{xp}\mathbf{u}_1\mathbf{p})\ddot{a}. \end{aligned} \tag{29}$$

One can see that the structure of terms in (29) is very similar to that in the perturbation formulae for a multiple zero eigenvalue of a matrix dependent on parameters [Seyranian and Mailybaev, 2004].

For zero eigenvalue of multiplicity k, the terms up to order ε^3 are

$$
\begin{aligned}
\mathbf{x}_{1+j/k} &= a^{(j)}\mathbf{u}_{j+1},\\
f_{2+j/k} &= \sum_{a=1}^{j}\sum_{b=1}^{a}\mathbf{v}_b\mathbf{F}_{xx}\mathbf{u}_a\mathbf{u}_{a-b+1}a^{(a)}a^{(b)}\\
&\quad+\sum_{a=1}^{j}\mathbf{v}_a\mathbf{F}_{xp}\mathbf{u}_{j-a+1}\mathbf{p}a^{(a)},\\
\mathbf{x}_{2+j/k} &= \sum_{a=1}^{j}\sum_{b=1}^{a}\mathbf{v}_b\mathbf{F}_{xx}\mathbf{u}_a\mathbf{u}_{a-b+1}a^{(a)}a^{(b)}\\
&\quad+\sum_{a=1}^{j}\mathbf{v}_a\mathbf{F}_{xp}\mathbf{u}_{j-a+1}\mathbf{p}a^{(a)},\\
&\quad j=0,\dots,k-1;
\end{aligned}
\tag{30}
$$

Higher order terms depend on third and higher order derivatives of the function $\mathbf{F}$.

4 Triple zero in vibrations of double pendulum with follower force

Let us consider a double pendulum with two massless rods of length l carrying point masses $2m$ and m, and loaded by a follower force F at the end, Fig. 1. Visco-elastic joints produce the linear force $-C\dot{\phi} - K\phi$, where ϕ is the deflection angle at the joint ($C > 0$, $K \geq 0$). Let ϕ_1 and ϕ_2 be the angles between the rods and vertical axis. Dimensionless equations of motion of the double pendulum are

$$
\begin{aligned}
&3\ddot{\phi}_1 + \cos(\phi_2-\phi_1)\ddot{\phi}_2\\
&-\sin(\phi_2-\phi_1)\dot{\phi}_2^2 + 2\dot{\phi}_1 - \dot{\phi}_2\\
&+k(2\phi_1-\phi_2) + f\sin(\phi_2-\phi_1) = 0,
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\ddot{\phi}_2 + \cos(\phi_2-\phi_1)\ddot{\phi}_1\\
&+\sin(\phi_2-\phi_1)\dot{\phi}_1^2 - \dot{\phi}_1\\
&+\dot{\phi}_2 + k(\phi_2-\phi_1) = 0,
\end{aligned}
\tag{32}
$$

where dimensionless parameters are

$$
k = \frac{K\tau^2}{ml^2},\;\; f = \frac{F\tau^2}{ml}, \tag{33}
$$

and the derivative is taken with respect to the dimensionless time $t^* = t/\tau$ with the time scale $\tau = ml^2/C$. The system (31), (32) can be transformed to the form (1) with $\mathbf{x} = (x_1, x_2, x_3, x_4)^T = (\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2)^T$

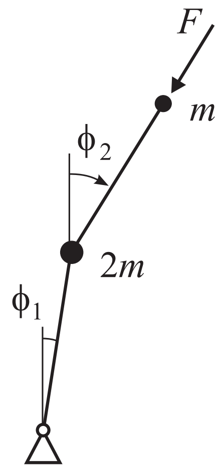

Figure 1. Double pendulum.

and $\mathbf{p} = (k, f)$, where $\mathbf{F}(\mathbf{q},\mathbf{p}) = (F_1, F_2, F_3, F_4)^T$:

$$
\begin{aligned}
&F_1 = x_2,\\
&F_2 = (f-3k)x_1/2 - 3x_2/2 + (2k-f)x_3/2 + x_4\\
&\quad + (k-f/3)x_1^3 + (f-11k/4)x_1^2x_3 - 3x_1^2x_4/4\\
&\quad + x_1^2x_2 - x_1x_2^2/2 - 2x_1x_2x_3 + (5k/2-f)x_1x_3^2\\
&\quad + 3x_1x_3x_4/2 - x_1x_4^2/2 + x_2^2x_3/2 + x_2x_3^2\\
&\quad + (f/3-3k/4)x_3^3 - 3x_3^2x_4/4 + x_3x_4^2/2\\
&\quad + O(\|\mathbf{x}\|^5),\\
&F_3 = x_4,\\
&F_4 = (5k-f)x_1/2 + 5x_2/2 + (f-4k)x_3/2 - 2x_4\\
&\quad + (7f/12 - 7k/4)x_1^3 + (19k-7f)x_1^2x_3/4\\
&\quad + 5x_1^2x_4/4 - 7x_1^2x_2/4 + 3x_1x_2^2/2 + 7x_1x_2x_3/2\\
&\quad + (7f-17k)x_1x_3^2/4 - 5x_1x_3x_4/2 + x_1x_4^2/2\\
&\quad - 3x_2^2x_3/2 - 7x_2x_3^2/4 + (5k/4-7f/12)x_3^3\\
&\quad + 5x_3^2x_4/4 - x_3x_4^2/2 + O(\|\mathbf{x}\|^5).
\end{aligned}
\tag{34}
$$

First, consider the problem with the parameters $\mathbf{p}_0 = (k, f) = (0, 1/2)$ linearized near the trivial equilibrium $\mathbf{q} = 0$. The Jacobian matrix becomes

$$
\mathbf{F}_x = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1/4 & -3/2 & -1/4 & 1\\ 0 & 0 & 0 & 1\\ -1/4 & 5/2 & 1/4 & -2 \end{pmatrix}. \tag{35}
$$

