ON PERMANENT ROTATIONS OF A STRING-DRIVEN RIGID BODY.

G.G. Besedin
Department of Mechanics and Mathematics
Lomonosov Moscow State University
Russia
genqua@mail.ru

T.S. Sumin
Institute of Mechanics
Lomonosov Moscow State University
Russia
sumin@imec.msu.ru

Abstract

Considered two problems of motion of a rigid body suspended on the rod. In the first part of report the body with an ellipsoidal cavity full-filled by viscous liquid is considered. By using Routh's method there were obtained steady motions of this mechanical system. It was proved that all steady motions were permanent rotations of the system as a whole about fixed upright. The stability and branching conditions of steady motions were also obtained.

In the second part it is considered the body suspended on the rod and immersed into viscous liquid. Basing on experimental and theoretical data there was constructed the mathematical model of interaction between rigid body and resistant media. Some properties of this dynamical system were pointed out.

Keywords:

Body with viscous liquid, body suspended on the rod, resistant media, steady motions, stability, bifurcation.

1 The body with viscous filling

1.1 Motion equations

Consider the problem on motion of the dynamically symmetrical body which has cavity full-filled with liquid. The body is suspended on the rigid inextensible rod O_1O, where O_1 is a fixed point, and O is the suspension point of the body. Ends of the rod are connected to the fixed point O_1 and the suspension point O by perfect hinges. Point O belongs to the dynamical symmetry axis of the body. The cavity inside body-shell has a shape of an ellipsoid of revolution. It is assumed that the axis of symmetry of the cavity coincides with the axis of dynamical symmetry of the body. The filling of the cavity is viscous. So walls of cavity and liquid interact. To describe this interaction the Samsonov's phenomenological model of internal friction is used [Samsonov, Dosaev, 1997].

Let C be the centre of mass of the system "body-liquid"; define movable axes $Cx_1x_2x_3$, which are connected with shell and directed along its principal axes of inertia. The unit orts of these axes are $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, where $\mathbf{i}_3$ defines the axis of dynamical symmetry of the body. Now, let m be mass of the whole system, $\mathbf{v}$ — vector of absolute velocity of point C, $\boldsymbol{\omega}$ — absolute angular velocity of the shell, $\boldsymbol{\Omega}$ — average vorticity vector of cavity filling. The system is in uniform field of the gravity force $-mg\boldsymbol{\gamma}$, where $\boldsymbol{\gamma}$ is the unit vector of ascending upright. Let $-N\mathbf{e}$ denote the rod tension force, here $\mathbf{e} = \mathbf{O_1O} / |\mathbf{O_1O}|$ is the unit vector, directed along the rod, N is the projection of tension force on the vector $-\mathbf{e}$; $l = |\mathbf{O_1O}|$ — length of the rod, $a = |\mathbf{CO}|$ — distance between centre of mass of the system and suspension point.

With respect to the movable axes the motion equations are of the following form:

$$\begin{aligned} & m\dot{\mathbf{v}} + [\boldsymbol{\omega}, m\mathbf{v}] = -mg\boldsymbol{\gamma} - N\mathbf{e}, \\ & J_*\dot{\boldsymbol{\omega}} + J'\dot{\boldsymbol{\Omega}} + [\boldsymbol{\omega}, J_*\boldsymbol{\omega} + J'\boldsymbol{\Omega}] = [a\mathbf{i}_3, -N\mathbf{e}], \\ & \dot{\boldsymbol{\gamma}} + [\boldsymbol{\omega}, \boldsymbol{\gamma}] = 0, \\ & l\dot{\mathbf{e}} + [\boldsymbol{\omega}, l\mathbf{e} - a\mathbf{i}_3] = \mathbf{v}, \\ & J'\dot{\boldsymbol{\Omega}} + [\boldsymbol{\omega} - \boldsymbol{\Omega}, L\boldsymbol{\Omega}] = D(\boldsymbol{\omega} - \boldsymbol{\Omega}), \\ & E = (\mathbf{e}, \mathbf{e}) = 1. \end{aligned} \tag{1}$$

