TIME-REVERSAL AND STRONG H-THEOREM FOR QUANTUM DISCRETE-TIME MARKOV CHANNELS

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Abstract
The time reversal of a completely-positive, nonequilibrium discrete-time quantum Markov evolution is derived via a suitable adjointness relation. Space-time harmonic processes are introduced for the forward and reverse-time transition mechanisms, and their role for relative entropy dynamics is discussed.

Key words
Quantum channel, time reversal, space-time harmonic process, operator Jensen inequality, H-theorem.

1 Introduction

In this paper, we develop a mathematical framework for discrete-time Markovian processes originating from Nelson's kinematics of diffusion processes [Nelson, 1958; Nelson, 1967]. *Time-reversal* for Markovian evolutions entails the Lagrange adjoint with respect to the (semi-definite) inner product induced by the flow of probability distributions. We show that this also holds for finite-dimensional, discrete time quantum Markov evolutions. Hence, the time-reversal of a discrete-time quantum Markov process emerges from a structure of kinematical nature that is common to all Markovian equilibrium and non-equilibrium evolutions. We discuss the structure Nelson's kinematics [Nelson, 1967] for discrete-time processes, and we apply these ideas to derive the reverse-time transition mechanism of a Markov chain via a certain adjointness relation on space-time. This is needed as a starting point for deriving the time reversal in the quantum case. Time reversal of Markov transitions is deeply involved in the solution of certain *maximum entropy problems* on path space [Pavon and Ticozzi, 2008], and in deriving a strong form of the *H-theorem* for Markov channels we illustrate in this paper. In proving this, we also show that a key role is played by a suitable class of quantum *space-time harmonic processes*, that are related to martingale processes. This paper is a shortened version of [Ticozzi and Pavon, 2008], to which we refer for the proofs and additional discussions.

2 Elements of Nelson's kinematics for discrete-time stochastic processes

Let $I = [t_0, t_1]$ be a discrete-time interval with $-\infty < t_0 < t_1 < \infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\mathcal{F}_t^-\}, t \in I$, be a nondecreasing family of σ-algebras of events (filtration) representing a flow of information. Let $X : I \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a second order stochastic process such that $X(t)$ is $\{\mathcal{F}_t^-\}$-measurable, for all $t \in I$. Then the *conditional forward difference* of X is defined by

$$\Delta^+ X(t) = \mathbb{E}(X(t+1) - X(t)|\mathcal{F}_t^-).$$

Consider now a nonincreasing family of σ-algebras of events $\{\mathcal{F}_t^+\}, t \in I$, and suppose that $X(t)$ is $\{\mathcal{F}_t^+\}$-measurable, $\forall t \in I$. Then the *conditional backward diference* of X is defined by

$$\Delta^- X(t) = \mathbb{E}(X(t-1) - X(t)|\mathcal{F}_t^+).$$

Observe that both $\Delta^+ X(t), \Delta^- X(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, $\forall t$. A process satisfying $\Delta^+ X(t) = 0, \forall t \in I$ is called a $\{\mathcal{F}_t^-\}$*-martingale* if $\Delta^+ X(t) = 0, \forall t \in I$, namely if

$$\mathbb{E}(X(t+1)|\mathcal{F}_t^-) = X(t), \quad a.s. \qquad (1)$$

It is called a *reverse-time,* $\{\mathcal{F}_t^+\}$*-martingale* if $\Delta^- X(t) = 0, \forall t \in I$, namely if

$$\mathbb{E}(X(t-1)|\mathcal{F}_t^+) = X(t), \quad a.s. \qquad (2)$$

If $\Delta^+ X(t) \geq 0$ or $\Delta^- X(t) \geq 0$, $\forall t \in I$ then $X(t)$ is called a $\{\mathcal{F}_t^+\}$ *submartingale* and a $\{\mathcal{F}_t^-\}$ *reverse-time submartingale*, respectively. We can say that a martingale is *conditionally constant* and a submartingale is *conditionally increasing*. Notice that, by iterated conditioning, if $X(t), t \in I$ is a $\{\mathcal{F}_t^-\}$-martingale and $Y(t), t \in I$ is a $\{\mathcal{F}_t^-\}$-submartingale, then

$$\begin{aligned} &\mathbb{E}X(s) = \mathbb{E}X(t), \quad \forall\, s, t \in I, \\ &\mathbb{E}Y(s) \le \mathbb{E}X(t), \quad \forall\, s < t \in I. \end{aligned} \tag{3}$$

Similarly, for reverse time (sub)martingales.

