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INVERSE PROBLEM OF ANISOTROPY-BASED PERFORMANCE ANALYSIS

Alexander Kurdyukov **Victor Timin** **Michael Tchaikovsky**
Institute of Control Sciences RAS
Moscow, Russia
akurd@ipu.ru; timin.victor@rambler.ru; mmtchaikovsky@hotmail.com

Abstract
This paper addresses the inverse problem of anisotropy-based performance analysis of linear discrete time-invariant system. The inverse problem consists in finding the maximum level of mean anisotropy of random input sequence corresponding to some given value of anisotropic norm of system. A unique solution is determined by solving a system of four cross-coupled nonlinear algebraic equations including Riccati, Lyapunov, and two scalar equations with respect to traces of symmetric matrices. A simple illustrative numerical example is considered.

Key words
Linear system, stochastic input, norm, anisotropy

1 Introduction

The main concept of the anisotropy-based approach to robust stochastic control developed in the mid 1990's by I. G. Vladimirov and presented in papers [Semyonov et al., 1994; Vladimirov et al., 1995; Vladimirov et al., 1996-1], is the *anisotropic norm* of systems which builds on the *anisotropy* of random signals. The anisotropy functional, considered there, is an entropy theoretic measure of the deviation of a probability distribution in Euclidean space from Gaussian distributions with zero mean and scalar covariance matrices. The *mean anisotropy* of a stationary random sequence is defined in [Vladimirov et al., 2006] as the anisotropy production rate per time step for long segments of the sequence. In application to random disturbances, the mean anisotropy describes the amount of statistical uncertainty which is understood as the discrepancy between the imprecisely known actual noise distribution and the family of nominal models which consider the disturbance to be a Gaussian white noise sequence with a scalar covariance matrix. The *a-anisotropic norm* quantifies the disturbance attenuation capabilities of a linear discrete time invariant (LDTI) system by the largest ratio of the power norm of the system output to that of the input, provided that the mean anisotropy of the input disturbance does not exceed a given nonnegative parameter a.

This paper is devoted to the inverse problem of anisotropy-based performance analysis of LDTI systems. The inverse problem consists in finding the maximum level a of the mean anisotropy of the random input sequence that corresponds to the case when the anisotropic norm of the system is equal to given value γ.

The paper is organized as follows. Some necessary theoretical background and problem statement are given in Section 2. The main result of the paper is a sufficient condition defining the maximum level of the input sequence mean anisotropy corresponding to some given value of the system anisotropic norm γ represented in Section 3. A simple numerical example is considered in Section 4.

2 Background and Problem Statement

Let us recall some background material on the anisotropy of signals and anisotropic norm of systems. An extended exposition of the anisotropy-based robust performance analysis, developed originally in [Vladimirov et al., 1995; Vladimirov et al., 1996-1], can be found in [Diamond et al., 2001]; see also [Vladimirov et al., 2006].

2.1 Anisotropy of Random Vector

Denote by $\mathbb{L}_2^m$ the class of square integrable $\mathbb{R}^m$-valued random vectors distributed absolutely continuously with respect to the m-dimensional Lebesgue measure mes_m. For any $W \in \mathbb{L}_2^m$ with PDF $f\colon \mathbb{R}^m \to \mathbb{R}_+$, the relative entropy of its distribution with respect to the Gaussian PDF

$$p_{m,\lambda}(w) \triangleq (2\pi\lambda)^{-m/2} \exp\left(-\frac{|w|^2}{2\lambda}\right), \quad w \in \mathbb{R}^m, \tag{1}$$

is computed as

$$\mathbf{D}(f\|p_{m,\lambda}) = \mathbf{E}\ln\frac{f(W)}{p_{m,\lambda}(W)} = \frac{m}{2}\ln(2\pi\lambda) + \frac{\mathbf{E}(|W|^2)}{2\lambda} - \mathbf{h}(W), \quad (2)$$

where $\mathbf{D}(f\|p_{m,\lambda})$ denotes the relative entropy (or Kullback-Leibler informational divergence) of f with respect to $p_{m,\lambda}$,

$$\mathbf{h}(W) \triangleq \mathbf{E}\ln f(W) = -\int_{\mathbb{R}^m} f(w)\ln f(w)dw$$

denotes the differential entropy of W with respect to mes_m; see [Cover and Thomas, 1991].