Here $J_* = \mathrm{diag}\{A_*, A_*, C_*\}$ is a sum of tensors of inertia of shell and equivalent body, $J' = \mathrm{diag}\{A', A', C'\}$ is a difference of tensors of inertia of liquid and of the equivalent body (see [Moiseev, Rumyantsev, 1965]). D is a diagonal positively definite tensor, characterizing friction between viscous filling and walls of cavity. In case of ideal liquid, i.e. when $D = 0$, the fifth equation of the system (1) coincides with well-known Helmholtz's equation for the vortex vector when liquid is in uniform vortex motion in ellipsoidal cavity. L is a diagonal subsidiary tensor.

The first equation of the system (1) expresses the theorem of centre of mass varying, the second one expresses the theorem of the system moment of momentum varying, the third one is the Poisson's equation, the fourth one is the kinematical condition connecting velocities of suspension point and centre of mass of the body, the fifth one describes varying of the average vorticity vector of liquid (compare with [Samsonov, Dosaev, 1997]), and the sixth equation is the condition of inextensibility of the rod.

1.2 General properties of steady motions.

It can be shown, that the system (1) admits area intergral K. At that

$$\dot{H} = (D(\boldsymbol{\omega}-\boldsymbol{\Omega}), \boldsymbol{\omega}-\boldsymbol{\Omega}) \le 0 ,$$

i.e. full mechanical energy H is non-increasing function. Hence, we can apply the modified Routh's theory for dissipative systems [Karapetyan, 1998]. The effective potential of the system (1) equals:

$$W_k(\boldsymbol{\gamma},\mathbf{e}) = \min_{\mathbf{v},\boldsymbol{\omega},\boldsymbol{\Omega}}\Big|_{K=k} H = \frac{1}{2}k^2 J^{-1} + \Pi,$$

where k is a constant of area integral, $J(\boldsymbol{\gamma},\mathbf{e}) = ((J_* + J')\boldsymbol{\gamma},\boldsymbol{\gamma}) + m[\boldsymbol{\gamma}, l\mathbf{e} - a\mathbf{i}_3]^2$, $\Pi = mg(l\mathbf{e} - a\mathbf{i}_3, \boldsymbol{\gamma})$ is potential energy of the system. Minimum of the energy function at the area integral level is achieved when $\boldsymbol{\omega} = kJ^{-1}\boldsymbol{\gamma}$, $\boldsymbol{\Omega} = kJ^{-1}\boldsymbol{\gamma}$, $\mathbf{v} = kJ^{-1}[\boldsymbol{\gamma}, l\mathbf{e} - a\mathbf{i}_3]$. So according to the modified Routh's theory, the critical points $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0, \mathbf{e} = \mathbf{e}_0$ of effective potential W_k are corresponded to the steady motions of the system (1)

$$\boldsymbol{\omega} = \frac{k}{J(\boldsymbol{\gamma}_0,\mathbf{e}_0)}\boldsymbol{\gamma}_0, \boldsymbol{\Omega} = \frac{k}{J(\boldsymbol{\gamma}_0,\mathbf{e}_0)}\boldsymbol{\gamma}_0, \quad \mathbf{v} = \frac{k}{J(\boldsymbol{\gamma}_0,\mathbf{e}_0)}[\boldsymbol{\gamma}_0, l\mathbf{e}_0 - a\mathbf{i}_3], \tag{2}$$

Moreover, minimum points are corresponded to stable steady motions. Obviously, the steady motions (2) are permanent rotations of the system as a whole about fixed upright. Hence, steady motions such as regular precessions do not exist in this problem.

