About the problem of cycle-slipping in discrete system with periodic nonlinear vector function

Vera B. Smirnova Alexander I. Shepeljavyi Natalia V. Utina

***Abstract*— A multidimensional discrete phase control system with periodic vector nonlinearity is investigated. By means of Lyapunov direct method and Yakubovich–Kalman theorem certain estimates for the phase error are obtained. The results are formulated as frequency-domain criteria.**

I. Introduction

In this paper we consider a multidimensional discrete phase system with vector nonlinearity:

$$\begin{aligned} z(n+1) &= Az(n) + B\xi(n), \\ \sigma(n+1) &= \sigma(n) + C^* z(n) + R\xi(n), \\ \xi(n) &= \varphi(\sigma(n)), \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{1}$$

Here A, B, C, R are real matrices of order $(m \times m)$, $(m \times l)$, $(m \times l)$, $(l \times l)$ respectively and the symbol * is used for Hermitian conjugation. We suppose that the pair (A, B) is controllable, the pair (A, C) is observable and all eigenvalues of A lie inside the open unit circle. We suppose also that $\varphi : \mathbf{R}^l \to \mathbf{R}^l$ is a vector-valued function with the property $\varphi(\sigma) = (\varphi_1(\sigma_1), ..., \varphi_l(\sigma_l))$ for $\sigma = (\sigma_1, ..., \sigma_l) \in \mathbf{R}^l$. We assume that every component $\varphi_j(\sigma_j)$ is Δ_j-periodic, belongs to $\mathbf{C}^1$, has a finite number of simple zeros on $[0, \Delta_j)$. Let $\Delta = (\Delta_1, ..., \Delta_l)$.

In this paper the subject of cycle–slipping for discrete phase systems is developed. This subject has already been investigated in published works [1],[2],[3] for the case of scalar nonlinear periodic function $\varphi(\sigma)$ $(l = 1)$. These works contain a number of assertions which guarantee that (in a case of $l = 1$)

$$|\sigma(n) - \sigma(0)| < k\Delta, \quad \text{for all} \quad n = 1, 2, ...,$$

where k is a positive integer. In this paper the results of [1]-[3] are extended to the case of $l > 1$. All the theorem of this paper are obtained by means of Lyapunov direct method and Yakubovich–Kalman theorem [4]. All the results are formulated as frequency-domain criteria, i.e. in terms of the transfer function of the linear part of system (1)

$$K(p) = C^*(A - pE_m)^{-1}B - R \quad (p \in \mathbf{C}),$$

where E_m is an $(m \times m)$-unit matrix.

A.I. Shepeljavyi is with Dept. of Mathematics and Mechanics, St.Petersburg State University, 28, Universitetsky av., 198504 St.Petersburg, Petrodvoretz, Russia; email: as@as1020.spb.edu

V.B. Smirnova and N.V. Utina are with Dept. of Mathematics, St. Petersburg State University of Architecture and Civil Engineering, 4, Vtoraja Krasnoarmeiskaja Str., 198005 St.Petersburg, Russia; email: unv74@mail.ru

II. Frequency-domain estimates for the phase error of discrete system

Let us suppose that

$$\int_0^{\Delta_j} \varphi_j(\sigma)\, d\sigma < 0 \quad (j = 1, ..., l).$$

Let α_{1j}, α_{2j} be such numbers that

$$\alpha_{1j} \le \frac{d\varphi_j(\sigma)}{d\sigma} \le \alpha_{2j} \quad \text{for all} \quad \sigma \in \mathbf{R}, \tag{2}$$

where $\alpha_{1j} < 0 < \alpha_{2j}$.

Let us introduce several notations $(j = 1, 2, ..., l)$:

$$\Omega_j^{(1)} = \{\sigma_j \in [0, \Delta_j) : \varphi_j(\sigma_j) > 0\},$$

$$\Omega_j^{(2)} = \{\sigma_j \in [0, \Delta_j) : \varphi_j(\sigma_j) < 0\},$$

$$\Gamma_j = \int_{\Omega_j^{(2)}} |\varphi_j(\sigma)|\, d\sigma,$$

$$\gamma_j = \int_{\Omega_j^{(1)}} \varphi_j(\sigma)\, d\sigma, \quad R_j = \frac{2\Gamma_j \gamma_j}{\Gamma_j + \gamma_j},$$

$$\mu_j^{(1)}(æ, k, w) = \frac{\gamma_j - \Gamma_j - \frac{w + \sum_{i=1}^{l} |æ_j| R_j}{æ_j k}}{\gamma_j + \Gamma_j},$$

$$\mu_j^{(2)}(æ, k, w) = \frac{\gamma_j - \Gamma_j + \frac{w + \sum_{i=1}^{l} |æ_j| R_j}{æ_j k}}{\gamma_j + \Gamma_j},$$

where $æ = diag\{æ_1, ..., æ_l\}$ is a real diagonal $(l \times l)$-matrix, $w \in \mathbf{R}$ and k is a natural number. We shall also need the following quadratic forms of $z \in \mathbf{R}^m$ and $\xi \in \mathbf{R}^l$:

$$F(z, \xi) = \xi^* æ(C^*z + R\xi) + \xi^*\eta\xi + (C^*z + R\xi)\varepsilon(C^*z + R\xi).$$

$$\Phi(z, \xi) = (Az + B\xi)^* H(Az + B\xi) - z^*Hz + F(z, \xi),$$

Here $H = H^*$ is a $(m \times m)$-matrix and $\varepsilon = diag\{\varepsilon_1, ..., \varepsilon_l\}$, $\eta = diag\{\eta_1, ..., \eta_l\}$, $æ = diag\{æ_1, ..., æ_l\}$ are real diagonal $(l \times l)$-matrices.

