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FULL SUPERPOSITION PRINCIPLE IS INCONSISTENT WITH NON-DETERMINISTIC VERSIONS OF QUANTUM PHYSICS

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Abstract
Many practical systems are non-deterministic, in the sense that available information about the initial states and control values does not uniquely determine the future states. For some such systems, it is important to take quantum effects into account. For that, we need to develop non-deterministic versions of quantum physics. In this paper, we show that for non-deterministic versions of quantum physics, we cannot require *superposition principle* – one of the main fundamental principles of modern quantum mechanics. Specifically, while we can consider superpositions of states corresponding to the same version of the future dynamics, it is not consistently possible to consider superpositions of states corresponding to different versions of the future.

1 Why Non-Deterministic Versions of Quantum Physics

Traditional control theory has been developed to control deterministic systems, in which the initial state and the control values uniquely determine all future states of the system. In many practical situations, however, physical systems are non-deterministic – e.g., as a result of partial observability of events. Special modifications of traditional control techniques have been invented for such non-deterministic systems; see, e.g., [Ramadge and Wonham, 1987; Overkamp, 1994; Heymann and Lin, 1998] and references therein.

In many practical situations, we need to take quantum effects into account. For non-deterministic systems, this means that we need to consider non-deterministic versions of quantum physics. The usual quantum physics is deterministic, in the sense that once we know the initial state $\psi(t_0)$, we can uniquely predict the state $\psi(t)$ at any future moment of time $t > t_0$ and thus, we can uniquely predict the probabilities of different future measurement results. It is therefore necessary to consider *non-deterministic* versions of quantum physics, in which for the same initial state $\psi(t_0)$ we may have several different possible states $\psi(t) \neq \psi'(t)$ at a future moment of time t.

2 How to Describe Non-Deterministic Versions of Quantum Physics: Superposition Principle

How can we describe non-deterministic versions of quantum physics? It is definitely necessary to make sure that this description satisfies fundamental principles of quantum physics. One of such fundamental principles is the *superposition principle*. There are many ways to formulate this principle. To be able to apply it to non-deterministic situations, let us formulate this principle in such a way that would not depend on the deterministic character of dynamics.

The traditional formulations of the superposition principle use the fact that the states ψ of a quantum system are unit vectors in a complex-valued Hilbert space. In the non-relativistic quantum mechanics, which studies systems with a fixed number of particles, states are complex-valued functions $\psi(x)$ for which $\int |\psi(x)|^2\,dx = 1$. In relativistic quantum mechanics, the basis of the corresponding Hilbert space include states corresponding to different number of particles; in quantum field theory, states are even more complicated – since they describe fields. In all these cases, we have a Hilbert space, i.e., a linear space in which addition of elements (vectors) and multiplication of its elements by a complex number are well defined, and there is a (bilinear) form $\langle x, y\rangle$ for which:

- $\langle a \cdot x + a' \cdot x', y\rangle = a \cdot \langle x, y\rangle + a' \cdot \langle x', y\rangle$,
- $\langle x, a \cdot y + a' \cdot y'\rangle = \overline{a} \cdot \langle x, y\rangle + \overline{a'} \cdot \langle x, y'\rangle$ (where $\overline{z}$ means complex conjugate),

- $\langle y, x\rangle = \overline{\langle x, y\rangle}$,
- $\langle x, x\rangle = 0$, and
- $\langle x, x\rangle > 0$ for $x \neq 0$.

The bilinear norm defines a *norm* $\|x\| \stackrel{\text{def}}{=} \sqrt{\langle x, x\rangle}$.

Definition 1. *Let H be a Hilbert space. By a* state, *we mean a unit vector in H.*

In these terms, the superposition principle can be formulated as follows. Let $\psi(t_0)$ and $\psi'(t_0)$ be states for which the quantum physics predicts future states $\psi(t)$ and $\psi'(t)$, and let a and a' be complex numbers for which

$$\psi''(t_0) \stackrel{\text{def}}{=} a \cdot \psi(t_0) + a' \cdot \psi'(t_0)$$

is also a state. Then, if we start with the initial state $\psi''(t_0)$, at the moment $t > t_0$, we get a state

$$\psi''(t) = a \cdot \psi(t) + a' \cdot \psi'(t).$$

In physical terms, superposition principle means that if we start with a superposition

$$\psi''(t_0) = a \cdot \psi(t_0) + a' \cdot \psi'(t_0)$$

of the states $\psi(t_0)$ and $\psi'(t_0)$, then at every future moment of time $t > t_0$, we still get a superposition

$$\psi''(t) = a \cdot \psi(t) + a' \cdot \psi'(t)$$

of the corresponding states $\psi(t_0)$ and $\psi'(t_0)$.