It can be shown that for any steady motion (2) vectors $\boldsymbol{\gamma}, \mathbf{e}, \mathbf{i}_3$ belong to the same vertical plane. So it is more comfortable to investigate permanent rotations in the coordinate system $O_1y_1y_2y_3$ which revolves about fixed upright with the angular speed $\eta = kJ^{-1}$. Let $e_{y_j}, i_{1j}, i_{2j}, i_{3j}$ denote the projections of vectors $\mathbf{e}, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ on the axis O_1y_j ($j = 1,2,3$). There are geometric conditions between these coordinates:

$$\pi_{\mathbf{e}_y} = e_{y_1}^2 + e_{y_2}^2 + e_{y_3}^2 - 1 = 0, \quad \pi_{\mathbf{i}_r} = i_{r1}^2 + i_{r2}^2 + i_{r3}^2 - 1 = 0, \quad \pi_{\mathbf{i}_s\mathbf{i}_t} = i_{s1}i_{t1} + i_{s2}i_{t2} + i_{s3}i_{t3} = 0 \ (s \ne t) \tag{3}$$

Magnitudes J and Π are the following in new coordinates:

$$J = A(i_{13}^2 + i_{23}^2) + Ci_{33}^2 + m[(le_{y_1} - ai_{31})^2 + (le_{y_2} - ai_{32})^2], \quad \Pi = mg(le_{y_3} - ai_{33}) \tag{4}$$

To obtain formulae for steady motions in new coordinates it is necessary to find critical points of effective potential (4) taking into account conditions (3). As far as set of the critical points of effective potential W_k coincides with set of the critical points of varied potential energy W_η in case when the system is in relative equilibrium [Moiseev, Rumyantsev, 1965], so we can assume rotational frequency to be constant, $\eta = \text{const}$. Although for determining the character of the critical points (and consequently the stability of steady motions) in general case we cannot assume rotational frequency to be constant. Let us introduce the function

$$W_* = W_\eta + \frac{1}{2}\eta^2(\lambda_{\mathbf{e}_y}\pi_{\mathbf{e}_y} + \sum_{j=1}^{3}\lambda_{\mathbf{i}_j}\pi_{\mathbf{i}_j} + \sum_{s,t=1,\, s\ne t}^{3}\lambda_{\mathbf{i}_s\mathbf{i}_t}\pi_{\mathbf{i}_s\mathbf{i}_t}),$$

where $\lambda_{\mathbf{e}_y}, \lambda_{\mathbf{i}_j}, \lambda_{\mathbf{i}_s\mathbf{i}_t}$ are indefinite multipliers of Lagrange. Then the steady motions could be obtained from the stationary conditions of function W_*.

$$\delta W_* = 0 \tag{5}$$

Equations (5) admit two kinds of steady motions: trivial and non-trivial. The first type includes steady motions on which the rod and the axis of symmetry of the shell coincide with fixed upright:

$$i_{33} = \pm 1, e_{y_3} = \pm 1, i_{11} = i_{22} = 1, k = C\eta, \quad i_{12} = i_{13} = i_{21} = i_{23} = i_{31} = i_{32} = e_{y_1} = e_{y_2} = 0 \tag{6}$$

From formulae (6) one can conclude, that there exist 4 types of trivial permanent rotations depending on the fixed point position with respect to suspension point position (lower or higher), and also on suspension point position with respect to the centre of mass position (lower or higher). These steady motions always exist, This means that they exist at any level of area integral set by angular speed η in this case.

Non-trivial steady motions are permanent rotations on which the rod and the axis of symmetry of the body deflect from the upright to some angles α and ϑ correspondingly:

$$i_{11}=1,\ i_{12}=i_{13}=i_{21}=i_{31}=e_{y_1}=0,\ i_{23}=-i_{32},$$
$$i_{22}=i_{33},\ e_{y_2}^2+e_{y_3}^2=1,\ i_{32}^2+i_{33}^2=1,\ e_{y_3}=\pm 1,$$
$$i_{33}=\frac{q(1+e_{y_3}w)}{w(p(1+e_{y_3}w)+1)},\ e_{y2}=\frac{e_{y_3}wi_{32}}{q(1+e_{y_3}w)}, \qquad (7)$$
$$k^2=\left\{C+ma^2\left[p+\frac{1}{(1+e_{y_3}w)^2}\right](1-i_{33}^2)\right\}^2\frac{g}{l}w$$

Here $w=\eta^2 l/g$ is a non-dimensional angular velocity squared; $p=\dfrac{A-C}{ma^2}, q=l/a$ — non-dimensional parameters of the system, A, C — equatorial and axial moments of inertia of tensor J_*+J'.