Theorem 1: Let there exist such diagonal matrices $\varepsilon > 0$, $\eta > 0$, æ and such positive integers $m_1, m_2, ..., m_l$ that the following hypotheses hold:

1) For all $p \in \mathbf{C}, |p| = 1$ the matrix

$$\Re e\{æK(p) - K^*(p)\varepsilon K(p) - \eta\} \tag{3}$$

(where the designation $\Re eA = (1/2)(A^* + A)$ is used) is positive definite.
2) The inequalities

$$4\eta_j \left[\varepsilon_j - \frac{æ_j \alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, m_j, z^*(0)Hz(0))|) \right] >$$

$$> \left[æ_j \mu_j^{(i)}(æ, m_j, z^*(0)Hz(0)) \right]^2 \quad (4)$$

$$(j = 1, 2, ..., l, i = 1, 2)$$

with $\alpha_{0j} = \alpha_{2j}$ if $æ_j > 0$, and $\alpha_{0j} = \alpha_{1j}$ if $æ_j < 0$ are true. Here $H = H^*$ is just such a $(m \times m)$-matrix that $\Phi(z, \xi) \leq 0$, $\forall z \in \mathbf{R}^m$, $\xi \in \mathbf{R}^l$.

Then for any solution $(z(n), \sigma(n))$ of (1) with initial data $(z(0), \sigma(0))$ the estimates

$$|\sigma_j(n) - \sigma_j(0)| < m_j \Delta_j (j = 1, 2, ..., l) \quad (5)$$

are true for all natural n.

Remark 1. Notice that of the hypothesis 1) of the theorem is fulfilled for certain matrices æ, $\varepsilon > 0$, $\eta > 0$ then according to Yakubovich–Kalman frequency–domain theorem [4] there exists a matrix $H = H^*$, which guarantees that the inequality $\Phi(z, \xi) \leq 0$ is valid for all $z \in \mathbf{R}^m$, $\xi \in \mathbf{R}^l$.

The proof of theorem 1 is base on a special Lyapunov-type lemma with Lyapunov functions of the form "a quadratic form plus integral of a nonlinearity". The nonlinearity in Lyapunov function is conctructed by Bakaev-Guzh technique [5] intended specially for phase control systems.

Let sequences $\sigma_1(n), ..., \sigma_l(n)$ and $W(n) \geq 0$ be defined. Let $\varphi_j(\sigma)$ $(j = 1, ..., l)$ be Δ_j-periodic functions with all the properties described in this paper.

Lemma 1: Suppose there exist such numbers $\varepsilon > 0$, $\eta > 0$, $æ \neq 0$ and natural m_j $(j = 1, ..., l)$ and functions $\varphi_j(\sigma)$, $\sigma_j(n)$ $(j = 1, ..., l)$, $W(n) \geq 0$ that the following hypotheses are fulfilled:
1) for all integer $n \geq 0$ the inequality

$$W(n+1) - W(n) + \sum_{j=1}^{l} \{æ_j \varphi(\sigma(n))[\sigma(n+1) - \sigma(n)] +$$

$$+ \varepsilon[\sigma(n+1) - \sigma(n)]^2 + \eta \varphi^2(\sigma(n))\} \leq 0$$

is valid;
2) functions $\mu_j^{(i)}(æ, k, w)$ satisfy inequalities

$$4\eta_j \left[\varepsilon_j - \frac{æ\alpha_{0j}}{2}(1 + |\mu_i^{(i)}(æ, m_j, W(0))|) \right] >$$

$$> \left[æ_j \mu_i^{(i)}(æ, m_j, W(0)) \right]^2, \quad j = 1, ..., l; i = 1, 2$$

where α_{0j} are defined in theorem 1.

Then for all natural n the estimates

$$|\sigma_j(n) - \sigma_j(0)| < m_j \Delta_j \quad (j = 1, ..., l) \quad (6)$$

are valid.

Proof: It follows from the requirement 2) that for a certain $\varepsilon_0 > 0$ and all integer $k_j > m_j$ inequalities

$$4\eta_j \left(\varepsilon_j - \frac{æ\alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, k_j, W(0) + \varepsilon_0)|) \right)$$

$$\geq \left(æ\mu_j^{(i)}(æ, k_j, W(0) + \varepsilon_0) \right)^2 \quad (j = 1, ..., l; i = 1, 2) \quad (7)$$

are true.