It is sufficient to restrict ourselves to the case when the states $\psi(t_0)$ and $\psi'(t_0)$ are orthogonal to each other: $\psi(t_0) \perp \psi'(t_0)$, i.e., $\langle \psi(t_0), \psi'(t_0)\rangle = 0$. In this case, the requirement that a linear combination

$$a \cdot \psi(t_0) + a' \cdot \psi'(t_0)$$

is also a state – i.e., that it is a unit vector – means that

$$|a|^2 \cdot \|\psi(t_0)\|^2 + |a'|^2 \cdot \|\psi'(t_0)\|^2 = |a|^2 + |a'|^2 = 1.$$

The above formulation assumes that the future state is uniquely determined by the original state. To be able to apply this principle to possible non-deterministic versions of quantum physics, we need to reformulate this principle in such a way that it does not depend on whether the underlying theory is deterministic or not.

In a non-deterministic theory, a state ψ_0 at the moment t_0 does not, in general, uniquely determine the state ψ_1 at the moment $t > t_0$; for each ψ_0, we may have different states ψ_1. A theory must then describe which pairs (ψ_0, ψ_1) are possible are which are not. The only restriction is that for each initial state ψ_0, we must have at least one possible future state ψ_1. Thus, we arrive at the following definition:

Definition 2. *Let $t_0 < t_1$ be two real numbers; these numbers will be called* moments of time.

- *By* dynamics $D(t_0 \to t_1)$ *corresponding to these two moments of time, we mean a set of pairs of states (ψ_0, ψ_1) such that for every state ψ_0, there is a state ψ_1 for which $(\psi_0, \psi_1) \in D(t_0 \to t_1)$.*
- *When $(\psi_0, \psi_1) \in D(t_0 \to t_1)$, we say that* it is possible to have a state ψ_0 at moment t_0 and a state ψ_1 at moment t_1, *or, in short, that* a transition from ψ_0 to ψ_1 is possible. *Alternatively, we will denote the possibility of such a transition as $\psi_0 \to \psi_1$.*

In the traditional (deterministic) quantum physics, where the next state ψ_1 is uniquely determined by the previous state ψ_0 as $\psi_1 = U\psi_0$ for an appropriate operator U, the above-defined dynamics takes the form $D(t_0 \to t_1) = \{(\psi_0, U\psi_0)\}$, i.e., it coincides with the (graph of) the operator U.

Definition 3. *We say that a dynamics $D(t_0 \to t_1)$ is* deterministic *if for every state ψ_0, there exists exactly one state ψ_1 for which a transition from ψ_0 to ψ_1 is possible.*

In the general (not necessarily deterministic) case, it is natural to formulate the superposition principle as follows:

Definition 4. *We say that a dynamics $D(t_0 \to t_1)$* satisfies the superposition principle *if it satisfies the following property: for every four states $\psi_0 \perp \psi_0'$, ψ_1, and ψ_1' for which transitions from ψ_0 to ψ_1 and from ψ_0' to ψ_1' are possible, and for every two complex numbers a and a' for which $|a|^2 + |a'|^2 = 1$, the combination $\psi_1'' = a \cdot \psi_1 + a' \cdot \psi_1'$ is also a state, and a transition from $\psi_0'' = a \cdot \psi_0 + a' \cdot \psi_0'$ to ψ_1'' is also possible.*

Comment. For the deterministic case, this formulation is equivalent to the above-presented usual formulation of the superposition principle.

3 Main Result

Here is our unexpected result:

Theorem. *If a dynamics $D(t_0 \to t_1)$ satisfies the superposition principle, then it is deterministic.*

Proof. Let us assume that the dynamics $D(t_0 \to t_1)$ satisfies the superposition principle. We will prove that for any state ψ_0, if there is a transition from ψ_0 to ψ_1 and a transition from ψ_0 to φ_1, then $\psi_1 = \varphi_1$.

To prove this, let us select any unit vector orthogonal to ψ_0 and denote it by ψ_0'. By definition of the dynamics, there exists at least one state for which a transition from ψ_0' to this state is possible; let us select one of these states and denote it by ψ_1'.

By superposition principle, since the vectors ψ_0 and ψ'_0 are orthogonal, and since it is possible to have transitions $\psi_0 \to \psi_1$ and $\psi'_0 \to \psi'_1$, the transition

$$\varphi_+ \stackrel{\rm def}{=} \frac{1}{\sqrt{2}}\cdot\psi_0+\frac{1}{\sqrt{2}}\cdot\psi'_0 \to \frac{1}{\sqrt{2}}\cdot\psi_1+\frac{1}{\sqrt{2}}\cdot\psi'_1 \quad (1)$$

is also possible. Similarly, since it is possible to have transitions $\psi_0 \to \varphi_1$ and $\psi'_0 \to \psi'_1$, the transition

$$\varphi_- \stackrel{\rm def}{=} \frac{1}{\sqrt{2}}\cdot\psi_0-\frac{1}{\sqrt{2}}\cdot\psi'_0 \to \frac{1}{\sqrt{2}}\cdot\varphi_1-\frac{1}{\sqrt{2}}\cdot\psi'_1 \quad (2)$$

is also possible.