The *anisotropy* $\mathbf{A}(W)$ is defined in [Vladimirov et al., 2006] as the minimal value of relative entropy (2) with respect to the Gaussian distributions in $\mathbb{R}^m$ with zero mean and scalar covariance matrices described by (1):

$$\mathbf{A}(W) \doteq \min_{\lambda>0}\mathbf{D}(f\|p_{m,\lambda}) = \frac{m}{2}\ln\left(\frac{2\pi\mathrm{e}}{m}\mathbf{E}(|W|^2)\right) - \mathbf{h}(W), \quad (3)$$

where the minimum is achieved at $\lambda = \mathbf{E}(|W|^2)/m$; see [Vladimirov et al., 2006].

Let $\mathbb{G}^m(\mu,\Sigma)$ denote the class of $\mathbb{R}^m$-valued Gaussian random vectors with mean $\mathbf{E}W = \mu$ and nonsingular covariance matrix $\mathbf{cov}(W) \triangleq \mathbf{E}((W-\mu)(W-\mu)^{\mathrm{T}}) = \Sigma$. Basic properties of the anisotropy of a random vector stated in [Vladimirov et al., 2006] are as follows:

1. $\mathbf{A}(\sigma UW) = \mathbf{A}(W)$ for any orthogonal matrix $U \in \mathbb{R}^{m\times m}$ and any $\sigma \in \mathbb{R}\setminus\{0\}$;
2. For any $\mathbb{R}^{m\times m} \ni \Sigma \succ 0$,

$$\min\left\{\mathbf{A}(W) : W \in \mathbb{L}_2^m,\ \mathbf{E}(WW^{\mathrm{T}}) = \Sigma\right\} = -\frac{1}{2}\ln\det\frac{m\Sigma}{\mathrm{tr}\,\Sigma}, \quad (4)$$

   where the minimum is only achieved at $W \in \mathbb{G}^m(0,\Sigma)$;
3. $\mathbf{A}(W) \geqslant 0$ for any $W \in \mathbb{L}_2^m$, and $\mathbf{A}(W) = 0$ if and only if $W \in \mathbb{G}^m(0,\lambda I_m)$ for some $\lambda > 0$.

2.2 Mean Anisotropy of Random Sequence

Let $W \triangleq (w_k)_{-\infty<k<+\infty}$ be a stationary sequence of square integrable random vectors with values in $\mathbb{R}^m$ which is interpreted as a discrete-time random signal. Assembling the elements of W, associated with a time interval $[s,t]$, into a random vector

$$W_{s:t} \triangleq \begin{bmatrix} w_s \\ \vdots \\ w_t \end{bmatrix}, \quad (5)$$

we assume that $W_{0:N}$ is absolutely continuously distributed for every $N \geqslant 0$. The *mean anisotropy* of the sequence W is defined in [Vladimirov et al., 2006] as the anisotropy production rate per time step by

$$\overline{\mathbf{A}}(W) \doteq \lim_{N\to+\infty}\frac{\mathbf{A}(W_{0:N})}{N}. \quad (6)$$

Now suppose the stationary random sequence W is Gaussian. Furthermore, let $V \triangleq (v_k)_{-\infty<k<+\infty}$ be an m-dimensional Gaussian white noise sequence, so that v_k are independent Gaussian random vectors with zero mean $\mathbf{E}v_k = 0$ and identity covariance matrix $\mathbf{cov}(v_k) = I_m$. Suppose $W = GV$ is generated from V by a shaping filter G as

$$w_j = \sum\nolimits_{k=0}^{+\infty} g_k v_{j-k}, \quad -\infty < j < +\infty. \quad (7)$$