Let us define functions

$$F_j^{(i)}(\sigma) = \varphi_j(\sigma) - \mu_j^{(i)}|\varphi(\sigma)|, \quad (j = 1, ..., l; i = 1, 2). \quad (8)$$

It follows from [2] that the following estimates are valid:

$$F_j^{(i)}(a)(u-a) + \frac{\alpha_{1j}}{2}(1 + |\mu_j^{(i)}|)(u-a)^2 \leq \int_a^u F_j^{(i)}(\sigma)\, d\sigma \leq$$

$$\leq F_j^{(i)}(a)(u-a) + \frac{\alpha_{2j}}{2}(1 + |\mu_j^{(i)}|)(u-a)^2. \quad (9)$$

In formula (8) and (9) we used the designation

$$\mu_j^{(i)} = \mu_j^{(i)}(æ, k_j, W(0) + \varepsilon_0), \quad (j = 1, ..., l; i = 1, 2).$$

Let us introduce Lyapunov sequences

$$V^I(n) = W(n) + \sum_{j=1}^{l} æ_j \int_{\sigma_j(0)}^{\sigma_j(n)} F_i^{i_j}(\sigma)\, d\sigma, \quad n = 0, 1, 2, ...,$$

where i_j is equal either to 1 or 2 and

$$I = \begin{pmatrix} i_1 \\ . \\ . \\ . \\ i_l \end{pmatrix}.$$

Their increments are as follows:

$$V^I(n+1) - V^I(n) =$$

$$= W(n+1) - W(n) + \sum_{j=1}^{l} æ_j \int_{\sigma_j(n)}^{\sigma_j(n+1)} F_j^{(i_j)}(\sigma)\, d\sigma. \quad (10)$$

Let us consider every summand in the right part of (10) separately. According to hypothesis 1) of the lemma we have

$$W(n+1) - W(n) \leq -\sum_{j=1}^{l} \{æ_j \varphi_j(\sigma_j(n))[\sigma_j(n+1) - \sigma_j(n)]$$

$$+ \varepsilon_j[\sigma_j(n+1) - \sigma_j(n)]^2 + \eta_j \varphi_j^2(\sigma_j(n))\}.$$

To estimate the other summand we use the formula (9). As a result we have

$$V^I(n+1) - V^I(n) \leq -\sum_{j=1}^{l} P_j^{(i_j)},$$

where

$$P_j^{(i_j)} = -æ_j \varphi_j(\sigma_j(n))[\sigma_j(n+1) - \sigma_j(n)] -$$

$$- \varepsilon_j[\sigma_j(n+1) - \sigma_j(n)]^2 - \eta_j \varphi_j^2(\sigma_j(n)) +$$

$$+ æ_j[F_j^{(i_j)}(\sigma_j(n))(\sigma_j(n+1) - \sigma_j(n)) +$$

$$+ \frac{\alpha_{0j}}{2}(1 + |\mu_j^{(i_j)}|)(\sigma_j(n+1) - \sigma_j(n))^2].$$

Note that

$$P_j^{(i_j)} = -\{(\varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1+|\mu_j^{(i_j)}|)(\sigma_j(n+1)-\sigma_j(n))^2+$$

$$+æ_j(\sigma_j(n+1)-\sigma_j(n))[\varphi_j(\sigma_j(n)) - F_j^{(i_j)}(\sigma_j(n))]+$$

$$+\frac{æ_j^2}{4(\varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1+|\mu_j^{(i_j)}|))}[\varphi_j(\sigma_j(n)) - F_j^{(i_j)}(\sigma_j(n))]^2\}+$$

$$+\frac{æ_j^2}{4(\varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1+|\mu_j^{(i_j)}|))}[\varphi_j(\sigma_j(n)) - F_j^{(i_j)}(\sigma_j(n))]^2-$$

$$-\eta_j\varphi_j^2(\sigma_j(n)).$$

So

$$P_j^{(i_j)} \le \frac{æ_j^2}{4(\varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1+|\mu_j^{(i_j)}|))}[\varphi_j(\sigma_j(n)) -$$

$$-F_j^{(i_j)}(\sigma_j(n))]^2 - \eta_j\varphi_j^2(\sigma_j(n)) =$$

$$= \frac{æ_j^2(\mu_j^{(i_j)})^2}{4(\varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1+|\mu_j^{(i_j)}|))} - \eta_j.$$

In virtue of hypothesis 2) of the lemma one can affirm that

$$V^{(I)}(n+1) - V^{(I)}(n) \le 0. \tag{11}$$

Hence

$$V^{(I)}(n) \le V^{(I)}(0) \quad (n \in \mathbf{N})$$

or

$$V^{(I)}(n) \le W(0). \tag{12}$$

Suppose now that for certain $n_0 \in \mathbf{N}$ several estimates (6) are false. Suppose there exits such $q_i \in [1, l]$ $(i = 1, 2, ..., k; k \le l)$ that

$$|\sigma_{q_i}(n_0) - \sigma_{q_i}(0)| \ge m_{q_i}\Delta_{q_i}. \tag{13}$$

Let for $(i = 1, 2, ..., k_1$ with $k_1 \le k)$

$$\sigma_{q_i}(n_0) = \sigma_{q_i}(0) + l_{q_i}\Delta_{q_i} + \beta_{1q_i}, \beta_{1q_i} \in [0, \Delta_{q_i}), l_{q_i} \ge m_{q_i} \tag{14}$$

and for $i = k+1+1, ..., k$

$$\sigma_{q_i}(n_0) = \sigma_{q_i}(0) - l_{q_i}\Delta_{q_i} - \beta_{2q_i}, \beta_{2q_i} \in [0, \Delta_{q_i}), l_{q_i} \ge m_{q_i} \tag{15}$$

Note that if j does not coincide with $q_1, ..., q_k$ we either

$$\sigma_j(n_0) = \sigma_j(0) - l_j\Delta_j + \beta_{1j}, \beta_{1j} \in [0, \Delta), 0 \le l_j < m_j \tag{16}$$

or

$$\sigma_j(n_0) = \sigma_j(0) - l_j\Delta_j - \beta_{2j}, \beta_{2j} \in [0, \Delta), 0 \le l_j < m_j \tag{17}$$

Let us now consider $V^{(I)}(n_0)$ and choose $i_j = 1$ for those j for which formulae (14) or (16) are true and $i_j = 2$ for those j for which formulae (15) or (17) take place. Further we choose $k_j = l_j$ if formulae (14) or (15) are true and $k_j = m_j$ if formulae (16) or (17) take place.