1.3 Stability of steady motions.

The stability of trivial steady motions is determined by the sign of the second quadratic form $\delta^2 W_*$ at manifold $M=\{\delta\pi_{\mathbf{e}_y}=0, \delta\pi_{\mathbf{i}_j}=0, \delta\pi_{\mathbf{i}_s\mathbf{i}_t}=0\}$.

Consider trivial solution I, which corresponds to such permanent rotation when the suspension point is lower then the fixed point, and centre of mass is lower then the suspension point. In this case non-zero coordinates are $i_{33}=1, e_{y_3}=-1, i_{11}=i_{22}=1$. The stability conditions of the solution I are written in the following system of inequalities:

$$-w^2+\frac{g}{l}>0$$
$$w^4p-w^2(p+1+q)+q>0 \qquad (8)$$

The analysis shows that if $p>0$ (the case of "dynamically extended" body), the solution I is stable when $w^2<w_-^2$, is unstable with degree of instability be equal to 2 when $w_-^2<w^2<w_+^2$, and is unstable with degree of instability be equal to 4 when $w^2>w_+^2$. If $p<0$ (the case of "dynamically flattened" body) the solution I is stable when $w^2<w_-^2$ and is unstable with degree of instability be equal to 2 when $w^2>w_-^2$. Here $w_\pm^2$ are roots of the squared trinomial from the second inequality of the system (8).

We can perform analogous calculations for rest three trivial permanent rotations. For the solution II ($i_{33}=-1, e_{y_3}=-1$), i.e. for such permanent rotation on which point O is lower then point O_1, and point C is higher then point O, the stability conditions are more complicated: the solution II is stable when $w_+^2<w^2<w_-^2$ in parameter domains $\{(p,q): p<-(1+\sqrt{q})^2\}$ and $\{(p,q): -(1-\sqrt{q})^2<p<0, q>1\}$; in other domains the solution II is unstable with degree of instability be equal to 2 or 4. Notice that the steady motions on which the rod is up ($e_{y_3}=1$) are always unstable.

According to the bifurcation theory the non-trivial ("skewed") steady motions must emerge in the points of degree of instability interchange from the trivial steady motions. Apply the small parameter method for investigating stability of germs of the "skewed" steady motions. Let angle α between the rod and the upright be the small parameter. Then

$$\vartheta=x_\pm\alpha+o(\alpha),\quad k^2=k_\pm+y_\pm\alpha^2+o(\alpha^2),$$
$$w=w_\pm+z_\pm\alpha^2+o(\alpha^2) \qquad (9)$$

Here ϑ is an angle between the axis of symmetry of the body and the upright., $x_\pm, y_\pm, z_\pm$ are some constants depending on the parameters of the problem. Index $\pm$ determines the point of branching, in which neighborhood the stability analysis of the germs of "skewed" steady motion is performed.

Consider the point of branching $(-1,1,k_-^2)$ in the space (e_{y_3}, i_{33}, k^2). It exists when p and q take on all physically possible values, i.e. when $p\in R, q>0$. In this point the interchange of degree of instability of the solution I occurs. Substituting formulae (9) into relations (7), we obtain the expressions for x_-, y_-, z_-.