This matrix possesses a triple zero eigenvalue with the right generalized eigenvectors

$$
[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 1 & -2 & -12\\ 0 & 1 & -2 \end{pmatrix}, \tag{36}
$$

and left generalized eigenvectors

$$\begin{bmatrix} \mathbf{v}_3 \\ \mathbf{v}_2 \\ \mathbf{v}_1 \end{bmatrix} = \frac{1}{686} \begin{pmatrix} 676 & 80 & 10 & -60 \\ 35 & 406 & -35 & 210 \\ 49 & -49 & -49 & -49 \end{pmatrix}. \quad (37)$$

Product of the matrices (37) and (36) gives the identity matrix, which means that conditions (26) are satisfied.

The method described above gives the reduced equation

$$\begin{aligned} a^{(3)} &= \beta_0 a + \beta_1 \dot{a} + \beta_2 \ddot{a} \\ &- \tfrac{1}{7}\dot{a}^3 - \tfrac{57}{49}\dot{a}^2\ddot{a} - \tfrac{1732}{343}\dot{a}\ddot{a}^2 - \tfrac{1}{7}ka\dot{a}^2 + \cdots, \end{aligned} \quad (38)$$

where $g = f - 1/2$ and

$$\begin{aligned} \beta_0 &= -\tfrac{1}{7}k^2 - \tfrac{90}{2401}k^3 + \tfrac{4}{343}k^2 g, \\ \beta_1 &= -\tfrac{2}{7}k - \tfrac{82}{2401}k^2 + \tfrac{8}{343}kg + \tfrac{14104}{823543}k^3 \\ &\quad + \tfrac{2032}{117649}k^2 g - \tfrac{64}{16807}kg^2, \\ \beta_2 &= -\tfrac{45}{49}k + \tfrac{2}{7}g - \tfrac{3886}{16807}k^2 + \tfrac{344}{2401}kg - \tfrac{8}{343}g^2 \\ &- \tfrac{95896}{823543}k^3 + \tfrac{88944}{823543}k^2 g - \tfrac{4064}{117649}kg^2 + \tfrac{64}{16807}g^3. \end{aligned} \quad (39)$$

In (38), (39) all terms of order lower than ε^5 are given. The central manifold parameterized by a, $\dot{a}$, $\ddot{a}$ is given up to ε^3 terms by

$$\begin{aligned} x_1 &= (1 + \tfrac{10}{343}k)a + (\tfrac{380}{2401}k - \tfrac{20}{343}g)\dot{a} \\ &\quad + (\tfrac{16040}{16807}k - \tfrac{800}{2401}g)\ddot{a} + \cdots, \\ x_2 &= (1 + \tfrac{10}{343}k)\dot{a} + (\tfrac{380}{2401}k - \tfrac{20}{343}g)\ddot{a} + \cdots, \\ x_3 &= (1 - \tfrac{676}{343}k)a + (-2 + \tfrac{1352}{343}g - \tfrac{5248}{2401}k)\dot{a} \\ &\quad + (-12 + \tfrac{62592}{2401}g - \tfrac{214680}{16807}k)\ddot{a} + \cdots, \\ x_4 &= (1 + \tfrac{500}{343}k)\dot{a} + (-2 + \tfrac{21212}{2401}k + \tfrac{176}{343}g)\ddot{a} + \cdots. \end{aligned} \quad (40)$$

Note that the linear part of equation (38) automatically gives the miniversal deformation of the linearized equations (31), (32) [Arnold, 1983; Mailybaev, 2001]. Using the Routh-Hurwitz stability criterion, we find the following conditions for stability of the trivial equilibrium $\phi_1 = \phi_2 = 0$:

$$\beta_{0,1,2} < 0, \quad \beta_0 + \beta_1\beta_2 > 0. \quad (41)$$

Fig. 2 shows stability domain in the parameter space (k, f) given by formulae (41) with asymptotic relations (39).

The normal form equation gives no nontrivial equilibria $a \equiv const \neq 0$ near the bifurcation point, at least up to the terms taken into account. This is supported by the stability diagram, where the stability boundary corresponds to two purely imaginary eigenvalues (Hopf bifurcation). Small amplitude periodic solutions may exist.

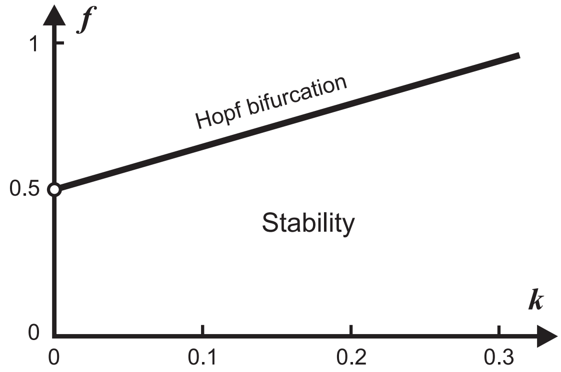

Figure 2. Stability domain of the trivial equilibrium.

5 Conclusion

In this paper, we presented a new method for finding normal form equation and invariant manifold in the case of multiple zero eigenvalue with a single Jordan block. The method utilizes the concept of fractional scale. This allows using a single scale parameter in the normal form reduction for systems with multiple variables and parameters. The use of fractional scales substantially simplifies the procedure of system reduction. The presented approach establishes the relation between the normal form theory and the multiple time-scale methods in nonlinear equations, as well as the methods of perturbation theory in linear algebra.

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