Suppose formula (14) or (16) is true. Then

$$F_j^{(i_j)}(\sigma) = F_j^{(1)}(\sigma) = \varphi_j(\sigma) - \mu_j^{(1)}(æ, l_j, W(0)+\varepsilon_0)|\varphi(\sigma)|,$$

and

$$æ_j \int_{\sigma_j(0)}^{\sigma_j(n_0)} F_j^{(1)}(\sigma)\, d\sigma =$$

$$= æ_j l_j \int_0^{\Delta_j} F_j^{(1)}(\sigma)\, d\sigma + æ_j \int_{\sigma_j(0)}^{\sigma_j(0)+\beta_{1j}} F_j^{(1)}(\sigma)\, d\sigma.$$

Futher as it follows from [1]

$$æ_j \int_{\sigma_j(0)}^{\sigma_j(0)+\beta_{1j}} F_j^{(1)}(\sigma)\, d\sigma =$$

$$= \frac{(\gamma_{0j} + \Gamma_{0j})(W(0) + \varepsilon_0 + \sum_{j=1}^{l} |æ_j| R_j}{l_j(\gamma_j + \Gamma_j))} +$$

$$+ \frac{2æ_j(\Gamma_j\gamma_{0j} - \Gamma_{0j}\gamma_j)}{\gamma_j + \Gamma_j}, \tag{18}$$

where

$$\int_{\sigma_j(0)}^{\sigma_j(0)+\beta_{1j}} \varphi_j(\sigma)\, d\sigma = \gamma_{0j} - \Gamma_{0j},$$

$$\int_{\sigma_j(0)}^{\sigma_j(0)+\beta_{1j}} |\varphi_j(\sigma)|\, d\sigma = \gamma_{0j} + \Gamma_{0j} \quad (\gamma_{0j}, \Gamma_{0j} \ge 0).$$

If $l_j \ge m_j$ (formula (14)) we have

$$æ_j l_j \int_0^{\Delta_j} F_j^{(1)}(\sigma)\, d\sigma = W(0) + \varepsilon_0 + \sum_{j=1}^{l} |æ_j| R_j$$

and if $0 \ge l_j < m_j$ (formula (16)) we have

$$æ_j l_j \int_0^{\Delta_j} F_j^{(1)}(\sigma)\, d\sigma = \frac{l_j}{m_j}\left(W(0) + \varepsilon_0 + \sum_{j=1}^{l} |æ_j| R_j\right).$$

Analogous by if formula (15) or formula (17) is true then

$$F_j^{(i_j)}(\sigma) = F_j^{(2)}(\sigma)$$

and

$$æ_j \int_{\sigma_j(0)}^{\sigma_j(n_0)} F_j^{(2)}(\sigma)\, d\sigma =$$

$$= -æ_j l_j \int_0^{\Delta_j} F_j^{(2)}(\sigma)\, d\sigma + æ_j \int_{\sigma_j(0)}^{\sigma_j(0)-\beta_{2j}} F_j^{(2)}(\sigma)\, d\sigma.$$

Note that

$$æ_j \int_{\sigma_j(0)}^{\sigma_j(0)-\beta_{2j}} F_j^{(2)}(\sigma)\, d\sigma =$$

$$= \frac{(\gamma_{1j} + \Gamma_{1j})(W(0) + \varepsilon_0 + \sum_{j=1}^{l} |æ_j| R_j}{l_j(\gamma_j + \Gamma_j))} +$$

$$+\frac{2æ_j(\Gamma_j\gamma_{1j}-\Gamma_{1j}\gamma_j)}{\gamma_j+\Gamma_j}, \tag{19}$$

where

$$\int\limits_{\sigma_j(0)-\beta_{2j}}^{\sigma_j(0)} \varphi_j(\sigma)\,d\sigma = \gamma_{1j}-\Gamma_{1j},$$

$$\int\limits_{\sigma_j(0)-\beta_{2j}}^{\sigma_j(0)} |\varphi_j(\sigma)|\,d\sigma = \gamma_{1j}+\Gamma_{1j} \quad (\gamma_{1j},\Gamma_{1j}>0).$$

If $l_j \geq m_j$ (formula (15)) then

$$-æ_j l_j \int\limits_0^{\Delta_j} F_j^{(2)}(\sigma)\,d\sigma = W(0)+\varepsilon_0+\sum_{j=1}^{l}|æ_j|R_j$$

and if $0 \geq l_j < m_j$ (formula (17)) then

$$-æ_j l_j \int\limits_0^{\Delta_j} F_j^{(2)}(\sigma)\,d\sigma = \frac{l_j}{m_j}\left(W(0)+\varepsilon_0+\sum_{j=1}^{l}|æ_j|R_j\right).$$