After that we can plot the germs of non-trivial steady motions in the neighborhood of the point of branching. We have

$$x_- = q(1 - w_-(p,q)) / w_-(p,q) .$$

This magnitude characterizes relative orientation of the rod and the axis of symmetry of the body at the branching steady motion. In this particular case, $x_- > 0$, as far as $w_-(p,q) < 1$; hence, the rod and the axis of symmetry are deflected to the same direction with respect to the upright. The magnitude z_- depending on its sign specifies branching direction of the "skewed" relative equilibriums of the system (1) in the space $(e_{y_3}, i_{33}, \eta^2)$. The calculations show that z_- also depends on two dimensionless parameters p and q. The magnitude y_- depending on its sign specifies branching direction of the "skewed" steady motions of the system (1) in the space (e_{y_3}, i_{33}, k^2). In other words, this means that in the neighborhood of the bifurcation point $(-1, 1, k_-^2)$ at the bifurcation Poincare-Chetaev's diagram in the space (e_{y_3}, i_{33}, k^2) one can plot the germ of the "skewed" steady motion and after that according to the bifurcation theory can deduce about the stability of branching motion.

The performed calculation of magnitude y_- showed, that it depends on three dimensionless parameters p, q and $r = \dfrac{C}{ma^2}$. However, y_- is positive when these parameters take on physically possible values. Hence, the "skewed" steady motion branching in this point according to the bifurcation theory [Karapetyan, 1998] is stable (in the neighborhood of the branching point at least).

We perform similar reasoning for all branching points of the trivial steady motions. In particular, in the point $(-1, 1, k_+^2)$, which exists only when $p > 0$, the degree of instability of the solution I changes. Substituting formulae (9) into relations (7), we obtain the expressions for x_+, y_+, z_+: $x_- = q(1 - w_+(p,q)) / w_+(p,q)$, $z_+ = z_+(p,q)$, $y_+ = y_+(p,q,r)$. As far as $w_+ > 1$ for the solution I, hence, the rod and the axis of symmetry of the body are deflected to the opposite directions with respect to the upright. The sign of y_+ in the domain of existence of the considered bifurcation point is positive. Hence, according to the bifurcation theory the branching non-trivial steady motion is unstable with degree of instability be equal to 2.

Consequently investigating the germs of non-trivial steady motions in the neighborhoods of bifurcation points, we can draw the atlas of typical Poincare-Chetaev's diagrams for the considered dynamical system.

2 The body is immersed into liquid.

Consider another problem on motion of a string-driven body. Let heavy axisymmetrical homogenous rigid body be suspended on the string to the fixed point O_1 (see Fig. 1). The other end of the string is mounted to the body in the point O_2, which belongs to the axis of its symmetry. Assume that the string is absolutely flexible, inextensible, weightless, always tense and it doesn't react to torsion. Then the string can be considered as geometrical constraint. $O_1O_2 = const = l$. The body is immersed into the vessel with liquid, represented by the cylinder with vertical axis of symmetry O_3O_1.

Define the unmovable coordinate system $\xi\eta\zeta$ with origin O_1. The axis $O\zeta$ is directed as an ascending upright (see Fig. 2). Put the origins of two other coordinate systems $\xi_1\eta_1\zeta_1$ and xyz in the centre of mass G. The axes of the first coordinate system are always parallel to the fixed system $\xi\eta\zeta$, the axes of the second one are rigidly bound with the body. At that the axis z of the system xyz is directed along the axis of dynamical symmetry of the body, so it is the principal central axis of inertia. The orientation of the body and the coordinate system xyz with respect to translational moving system $\xi_1\eta_1\zeta_1$ is determined by three Euler-Krylov's angles α, β and γ. Define the orientation of the string O_1O_2 with respect to the fixed system $\xi\eta\zeta$ by angles α_1 and β_1.