Consider now the family $\mathcal{H}(t_0, t_1)$ of second order stochastic processes X such that $X(t)$ is simultaneously $\{\mathcal{F}_t^-\}$-measurable and $\{\mathcal{F}_t^+\}$-measurable, $\forall t \in I$. We then have the discrete-time analogue of Nelson's integration by parts formula [Nelson, 1967, p.80].

Theorem 2.1. *Let* $X, Y \in \mathcal{H}(t_0, t_1)$. *Then*

$$\begin{aligned} &\mathbb{E}\left(X(t_1)Y(t_1) - X(t_0)Y(t_0)\right) \\ &= \sum_{t_0}^{t_1 - 1} \mathbb{E}\left(\Delta^+ X(t) Y(t) - X(t+1)\Delta^- Y(t+1)\right). \end{aligned} \tag{4}$$

3 Kinematics of Markov chains and space-time harmonic processes

Consider a Markov chain $\{X(t), t \in \mathbb{Z}\}$ taking values in the finite set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ which we identify from here on with the set of the indexes $\{1, 2, \ldots, n\}$. We denote by π_t the probability distribution of $X(t)$ over $\mathcal{X}$. In the following, π_t is always intended as a column vector, with i-th component $\pi_t(i) = \mathbb{P}(X(t) = i)$. Let $P(t)$ denote the *transition matrix* with elements $p_{ij}(t) = \mathbb{P}(X(t+1) = j | X(t) = i), i, j = 1, \ldots, n$. The matrix $P(t)$ is *stochastic*, namely

$$p_{ij}(t) \ge 0, \forall i, \forall j, \quad \sum_j p_{ij}(t) = 1, \forall i.$$

Let us agree that troughout the paper † indicates adjoint with respect to the natural inner product. Hence, in the case of matrices, it denotes transposition and, in the complex case below, transposition plus conjugation. The evolution is then given by the *forward* equation

$$\pi_{t+1} = P^\dagger(t)\pi_t. \tag{5}$$

When P does not depend on time, the chain is called *time-homogeneous*. A distribution $\bar{\pi}$ is called *stationary* for the time-homogeneous Markov chain X with transition matrix P if it satisfies

$$\bar{\pi} = P^\dagger \bar{\pi}. \tag{6}$$

For x and y n-dimensional column vectors, we define the semi-definite form:

$$\langle x, y\rangle_{\pi_t} = x^\dagger D_{\pi_t} y, \tag{7}$$

which is an inner product if $D_\pi = \operatorname{diag}(\pi_t(1), \pi_t(2), \ldots, \pi_t(n))$ is positive definite. It represents the expectation of the random variable Z defined on $(\mathcal{X}, \pi_t)$ by $Z(i) = x_i y_i$. In what follows, whenever a matrix M is not invertible, M^{-1} is to be understood as the generalized (Moore-Penrose) inverse $M^\#$, cf. [Horn and Johnson, 1990].