As a result

$$V^{(I)}(n_0) \geq W(n_0)+(W(0)+\varepsilon_0+\sum_{j=1}^{l}|æ_j|R_j)k+$$

$$+\sum_{j=1}^{k_1}\frac{2æ_j}{\gamma_j+\Gamma_j}(\Gamma_j\gamma_{0j}-\Gamma_{0j}\gamma_j)+$$

$$+\sum_{j=k_1+1}^{k}\frac{2æ_j}{\gamma_j+\Gamma_j}(\Gamma_{1j}\gamma_j-\Gamma_j\gamma_{1j}).$$

Since $k \geq 1$ and for $r=0,1$

$$|æ_j|R_j+\frac{2æ_j(-1)^r}{\gamma_j+\Gamma_j}(\Gamma_j\gamma_{rj}-\Gamma_{rj}\gamma_j) \geq$$

$$\geq \frac{2|æ_j|}{\gamma_j+\Gamma_j}(\gamma_j\Gamma_j-|\Gamma_j\gamma_{rj}-\Gamma_{rj}\gamma_j)\geq 0,$$

we obtain than

$$V^{(I)}(n_0) \geq W(n_0)+W(0)+\varepsilon_0$$

and in virtue of (12)

$$W(0) \geq W(n_0)+W(0)+\varepsilon_0.$$

Hence

$$W(n_0) \leq -\varepsilon_0 \quad (\varepsilon_0>0)$$

which contradict the fact that $W(n) \geq 0$. Lemma is proved. ■

Proof: (theorem 1) Let us consider the quadratic form $\Phi(z,\xi)$ ($z\in\mathbf{R}^m$, $\xi\in\mathbf{R}^l$). First of all we shell prove that there exists a matrix $H=H^*$ such that the inequality $\Phi(z,\xi)\leq 0$ is valid for all $z\in\mathbf{R}^m$, $\xi\in\mathbf{R}^l$. Let $\tilde{F}(z,\xi)$ and $\tilde{\Phi}(z,\xi)$ be the Hermitian extensions of the forms F and Φ to complex arguments. According to Yakubovich–Kalman frequency-domain theorem [4] the inequality

$$\tilde{\Phi}(z,\xi)\leq 0 \tag{20}$$

is valid for all $z\in\mathbf{R}^m$, $\xi\in\mathbf{R}^l$ iff for all $p\in\mathbf{C}, |p|=1$ the inequality

$$\tilde{F}(-(A-pE_m)^{-1}B\xi,\xi)\leq 0 \tag{21}$$

is true. We have

$$\tilde{F}(-(A-pE_m)^{-1}B\xi,\xi)=$$

$$=\Re e\{\xi^*æ(c^*(pE_m-A)^{-1}B\xi+R\xi)+\xi^*\eta\xi+$$

$$+(c^*(pE_m-A)^{-1}B\xi+R\xi)^*\varepsilon(c^*(pE_m-A)^{-1}B\xi+R\xi)\}=$$

$$=\Re e\{-æK(p)+\eta+K(p)^*\varepsilon K(p)\}|\xi|^2.$$

By virtue of hypothesis 1) of the theorem the inequality (21) is correct. Thus we have proved the existance of matrix $H=H^*$ with which (20) is correct.

Moreover as all eigenvalues of matrix A are situated inside the unit circle matrix H is positive define. Indeed

$$\Phi(z,0)=(Az)^*H(Az)-z^*Hz+z^*C\varepsilon C^*z.$$

Since $\Phi(z,0)\leq 0$ we have

$$z^*(A^*HA-H)z\leq -z^*C\varepsilon C^*z\leq -\bar{\varepsilon}|C^*z|^2, \tag{22}$$

where $\bar{\varepsilon}=min\{\varepsilon_1,\varepsilon_2,...,\varepsilon_m\}$. Hence and from the fact that (A,C) is observable it follows that $H>0$ [4].

We choose now $W(n)=z^*(n)Hz(n)$. It satisfies all hypotheses of lemma 1. Really on the one hand $W(n)\geq 0$ for all $n\geq 0$. On the other hand by virtue of system (1) we have

$$W(n+1)-W(n)+\sum_{j=1}^{l}\{æ_j\varphi_j(\sigma_j(n))(\sigma_j(n+1)-\sigma_j(n))+$$

$$+\varepsilon_j(\sigma_j(n+1)-\sigma_j(n))^2+\eta_j\varphi_j^2(\sigma_j(n))\}=$$

$$=(Az(n)+B\varphi(\sigma(n)))^*H(Az(n)+B\varphi(\sigma(n)))-$$

$$-z^*(n)Hz(n)+\varphi^*(\sigma(n))æ(C^*z(n)+R\varphi(\sigma(n)))+$$

$$+(C^*z(n)+R\varphi(\sigma(n)))^*\varepsilon(C^*z(n)+R\varphi(\sigma(n)))+$$

$$+\varphi^*(\sigma(n))\eta\varphi(\sigma(n))=\Phi(z(n),\varphi(\sigma(n))).$$

Since $\Phi(z(n),\varphi(\sigma(n)))\leq 0$ the hypothesis 1) of lemma 1 is valid. Hypothesis 2) of lemma 1 and hypothesis 2) of theorem 1 coincide. Thus the estimate (6) is true. It coincide with the conclusion of theorem 1. Theorem 1 is proved. ■