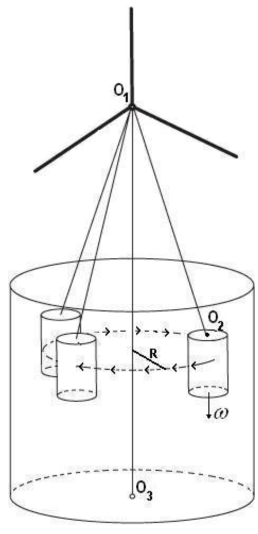

Fig. 1

Expressions for coordinates ξ_G, η_G, ζ_G of the centre of mass G with respect to the axes $\xi\eta\zeta$ are the following:

$$\xi_G = -l\sin\beta_1 - a\sin\beta,$$
$$\eta_G = l\sin\alpha_1\cos\beta_1 + a\sin\alpha\cos\beta,$$
$$\zeta_G = -l\cos\alpha_1\cos\beta_1 - a\cos\alpha\cos\beta.$$

It is known from experimental investigations that this mechanical system has the steady motion with centre of mass going round the circle of a constant radius R in a horizontal plane. For the quality description of interaction between the body and liquid (in case of steady motion represented at Fig. 1) it is necessary to construct adequate model of motion of liquid in the vessel. The presence of the steady solution of obtained equations will serve the criteria of adequacy of mathematical model.

On base of theoretical and experimental data for description of liquid motion in vessel there is assumed the mathematical model with following assumptions:

- liquid is ideal and incompressible,
- liquid performs vortex motion with circulation Γ (vortex fiber coincides with the axis of symmetry of the vessel).

In this case function of pressure in liquid is of the following form $p = -\dfrac{\rho\Gamma^2}{8\pi^2R^2} + const$, where ρ is density of media. Integrating pressure by the cylinder surface we obtain that the force of pressure at the body from the liquid has non-zero component F_r which is the projection of the force on the axis connecting interval O_3O_1 with centre of mass of the body: $F_r = -a\dfrac{\rho\Gamma^2}{4\pi}$, where a is the non-dimensional combination of geometric body parameters and radius R.

Express generalized forces corresponding to the generalized coordinates α, β, γ, α_1, β_1 and make substitution in motion equations of the body suspended on the string. We obtain the differential equations system describing the motion of the axisymmetrical body suspended on the string and immersed into resistant media that occupies limited volume, and velocity distribution will be the same as for a rectilinear vortex (compare with [Ishlinsky, Temchenko, Storozhenko, 1991]):

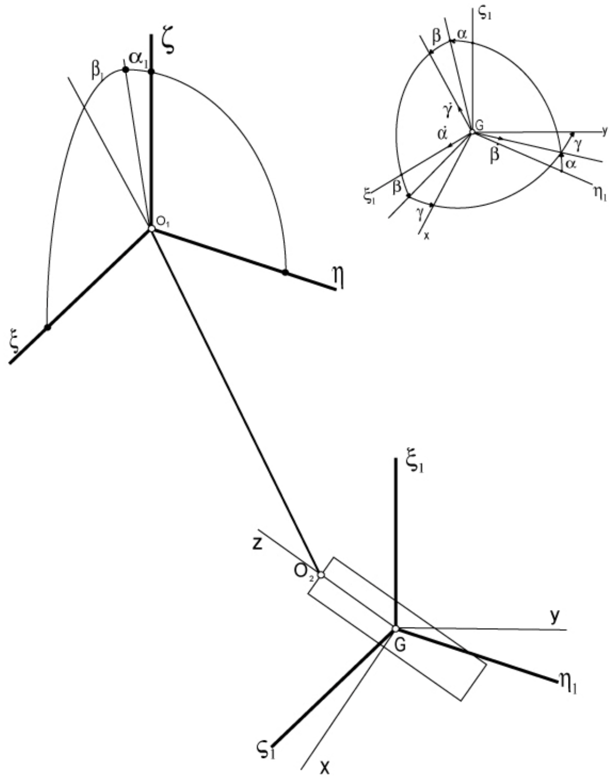

Fig. 2

$$A\cos\beta\left(\ddot{\alpha}\cos\beta - 2\dot{\alpha}\dot{\beta}\sin\beta\right) + C\left(\ddot{\alpha}\sin\beta + \dot{\alpha}\dot{\beta}\cos\beta + \ddot{\gamma}\right)\sin\beta + C\left(\dot{\alpha}\sin\beta + \dot{\gamma}\right)\dot{\beta}\cos\beta + ma\cos\beta\left(\ddot{\eta}_G\cos\alpha + \ddot{\zeta}_G\sin\alpha\right) = -mga\sin\alpha\cos\beta + F_r\frac{\eta_G}{d}a\cos\alpha\cos\beta,$$