The impulse response of the filter $g_k \in \mathbb{R}^{m\times m}$ is assumed to be square summable over $k \geqslant 0$, thus ensuring the mean square convergence of the series in (7). The spectral density of W is given by

$$S(\omega) \triangleq \widehat{G}(\omega)\widehat{G}(\omega)^*, \quad -\pi \leqslant \omega < \pi, \quad (8)$$

where $(\cdot)^* = \overline{(\cdot)}^{\mathrm{T}}$ denotes the complex conjugate transpose of a matrix, and $\widehat{G}(\omega) \triangleq \lim_{r\to 1-} G(re^{i\omega})$ is the boundary value of the transfer function $G(z) \triangleq \sum_{k=0}^{+\infty} g_k z^k$. The latter encodes all the properties of the filter as an input-output operator and belongs to the Hardy space $\mathcal{H}_2^{m\times m}$ of $(m\times m)$-matrix-valued functions, analytic in the disc $|z| < 1$ of the complex plane.

As is shown in [Vladimirov et al., 1996-1; Diamond et al., 2001], the mean anisotropy of the stationary Gaussian random sequence $W = GV$ can be computed in terms of the spectral density (8) and the associated $\mathcal{H}_2$-norm of the shaping filter G as

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\frac{mS(\omega)}{\|G\|_2^2}d\omega. \quad (9)$$

Since the probability law of the sequence W is completely determined by the shaping filter G or by the spectral density S, the alternative notations $\overline{\mathbf{A}}(G)$ and $\overline{\mathbf{A}}(S)$ will also be used instead of $\overline{\mathbf{A}}(W)$.

Mean anisotropy functional (9), which is always nonnegative, takes a finite value if the shaping filter G is of full rank, that is, if $\mathrm{rank}\,\widehat{G}(\omega) = m$ for almost all $\omega \in [-\pi,\pi)$. Otherwise, $\overline{\mathbf{A}}(G) = +\infty$; see [Vladimirov et al., 1996-1; Diamond et al., 2001]. The equality $\overline{\mathbf{A}}(G) = 0$ holds true if and only if G is an all-pass system up to a nonzero constant factor. In this case, the spectral density (8) is described by $S(\omega) = \lambda I_m$, $-\pi \leqslant \omega < \pi$, for some $\lambda > 0$, so that W is a Gaussian white noise sequence with zero mean and a scalar covariance matrix.

2.3 Anisotropic Norm of Linear System

Let $F \in \mathcal{H}_\infty^{p\times m}$ be an LDTI system with an m-dimensional input $W = GV$ and a p-dimensional output $Z = FW$, where, as before, V is a m-dimensional Gaussian white noise sequence with zero mean and identity covariance matrix. Let

$$\mathcal{G}_a \triangleq \{G \in \mathcal{H}_2^{m\times m} : \overline{\mathbf{A}}(G) \leqslant a\} \tag{10}$$

denote the set of shaping filters G which generate Gaussian random sequences W with mean anisotropy (9) bounded by a given parameter $a \geqslant 0$. The *a-anisotropic norm* ([Vladimirov et al., 1996-1; Diamond et al., 2001]) of the system F is defined by

$$|||F|||_a \doteq \sup_{G\in\mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2}. \tag{11}$$

The fraction $\|FG\|_2/\|G\|_2$ on the right-hand side of (11), which describes a "stochastic gain" of the system F with respect to $W = GV$, will also be referred to as the *power norm ratio*. As is shown in [Vladimirov et al., 1995; Diamond et al., 2001], the a-anisotropic norm (11) of a given system $F \in \mathcal{H}_\infty^{p\times m}$ is a nondecreasing continuous function of the mean anisotropy level a which satisfies

$$\frac{1}{\sqrt{m}}\|F\|_2 = |||F|||_0 \leqslant \lim_{a\to+\infty} |||F|||_a = \|F\|_\infty. \tag{12}$$

These relations show that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$-norms are the limiting cases of the a-anisotropic norm as $a \to 0, +\infty$, respectively.