III. Extension of frequency-domain criteion for the phase error

Let us extend the state space of system (1) [5], [6]. For the purpose we introduce the notations

$$y=\left\|\begin{array}{c} z \\ \varphi(\sigma)\end{array}\right\|, \quad P=\left\|\begin{array}{cc} A & B \\ 0 & E_l\end{array}\right\|, \quad L=\left\|\begin{array}{c} 0 \\ E_l\end{array}\right\|,$$

$C_1^*=||C^*,R||$, $\xi_1(n)=\varphi(\sigma(n+1))-\varphi(\sigma(n))$. Here P is a $((m+l)\times(m+l))$ - matrix, L is a $((m+l)\times l)$ - matrix, C_1^* is a $(l\times(m+l))$ - matrix, y is a $(m+l)$-vector and ξ_1 is a l-vector. Then system (1) can be written as follows

$$\begin{array}{rcl} y(n+1) & = & Py(n)+L\xi_1(n), \\ \sigma(n+1) & = & \sigma(n)+C_1^*y(n), \qquad n=0,1,2,\ldots \end{array} \tag{23}$$

Consider the forms of $y \in \mathbf{R}^{m+l}$ and $\xi_1 \in \mathbf{R}^l$

$$\Phi_1(y,\xi_1) = (Py + L\xi_1)^* H(Py + L\xi_1) - y^* Hy + F_1(y,\xi_1),$$

$$F_1(y,\xi_1) = y^* L æ C_1^* y + y^* C_1 \varepsilon C_1^* y + y^* L\eta L^* y +$$

$$+(A_1 C_1^* y - \xi_1)^* \tau(\xi_1 - A_2 C_1^* y),$$

where $A_i = diag\{\alpha_{i1}, \alpha_{i2}, ..., \alpha_{il}\}$ $(i = 1,2)$, $H = H^*$ is a $((m+l) \times (m+l))$ - matrix, and $\varepsilon, \eta, æ, \tau$ are real diagonal matrices with varied elements.

Remark 2. [5], [6] If (A, b) is controllable then (P, L) is controllable.

Remark 3. [5], [6] If $p \neq 1$ we have

$$C_1^*(P - pE)^{-1}L = \frac{1}{p-1}K(p), \qquad (24)$$

$$L^*(P - pE)^{-1}L = -\frac{1}{p-1}E_l. \qquad (25)$$

Lemma 2: Suppose all eigenvalues of matrix A are situated inside the unit circle. Suppose there exist such diagonal matrices $\varepsilon > 0$, $\eta, > 0$, $\tau > 0$ and $æ$ that for all $p \in \mathbf{C}, |p| = 1$ the frequency-domain inequality

$$\Re e\{æK(p) + (A_1 K(p) + (p-1)E_l)^* \tau((p-1)E_l + A_2 K(p))\} -$$

$$-K(p)^* \varepsilon K(p) - \eta \geq 0, \qquad (26)$$

is valid. Then there exist such $((m+l) \times (m+l))$ - matrix $H_1 = H_1^*$ that

$$\Phi_1(y,\xi_1) \leq 0 \quad \forall \quad y \in \mathbf{R}^{m+l}, \xi_1 \in \mathbf{R}^l. \qquad (27)$$

Proof: Let $\tilde{F}_1(y,\xi_1)$ and $\tilde{\Phi}_1(y,\xi_1)$ be the Hermitian extensions of the forms F_1 and Φ_1 to complex arguments. According to Yakubovich–Kalman frequency-domain theorem [4] the inequality $\tilde{\Phi}_1(y,\xi_1) \leq 0$ is valid for all $y \in \mathbf{C}^m$, $\xi_1 \in \mathbf{C}^l$ for certain matrix $H_1 = H_1^*$ iff

$$\tilde{F}_1(-(P - pE)^{-1}L\xi_1, \xi_1) \leq 0. \qquad (28)$$

We have

$$\tilde{F}_1(-(P - pE)^{-1}L\xi_1, \xi_1) =$$

$$= \Re e\{\xi_1^*[L^*((P - pE)^{-1})^* L æ C_1^*(P - pE)^{-1}L +$$

$$+L^*((P - pE)^{-1})^* C_1 \varepsilon C_1^*(P - pE)^{-1}L +$$

$$+L^*((P - pE)^{-1})^* L\eta L^*(P - pE)^{-1}L -$$

$$-(A_1 C_1^*((P - pE)^{-1})^* L + E)^* \tau(EA_2 C_1^*(P - pE)^{-1}L)]\xi_1\}.$$

Let us use (24) and (25). Then

$$\tilde{F}_1(-(P - pE)^{-1}L\xi_1, \xi_1) =$$

$$= -\frac{1}{|p-1|^2}\xi_1^* \Re e\{æK(p) - K(p)^* \varepsilon K(p) - \eta +$$

$$+(A_1 K(p) + (p-1)E_l)^* \tau((p-1)E_l + A_2 K(p))\}\xi_1.$$

They in (26) is valid then (27) is valid too. So there exist such matrix $H_1 = H_1^*$ that inequality (27) is fulfilled. Lemma 2 is proved. ■

Remark 4. Suppose all the hypotheses of lemma 2 are fulfilled. Then we can consider the sequence

$$W + 1(n) = y^*(n) H_1 y(n),$$

where $y(n)$ is a solution of system (23). As all eigenvalues of matrix A are situated inside the unit circle and functions $\varphi_j(\sigma_j)$ $(j = 1, ..., l)$ are bounded we can affirm that $|y(n)| < const$ for all $n \geq 0$. So the quadratic form $W_1(n)$ is bounded for all $n \geq 0$.