$$A\left(\ddot{\beta} + \dot{\alpha}^2\sin\beta\cos\beta\right) - C\left(\dot{\alpha}\sin\beta + \dot{\gamma}\right)\dot{\alpha}\cos\beta + ma\left(-\ddot{\xi}_G\cos\beta - \sin\alpha\sin\beta\ddot{\eta}_G + \cos\alpha\sin\beta\ddot{\xi}_G\right) = -mgl\sin\alpha_1\cos\beta_1 + F_r\left(\frac{\xi_G}{d}(-\cos\alpha) + \frac{\eta_G}{d}(-a\sin\alpha\sin\beta\right),$$

$$\cos\beta_1\left(\ddot{\eta}_G\cos\alpha_1 + \ddot{\zeta}_G\sin\alpha_1\right) = -mgl\sin\alpha_1\cos\beta_1 + F_r\frac{\eta_G}{d}l\cos\alpha_1\cos\beta,$$

$$ml\left(-\ddot{\xi}_G\cos\beta_1 - \ddot{\eta}_G\sin\alpha_1\sin\beta_1 + \ddot{\zeta}_G\cos\alpha_1\sin\beta_1\right) = -mg\cos\alpha_1\sin\beta_1 + F_r\left(\frac{\xi_G}{d}(-l\cos\beta_1) + \frac{\eta_G}{d}(-l\sin\alpha_1\sin\beta_1)\right),$$

$$C\left(\dot{\alpha}\sin\beta + \dot{\gamma}\right)^{\cdot} = 0,$$

where A and C are the principal central moments of inertia of the body, $d = \sqrt{x^2 + y^2}$.

After adding generalized forces to the equations the trivial particular solution on which the body rotates uniformly round fixed upright does not satisfy motion equation system. Consequently, the presence of liquid in the vessel changes the behavior of the dynamical system. This fact is also confirmed by the experimental data.

Define by κ^2 the multiplication of $\dfrac{\rho}{4\pi m}$ and dimensionless combination of geometrical parameters of the body. Hence, it can be shown that the new steady motion appears in the system

$$\alpha = \beta = 0,$$

$$\dot{\gamma} = \omega = const,\ \eta_G = R\cos\Gamma\kappa t\ , \xi_G = R\sin\Gamma\kappa t\ ,$$

On this motion the axis of symmetry of the body is vertical, the centre of mass (η_G, ξ_G) goes round the circle of a constant radius R with the centre belonging to the vertical axis; the velocity distribution is the same as for a rectilinear vortex. Rotational frequency of the body round fixed upright equals $\Gamma\kappa$. This means there is a direct dependence between frequency and circulation Γ of vortex motion. Circulation is proportional to the angular velocity ω of revolution of the body which perturbs the liquid and, hence, creates vortex in the vessel, i.e. $\Gamma = \Gamma(\omega)$. The problem of determination Γ as a function of ω is of separate interest and must be considered under the theory of viscous liquid.

Consequently, controlling the angular velocity ω, we get the opportunity to control the body motion in resistant media.

The work is supported by RFBR (06-01-00079 and 05-08-01378).

References

Ishlinsky A.Yu., Temchenko M.E., Storozhenko V.A. Revolution of a rigid body on a string and adjoining problems. Moscow: Nauka, 1991, 330 p. (in russian)

Karapetyan A.V. Stability of steady motions. Moscow: Editorial URSS. 1998, 168 p. (in russian)

Moiseev N.N., Rumyantsev V.V. Dynamics of a body with cavities containing liquid. Moscow: Nauka, 1965, 439 p. (in russian)

Samsonov V.A., Dosaev M.Z. The model of motion of the top with viscous filling on a rough plane. Report 4485 of Institute of Mechanics of MSU. 1997. 31 p. (in russian)