2.4 Problem Statement

Let $F \in \mathcal{H}_\infty^{p\times m}$ be a stable linear discrete time-invariant system with m-dimensional input W related with the n-dimensional internal state X and p-dimensional output $Z = FW$ by the equations

$$F(z) : \begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}, \ -\infty < k < +\infty, \tag{13}$$

where $A, B, C,$ and D are some appropriately dimensioned matrices. The only prior information on the probability distribution of the random input sequence $W = (w_k)_{-\infty\leqslant k\leqslant+\infty}$ is assumed to be that it is a stationary Gaussian sequence of random vectors w_k with zero mean $\mathbf{E}(w_k) = 0$, unknown covariance matrix $\mathbf{cov}(w_k w_k^{\mathrm{T}}) = \Sigma_W$. At that it is supposed that the mean anisotropy of the sequence W is upper-bounded by an unknown nonnegative parameter a. This means that W is produced from m-dimensional Gaussian white noise $V = (v_k)_{-\infty\leqslant k\leqslant+\infty}$ with zero mean $\mathbf{E}(v_k) = 0$ and scalar (possibly, identity) covariance matrix by an unknown shaping filter G belonging to family (10). Let us formulate the inverse problem of anisotropy-based performance analysis as follows.

Problem 1. *Let a stable linear system $F \in \mathcal{H}_\infty^{p\times m}$ be defined by equations (13), and let a real number $\gamma \in [m^{-1/2}\|F\|_2, \|F\|_\infty)$ be given. We are interested in finding an input mean anisotropy level $a \geqslant 0$ guaranteeing that the equality $|||F|||_a = \gamma$ holds true.*

3 Main Result

Let us formulate the main result of this paper. The following theorem gives sufficient conditions for computing the maximum level of the mean anisotropy of Gaussian random input sequence conforming to a given value of the anisotropic norm of the system.

Theorem 1. *For given stable linear system $F \in \mathcal{H}_\infty^{p\times m}$, any real number $\gamma \in [m^{-1/2}\|F\|_2, \|F\|_\infty)$, and some level a of the mean anisotropy of the random input signal W, the equality*

$$|||F|||_a = \gamma \tag{14}$$

holds true if there exists a solution (q, R, P), $q \in [0, \|F\|_\infty^{-2})$, $R \succ 0$, $P \succ 0$, to the system of the cross-coupled nonlinear matrix algebraic equations

$$R = A^{\mathrm{T}}RA + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L, \tag{15}$$
$$L \triangleq \Sigma(B^{\mathrm{T}}RA + qD^{\mathrm{T}}C), \tag{16}$$
$$\Sigma \triangleq (I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1}, \tag{17}$$

$$P = (A + BL)P(A + BL)^{\mathrm{T}} + B\Sigma B^{\mathrm{T}}, \tag{18}$$

$$\mathrm{tr}((C + DL)P(C + DL)^{\mathrm{T}} + D\Sigma D^{\mathrm{T}}) = \gamma^2, \tag{19}$$
$$\mathrm{tr}(LPL^{\mathrm{T}} + \Sigma) = 1. \tag{20}$$

At that, the level of the mean anisotropy of the random input sequence W is defined by formula

$$a = -\frac{1}{2}\ln\det(m\Sigma). \tag{21}$$

Proof. Consider the stable linear system $F \in \mathcal{H}_\infty^{p\times m}$ defined by equations (13). Let the anisotropic norm of the system F is equal to γ for some unknown level a of the mean anisotropy of the random input sequence W. Recall that the anisotropic norm of the system F is defined at the half-open interval $\gamma \in [m^{-1/2}\|F\|_2, \|F\|_\infty)$ for any level $a \in [0, \infty)$ of the mean anisotropy of the input sequence; see [Semyonov et al., 1994; Vladimirov et al., 1996-1]. Let us show that if the condition (14) holds true then there exist the stabilizing solution $R \succ 0$ to algebraic Riccati equation (15), the solution $P \succ 0$ to Lyapunov equation (18), and the value of the parameter $q \in$

$[0, \|F\|_\infty^{-2})$ such that (19), (20) hold true. Then the maximum level of the mean anisotropy of the input sequence produced by a worst-case shaping filter G equals to a and is defined by formula (21). By definition (11), $|\!|\!|F|\!|\!|_a = \sup_{G\in\mathcal{G}_a}\|FG\|_2/\|G\|_2 = \gamma$, where $\mathcal{G}_a$ is a set of the shaping filters with bounded mean anisotropy $\overline{\mathbf{A}}(G) \leqslant a$. Then for all $G \in \mathcal{G}_a$ the inequality

$$\|FG\|_2/\|G\|_2 \leqslant \gamma \tag{22}$$

holds true. Since $\|FG\|_2$, $\|G\|_2$, and γ are positive real numbers, inequality (22) holds true for all $G \in \mathcal{G}_a$ producing the output sequences with mean anisotropy levels not exceeding a.