Theorem 2: Let all eigenvalues of matrix A be situated inside the unit circle. Let pair (A, B) is controllable and pair (A, C) is observable. Suppose there exist such diagonal matrices $\varepsilon > 0$, $\tau > 0$, $\eta > 0$, $æ$ and such positive integers $m_1, m_2, ..., m_l$ that the following hypotheses hold:
1) The frequency-domain inequality (26) is valid.
2) The inequalities

$$4\eta_j \left[\varepsilon_j - \frac{æ_j \alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, m_j, y^*(0)H_1 y(0) - r)|) \right] >$$

$$> \left[æ_j \mu_j^{(i)}(æ, m_j, y^*(0)H_1 y(0) - r) \right]^2 \qquad (29)$$

$$(j = 1, 2, ..., l, i = 1, 2)$$

are valid, where $H_1 = H_1^*$ is such a $((m+l) \times (m+l))$-matrix that $\Phi_1(y,\xi_1) \leq 0$ $(y \in \mathbf{R}^{m+l}, \xi_1 \in \mathbf{R}^l)$ and

$$r \leq \inf_{n=0,1,2,...} y^*(n) H_1 y(n).$$

Then for solution $(z(n), \sigma(n))$ of (1) with initial data $(z(0), \sigma(0))$ the estimates (5) are true for all natural n.

Proof: The proof is based on lemma 1. Let us consider the sequence

$$W(n) = y^*(n) H_1 y(n) - r.$$

Note that $W(n) \geq 0$ for all $n \geq 0$. Let us prove this sequence satisfies all the hypotheses of lemma 1. Consider

$$z(n) = W(n+1) - W(n) +$$

$$+\sum_{j=1}^{l}\{æ_j \varphi_j(\sigma_j(n))[\sigma_j(n+1) - \sigma_j(n)] +$$

$$+\varepsilon_j[\sigma_j(n+1) - \sigma_j(n)]^2 + \eta_j \varphi_j^2(\sigma_j(n))\}$$

and transform it in virtue of system (1).

$$z(n) = (Py(n) + L\xi_1(n))^* H_1(Py()n + L\xi_1(n)) -$$

$$-y^*(n) H_1 y(n) + y^*(n) L æ C_1^* y(n) + y^*(n) C_1 \varepsilon C_1^* y(n) +$$

$$+y^*(n) L\eta L^* y(n) = \Phi_1(y(n), \xi_1(n)) -$$

$$-(A_1 C_1^* y(n) - \xi_1(n))^* \tau(\xi_1(n) - A_2 C_1^* y(n)).$$

Futher

$$A_1 C_1^* y(n) - \xi_1(n) =$$

$$= A_1(C^* x(n) + R\varphi(\sigma(n))) - \varphi(\sigma(n+1)) + \varphi(\sigma(n)) =$$

$$= A_1(\sigma(n+1) - \sigma(n)) - (\varphi(\sigma(n+1)) - \varphi(\sigma(n))).$$

$$\xi_1(n) - A_2 C_1^* y(n) =$$

$$= (\varphi(\sigma(n+1)) - \varphi(\sigma(n))) - A_2(\sigma(n+1) - \sigma(n)).$$

Let us take into account that

$$\varphi_j(\sigma_j(n+1)) - \varphi_j(\sigma_j(n)) = \varphi_j'(\sigma_j')(\sigma_j(n+1)) - (\sigma(n)),$$

where σ_j' lies between $(\sigma_j(n)$ and $\sigma_j(n+1)$. Then in virtue of (2) we have

$$(A_1C_1^*y(n) - \xi_1(n))^*\tau(\xi_1(n) - A_2C_1^*y(n)) =$$
$$= \sum_{j=1}^{l}\tau_j(\varphi_j'(\sigma_j'))^2(\sigma_j(n+1)) - (\sigma(n))^2 \geq 0.$$

As a result

$$z(n) \leq \Phi_1(y(n), \xi_1(n)).$$

In virtue of hypothesis 1) of theorem 2 we can establishe by lemma 2 that $z(n) \leq 0$. This fact is equivalent to hypothesis 1) of lemma 1. Hypothesis 2) of theorem 2 coincide with hypothesis 2) of lemma 1. So estimates (6) are valid, and theorem 2 is proved. ■

Let as now reject the requirement of $W(n) \geq 0$.