One can easily check that the vectors

$$\varphi_+ = \frac{1}{\sqrt{2}}\cdot\psi_0+\frac{1}{\sqrt{2}}\cdot\psi'_0 \text{ and } \varphi_- = \frac{1}{\sqrt{2}}\cdot\psi_0-\frac{1}{\sqrt{2}}\cdot\psi'_0$$

are orthogonal, and that

$$\frac{1}{\sqrt{2}}\cdot\varphi_+ + \frac{1}{\sqrt{2}}\cdot\varphi_- = \frac{1}{\sqrt{2}}\cdot\left(\frac{1}{\sqrt{2}}\cdot\psi_0+\frac{1}{\sqrt{2}}\cdot\psi'_0\right)+$$

$$\frac{1}{\sqrt{2}}\cdot\left(\frac{1}{\sqrt{2}}\cdot\psi_0-\frac{1}{\sqrt{2}}\cdot\psi'_0\right)=$$

$$\left(\frac{1}{2}+\frac{1}{2}\right)\cdot\psi_0+\left(\frac{1}{2}-\frac{1}{2}\right)\cdot\psi'_0=\psi_0.$$

Thus, from the possibility of the transitions (1) and (2), by using the superposition principle, we can conclude that

$$\psi_0 \to \frac{1}{\sqrt{2}}\cdot\left(\frac{1}{\sqrt{2}}\cdot\psi_1+\frac{1}{\sqrt{2}}\cdot\psi'_1\right)+$$

$$\frac{1}{\sqrt{2}}\cdot\left(\frac{1}{\sqrt{2}}\cdot\varphi_1-\frac{1}{\sqrt{2}}\cdot\psi'_1\right)=$$

$$\frac{1}{2}\cdot\psi_1+\frac{1}{2}\cdot\varphi_1+\frac{1}{2}\cdot\psi'_1-\frac{1}{2}\cdot\psi'_1=\frac{1}{2}\cdot\psi_1+\frac{1}{2}\cdot\varphi_1.$$

Thus, the combination

$$s \stackrel{\rm def}{=} \frac{1}{2}\cdot\psi_1+\frac{1}{2}\cdot\varphi_1$$

should be a state, i.e., a unit vector in the Hilbert space. It is known that in a Hilbert space (just like in a Euclidean space), for every two vectors x and y, we have $\|x+y\| \le \|x\|+\|y\|$, and the only possibility to have $\|x+y\|=\|x\|+\|y\|$ is when the vectors are collinear, i.e., when $y=\lambda\cdot x$ for some $\lambda>0$. For $x=\frac{1}{2}\cdot\psi_1$ and $y=\frac{1}{2}\cdot\varphi_1$, we have

$$\|x\|=\frac{1}{2}\cdot\|\psi_1\|=\frac{1}{2},\quad \|y\|=\frac{1}{2}\cdot\|\varphi_1\|=\frac{1}{2},$$

and thus, $1=\|s\|=\|x+y\|=\|x\|+\|y\|$. So, we conclude that $y=\lambda\cdot x$ for some $\lambda>0$. For the norms, we thus have $\|y\|=\lambda\cdot\|x\|$ and, since $\|x\|=\|y\|=\frac{1}{2}$, we conclude that $\lambda=1$ and $y=x$. From $y=\frac{1}{2}\cdot\varphi_1=\frac{1}{2}\cdot\psi_1=x$, we conclude that $\psi_1=\varphi_1$. The statement is proven.

4 Discussion

In the traditional (deterministic) quantum physics, all the future states correspond to a single version of the future. Superposition principle enables us to consider superpositions of such states. The fact that numerous experiments confirm the predictions of quantum physics support such superpositions.

When we go from the traditional (deterministic) quantum physics to a non-deterministic version, we also add states corresponding to alternative versions of the future. At first glance, it seems reasonable to extend the usual superposition principle to such states, and to allow not only superpositions of states from the same version of the future, but also superpositions of states from different alternative futures. Our result shows that such an extension is not possible: it is not possible to consider superpositions of states corresponding to different alternative futures. In other words, to consider non-deterministic version of quantum physics, we have to impose restrictions on the superposition principle.

5 Philosophical Comment

Our motivation was based on potential control applications; however, this result may also be of foundational interest, since some researchers consider non-determinism to be a natural consequence of the intuitive idea of freedom of will.

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