Let us find the maximum value of the mean anisotropy level a defined by (21) under norm ratio constraint (22). Since $\overline{\mathbf{A}}(G)$ and the power norm ratio $\|FG\|_2/\|G\|_2$ are invariant with respect to scalar multiplication of G [Diamond et al., 2001], we can assume $\|G\|_2 = 1$ without loss of generality. Consider the problem

$$\overline{\mathbf{A}}(G) \to \sup_{G\in\mathcal{G}_a}, \tag{23}$$

$$\|FG\|_2^2 - \gamma^2\|G\|_2^2 \leqslant 0, \tag{24}$$

$$\|G\|_2 = 1. \tag{25}$$

By virtue of Karush-Kuhn-Tucker Theorem, the problem of searching conditional extremum (23), (24) is equivalent to the problem of finding unconditional extremum of corresponding Lagrangian. Let us write the Lagrangian for this problem as

$$\begin{aligned}\mathcal{L}(F,S,\lambda) &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det(m\widehat{G}(\omega)\widehat{G}^*(\omega)(\omega))d\omega \\ &+ \lambda(\|FG\|_2^2 - \gamma^2\|G\|_2^2) \\ &+ \mu(\|G\|_2^2 - 1) \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det(mS(\omega))d\omega \\ &+ \frac{\lambda}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\,(\Lambda(\omega)S(\omega))\,d\omega \\ &- \frac{\lambda\gamma^2}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\,S(\omega)d\omega \\ &+ \frac{\mu}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\,S(\omega)d\omega,\end{aligned} \tag{26}$$

where $\Lambda(\omega) = \widehat{F}^*(\omega)\widehat{F}(\omega)$, $S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega)$ is the spectral density of the random input sequence, $\lambda, \mu \geqslant 0$ are the Langrange multipliers. The first Frechet variation of the functional $\mathcal{L}(F,S,\lambda)$ is given by

$$\begin{aligned}\delta\mathcal{L}(F,S,\lambda) &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\mathrm{tr}\left(S^{-1}(\omega)\delta S(\omega)\right)d\omega \\ &+ \frac{\lambda}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}(\Lambda(\omega)\delta S(\omega))d\omega \\ &- \frac{\lambda\gamma^2-\mu}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\,\delta S(\omega)d\omega,\end{aligned} \tag{27}$$

where $\delta S(\omega)$ is the spectral density variation represented by a Hermitian $(m\times m)$-matrix. The necessary condition of functional (26) extremum is given by

$$\frac{1}{2}S^{-1}(\omega) + \lambda\Lambda(\omega) - (\lambda\gamma^2 - \mu)I_m = 0. \tag{28}$$

The second Frechet variation of functional (26) is

$$-\frac{1}{4\pi}\int_{-\pi}^{\pi}\mathrm{tr}(\delta S(\omega)S^{-2}(\omega)\delta S(\omega))d\omega < 0,$$

i.e. only local maximum can be attained in stationary points.

From (28) we obtain

$$\begin{aligned}S^{-1}(\omega) &= 2(\lambda\gamma^2 - \mu)I_m - 2\lambda\Lambda(\omega) \\ &= \frac{2\lambda}{q}(I_m - q\Lambda(\omega)),\end{aligned}$$

where $q \triangleq \left(\gamma^2 - \frac{\mu}{\lambda}\right)^{-1} = \lambda(\lambda\gamma^2-\mu)^{-1}$. Denoting $\sigma \triangleq \frac{q}{2\lambda}$, we can define the worst-case spectral density of the random input sequence W :

$$S_\star(\omega) = \sigma\left(I_m - q\Lambda(\omega)\right)^{-1}, \quad -\pi \leqslant \omega < \pi. \tag{29}$$

Note that for any $q \in [0, \|F\|_\infty^{-2})$ the function $S_\star(\omega)$ exists and is positive definite.