Lemma 3: Let $\sigma_1(n), ..., \sigma_l(n)$, $W(n) \geq 0$ be sequences and $\varphi_j(\sigma)$ $(j = 1, ..., l)$ be Δ_j-periodic functions which have all the properties of nonlinear functions of system (1). Suppose there exist such numbers $\varepsilon_j > 0$, $\eta_j > 0$, $æ_j \neq 0$ $j = 1, 2, ..., l$ and natural numbers m_J $j = 1, 2, ..., l$ that the following hypotheses are fulfilled:
1) hypothesis 1) of lemma 1;
2) inequalities

$$4\eta_j\left[\ \varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, m_j, |W(0)|)|)\right] >$$
$$> \left[\ æ_j\mu_j^{(i)}(æ, m_j, |W(0)|)\right]^2 (j = 1, 2, ..., l, i = 1, 2)$$

are true.

Then for those natural n for which $W(n) \geq 0$ the estimates (6) are true.

Proof of the lemma 3 is analogous to those of lemma 2. Instead of inequality (20) we pbtain inequality

$$W(0) \geq W(n_0) + |W(0)| + \varepsilon_0 \quad (\varepsilon_0 > 0) \tag{30}$$

Hence

$$W(n_o) \leq -\varepsilon_0 \quad (\varepsilon_0 > 0),$$

which contradict the fact $W(n_o) \geq 0$.

Theorem 3: Let all the hypotheses of theorem 2 be fulfilled, exept hypothesis 2) which is substituted by the requirement
2') inequalities

$$4\eta_j\left[\ \varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, m_j, |y^*(0)H_1y(0)|)|)\right] > \tag{31}$$
$$> \left[\ æ_j\mu_j^{(i)}(æ, m_j, |y^*(0)H_1y(0)|)\right]^2 \quad (j = 1, 2, ..., l, i = 1, 2)$$

are valid with $H_1 = H_1^*$ satisfying (27).

Then for any solution $(z(n), \sigma(n))$ of (1) with initial data $(z(0), \sigma(0))$ the following limit relations are true:

$$z(n) \to 0 \text{ as } n \to +\infty, \tag{32}$$

$$\sigma_j(n) \to \hat{\sigma}_j \text{ as } n \to +\infty \quad (j = 1, 2, ..., l), \tag{33}$$

$$\varphi_j(\sigma_j(n)) \to 0 \text{ as } n \to +\infty, \tag{34}$$

where $\varphi_j(\hat{\sigma}_j) = 0$, and

$$|\sigma_j(0) - \hat{\sigma}_j| < m_j\Delta_j. \tag{35}$$

Proof: Inequalities (31) imply the inequalities

$$4\eta_j\left[\varepsilon_j - \frac{æ_j\alpha_{0j}}{2}\left(1 + \frac{\Gamma_j - \gamma_j}{\Gamma_j + \gamma_j}\right)\right] > \left(æ_j\frac{\Gamma_j - \gamma_j}{\Gamma_j + \gamma_j}\right)^2 \tag{36}$$

Then all the hypotheses of theorem 5.4.1 [5] are fulfilled. According to this theorem the limit relations (32), (34), (33) take place.

It follows from hypothesis 2') that for a certain $\varepsilon_0 > 0$ inequalities

$$4\eta_j\left[\ \varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, m_j, |y^*(0)H_1y(0) + \varepsilon_0|)|)\right] > \tag{37}$$
$$> \left[\ æ_j\mu_j^{(i)}(æ, m_j, |y^*(0)H_1y(0) + \varepsilon_0|)\right]^2$$
$$(j = 1, 2, ..., l, i = 1, 2)$$

are valid. Let

$$W(n) = y^*(n)H_1y(n) + \varepsilon_0.$$

Since (32) and (33) are true, the sequence $W(n)$ becomes positive for $n > N_0$, where N_0 is sufficientli great. Further we can repeat the proof of theorem 2 up to the moment when the correctness of hypothesis 1) of lemma 1 is established. The latter coincides with hypothesis 1) of lemma 3. The hypotheses of lemma 3 and theorem 3 coincide. So according lemma 3 estimate (6) is true. In virtue of (6) and (33) estimates (35) is true. Thus theorem 3 is proved. ■

REFERENCES

[1] V. B. Smirnova, A. I. Shepeljavyi and N. V. Utina, " Frequency-domain conditions for cycle-slipping in discrete systems with periodic nonlinearity," *International Conference "Physics and Control". Proceedings, Saint-Petersburg, August 20-22, 2003, P. 607-610.*

[2] V. B. Smirnova, A. I. Shepeljavyi and N. V. Utina, "Frequency-domain estimates for transient attributes of discrete phase systems," *Second International Conference "Physics and Control". Proceedings, Saint-Petersburg, Russia, August 24-26, 2005, P. 469-473..*

[3] V. B. Smirnova, A. I. Shepeljavyi and N. V. Utina, "Asymptotic frequency-domain estimaes for the amplitude of the output in discrete phase systems". *Vestnik SPbGU*, ser. 1, vyp. 1, pp. 60–68, 2006. (in Russian)

[4] V. A. Yakubovich, "A frequency–domain theorem in the control theory," *Sibirsk. Mat. Zh.* v. 14, no 2, pp. 265–289, 1973. (in Russian)

[5] G. A. Leonov and V. B. Smirnova, *Mathematical problems of phase synchronization theory*. Nauka, St.Petersburg, 2000. (in Russian)

[6] Yu.A. Koryakin, G.A. Leonov. "The Bakaev-Guzh technique for systems with several angular coordinates," *Izvestya Akad. Nauk Kazakhskoy SSR*, N 3, p. 41-46, 1976. (in Russian)