From the sufficient complementary slackness conditions for optimization problem (23)–(25) it follows that for $\lambda, \mu > 0$

$$\|FG_\star\|_2^2 = \gamma^2, \tag{30}$$

$$\|G_\star\|_2^2 = 1. \tag{31}$$

Now obtain a representation of the worst-case input random sequence in the state space. Let us factorize expression (28). It is not hard to see that (28) can be written as

$$\Theta^*(\omega)\Theta(\omega) = I_m, \tag{32}$$

where

$$\Theta = \begin{bmatrix}\sqrt{q}\widehat{F}(\omega) \\ \sqrt{\sigma}\widehat{G}_\star^{-1}(\omega)\end{bmatrix} \in \mathcal{H}_2^{2m\times m}$$

and $\widehat{G}_\star(\omega)\widehat{G}_\star^*(\omega) = \widehat{G}_\star^*(\omega)\widehat{G}_\star(\omega) = S_\star(\omega)$ is the factorization of the worst-case spectral density $S_\star(\omega)$. Condition (32) means that the transfer function $\Theta(z)$ is an inner one, see e.g. [Zhou et al., 1996].

Let the worst-case input random sequence $W_\star$ with spectral density $S_\star(\omega)$ defined by formula (29) have the state-space representation $w_k^\star = Lx_k + \sigma^{-1/2}\widetilde{\Sigma}^{1/2}v_k$, where the matrices L and $\widetilde{\Sigma} \succ 0$ are to be determined. Then the realization of the worst-case shaping filter $G_\star(z) \in \mathcal{H}_2^{m\times m}$ is given by

$$G_\star(z) \sim \left[\begin{array}{c|c} A+BL & \sigma^{-1/2}B\widetilde{\Sigma}^{1/2} \\ \hline L & \sigma^{-1/2}\widetilde{\Sigma}^{1/2} \end{array}\right].$$

It is not hard to determine by straightforward calculation that the realization of the system $\Theta(z)$ is given by

$$\Theta \sim \left[\begin{array}{c|c} A & B \\ \hline \Gamma & \Delta \end{array}\right], \qquad (33)$$

where $\Gamma = \begin{bmatrix} \sqrt{q}C \\ -\sqrt{\sigma}\widetilde{\Sigma}^{-1/2}L \end{bmatrix}, \Delta = \begin{bmatrix} \sqrt{q}D \\ \sqrt{\sigma}\widetilde{\Sigma}^{-1/2} \end{bmatrix}$.

Since the system $\Theta(z)$ is inner, its realization (33) obeys the equation system

$$A^{\mathrm{T}}RA - R + \Gamma^{\mathrm{T}}\Gamma = 0, \qquad (34)$$
$$B^{\mathrm{T}}RA + \Delta^{\mathrm{T}}\Gamma = 0, \qquad (35)$$
$$B^{\mathrm{T}}RB + \Delta^{\mathrm{T}}\Delta = I_m, \qquad (36)$$

see [Gu et al., 1989]. Substituting expressions

$$\begin{aligned} \Delta^{\mathrm{T}}\Delta &= qD^{\mathrm{T}}D + \sigma\widetilde{\Sigma}^{-1}, \\ \Gamma^{\mathrm{T}}\Gamma &= qC^{\mathrm{T}}C + \sigma L^{\mathrm{T}}\widetilde{\Sigma}^{-1}L, \\ \Delta^{\mathrm{T}}\Gamma &= qD^{\mathrm{T}}C - \sigma\widetilde{\Sigma}^{-1}L \end{aligned}$$

to equations (34)–(36), we obtain

$$\begin{aligned} R &= A^{\mathrm{T}}RA + qC^{\mathrm{T}}C + \sigma L^{\mathrm{T}}\widetilde{\Sigma}^{-1}L, \\ L &= \sigma^{-1}\widetilde{\Sigma}\left(B^{\mathrm{T}}RA + qD^{\mathrm{T}}C\right), \\ \sigma^{-1}\widetilde{\Sigma} &= \left(I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D\right)^{-1}. \end{aligned}$$

These equations can be rewritten in form (15)–(17) using notation $\Sigma(q) \triangleq \sigma(q)^{-1}\widetilde{\Sigma}(q)$. Note that in conditions of the theorem for any $q \in [0, \|F\|_\infty^{-2})$ there exists a unique stabilizing solution $R \succ 0$ to algebraic Riccati equation (15)–(17) such that the matrix $A + BL$ is stable ($\rho(A + BL) < 1$) and the matrix $\Sigma \succ 0$; see [Vladimirov et al., 1996-1; Diamond et al., 2001] for details.

Then, let us express equalities (30), (31) in terms of realization matrices of the weighted system $F(z)G_\star(z) = (A + BL, B\Sigma^{1/2}, C + DL, D\Sigma^{1/2})$ and the worst-case shaping filter $G_\star(z) = (A + BL, B\Sigma^{1/2}, L, \Sigma^{1/2})$. As is well known, see e.g. [Zhou et al., 1996],

$$\begin{aligned} \|FG_\star\|_2^2 &= \mathrm{tr}((C + DL)P(C + DL)^{\mathrm{T}} \qquad (37) \\ &\quad + D\Sigma D^{\mathrm{T}}), \\ \|G_\star\|_2^2 &= \mathrm{tr}\left(LPL^{\mathrm{T}} + \Sigma\right), \qquad (38) \end{aligned}$$

where $P \succ 0$ is the controllability Gramian satisfying the Lyapunov equation (18). Since the matrix $A + BL$ is stable and $\Sigma \succ 0$, then there exists a unique solution $P \succ 0$ to Lyapunov equation (18), see e.g. [Zhou et al., 1996]. From the fact that $\|G\|_2 > 0$ it follows that substituting expressions (37), (38) to equalities (30), (31) yields equations (19), (20).

As is known from [Vladimirov et al., 1996-1; Diamond et al., 2001], the level a of the mean anisotropy of the worst-case input sequence $W_\star$ generated by the worst-case shaping filter $G_\star(z)$ is defined by formula (21) that completes the proof.

Remark 1. *The levels a_s and a_t of the spatial and temporal parts [Diamond et al., 2001] of the mean anisotropy a of the worst-case input sequence $W_\star$ generated by the worst-case shaping filter $G_\star(z)$ are defined by formulas*

$$a_s = -\frac{1}{2}\ln\det((LPL^{\mathrm{T}} + \Sigma)(LRL^{\mathrm{T}} + \Sigma)^{-1} \qquad (39)$$
$$a_t = \frac{1}{2}\ln\det(m(LPL^{\mathrm{T}} + \Sigma)), \qquad (40)$$

respectively.

4 Numerical Example

To illustrate application of Theorem 1, let us briefly consider a simple numerical example. Let the state-space realization of the stable second-order system be given by

$$F(z) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0.3 & 0.6 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]. \qquad (41)$$

According to (12), the anisotropic norm of system (41) varies from $|||F|||_0 = \frac{1}{\sqrt{m}}\|F\|_2 = 2.4525$ for $a = 0$ to $\|F\|_\infty = 15.891$ with $a \to +\infty$. Therefore it makes sense to search the numerical solution to equation system (15)–(20) only for $\gamma \in \left[m^{-1/2}\|F\|_2, \|F\|_\infty - \epsilon\right]$, where ϵ is rather small. In practical computations we choose $\epsilon = 10^{-3}$. To solve the resulting equation system, the quite standard numerical methods of the Matlab Control System Toolbox is applied for consecutively increasing γ starting from $\gamma = m^{-1/2}\|F\|_2$. The mean anisotropy level to be defined is computed by formula (21), the spatial and temporal parts of the mean anisotropy are calculated by formulas (39) and (40).

The results of numerical solution are represented in Fig. 1 and 2. The mean anisotropy $a(\gamma)$, as well as its temporal and spatial parts $a_t(\gamma)$, $a_s(\gamma)$ are plotted in the top of Fig. 1, where the black coloured line corresponds to $a(\gamma)$, blue and red lines correspond to $a_t(\gamma)$ and $a_s(\gamma)$, respectively. In the bottom of Fig. 1, the parameter $q(\gamma)$ is shown. Fig. 2 illustrates changing of

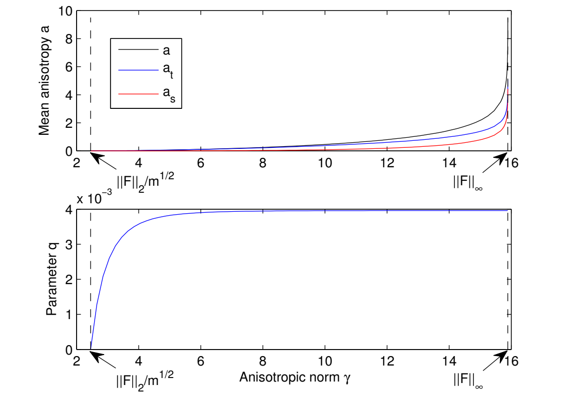

Figure 1. Mean anisotropy $a(\gamma)$, its temporal and spatial parts $a_t(\gamma)$, $a_s(\gamma)$, and parameter $q(\gamma)$

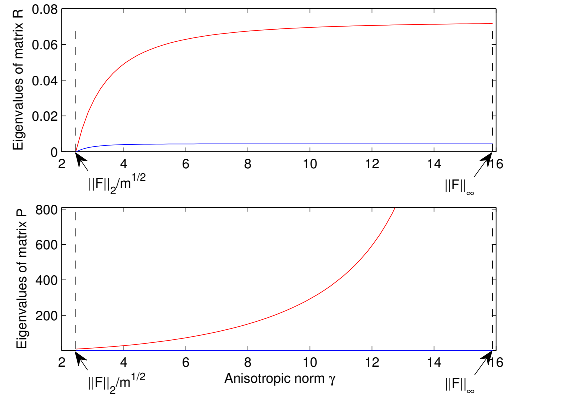

Figure 2. Eigenvalues of matrices R and P

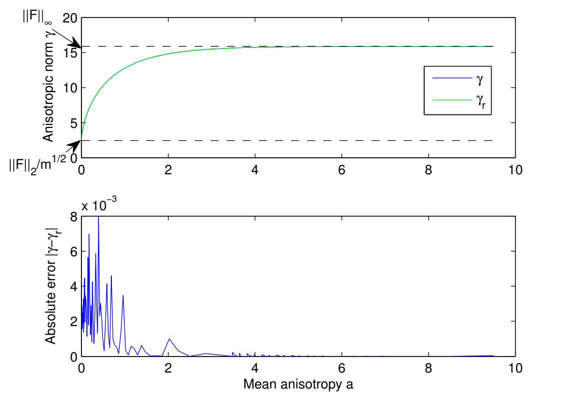

Figure 3. Anisotropic norm values $\gamma(a)$, $\gamma_r(a)$, and absolute error $|\gamma - \gamma_r|$

the eigenvalues of the matrices R (top diagram) and P (lower diagram) depending on γ.

The obtained values a_i of the mean anisotropy level were used for computing the anisotropic norm $\|F\|_{a_i} = \gamma_r(a_i)$ according to [Vladimirov et al., 1996-1; Diamond et al., 2001]. The graphs of $\gamma_r(a)$ and $\gamma(a)$ are plotted in the top of Fig. 3. The graph of absolute error $|\gamma(a) - \gamma_r(a)|$ is given in the bottom of Fig. 3.

5 Conclusion

A solution to the inverse problem of the anisotropy-based performance analysis has been obtained. The inverse problem consists in finding the level of the mean anisotropy of the random input sequence that corresponds to some given value of the system anisotropic norm. It is shown that the system anisotropic norm equals to a given value belonging to some admissible range if there exists a solution to a system of four cross-coupled matrix nonlinear algebraic equations. The solution to the inverse problem is uniquely defined by the solution to this equation system.

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