3.1 Space-time inner product and time-reversal

Let $\mathcal{F}_t^-, t \in \mathbb{Z}$ be the σ-algebra generated by $\{X(s), s \le t\}$ and $\mathcal{F}_t^+$ to be the σ-algebra generated by $\{X(s), s \ge t\}$. Let $f : \mathbb{Z} \times \mathcal{X} \to \mathbb{R}$. Let us compute the *forward difference* $\Delta^+ f(t, X(t))$ with respect to the family $\{\mathcal{F}_t^-\}, t \ge 0$, following Appendix 2. We have that

$$\begin{aligned} &\Delta^+ f(t, X(t))_{|X(t)=i} \\ &= \mathbb{E}(f(t+1, X(t+1)) - f(t, X(t)) | X(t) = i) \\ &= \sum_j f(t+1, j) p_{ij}(t) - f(t, i). \end{aligned} \tag{8}$$

Henceforth, we shall denote by f_t and $\Delta^+ f_t$ the column vectors with i-th component $f(t, i)$ and $\Delta^+ f(t, X(t))_{|X(t)=i}$, respectively. We can then rewrite (8) in the compact form

$$\Delta^+ f_t = P(t) f_{t+1} - f_t. \tag{9}$$

Consider now the vector space $\mathcal{K} = \{f : \mathbb{Z} \times \mathcal{X} \to \mathbb{R} \,|\, \exists\, t_0, t_1, t_0 \le t_1 \text{ s. t. } f(t, i) = 0, \forall i, t \notin [t_0, t_1]\}$, namely the set of functions with finite support. For $f, g \in \mathcal{K}$, we define the semi-definite *space-time inner product* as

$$\begin{aligned} \langle f, g\rangle_\pi &= \sum_{t=-\infty}^{\infty} \langle f_t, g_t\rangle_{\pi_t} = \sum_{t=-\infty}^{\infty} f_t^\dagger D_{\pi_t} g_t, \\ &= \sum_{t=-\infty}^{\infty} \mathbb{E}(f(t, X(t))\, g(t, X(t))), \end{aligned} \tag{10}$$

where $\pi \sim \{\pi_t, t \in \mathbb{Z}\}$ denotes the family of the Markov chain distributions. We then have the following Corollary to the "integration by parts" formula of Theorem 2.1.

Corollary 3.1. *Let* $f, g \in \mathcal{K}$. *Then*

$$\langle \Delta^+ f, g\rangle_\pi = \langle f, \Delta^- g\rangle_\pi \tag{11}$$

In view of relation (11), we call Δ^- a $\langle\cdot,\cdot\rangle_\pi$-adjoint of Δ^+. Hence, the two conditional differences are *adjoint* with respect to the semi-definite *space-time inner product*. On the other hand, by using (9) and some straightforward calculations, we get

$$\sum_{t=-\infty}^{\infty} \mathbb{E}(\Delta^+ f(t,X(t))\, g(t,X(t))) =$$
$$= \sum_{t=-\infty}^{\infty} \langle f_{t+1}, D_{\pi_{t+1}}^{-1} P^\dagger(t) D_{\pi_t} g_t - g_{t+1}\rangle_{\pi_{t+1}}$$

Let $\pi_t(i) > 0$ for all t, i. In this case, (10) is an inner product and the corresponding adjoint is unique. We conclude that $\Delta^- g_{t+1} = D_{\pi_{t+1}}^{-1} P^\dagger(t) D_{\pi_t} g_t - g_{t+1}$. More explicitly, defining the matrices $Q(t) = D_{\pi_{t+1}}^{-1} P^\dagger(t) D_{\pi_t}$, we have that (component-wise):

$$\begin{aligned}&\Delta^- g(t+1, X(t+1))_{|X(t+1)=j}\\ &= \mathbb{E}(g(t,X(t) - g(t+1,X(t+1))|X(t+1)=j)\\ &= \sum_i g(t,i) q_{ji}(t) - g(t+1,j). \qquad (12)\end{aligned}$$

Hence, $Q(t)$ is simply the matrix of the reverse-time transition probabilities.*.

Two remarks are in order: (i) The backward transitions are time-dependent even when the forward are not. (ii) When $\pi_{t+1}(j) = 0$, $q_{ji}(t)$ may be defined arbitrarily to be any number between zero and one without actually affecting relation (13), provided it satisfies the normalization condition $\sum_i q_{ji}(t) = 1$. Notice then that (12) leads to the correct form of the time-reversal even if the distributions $\{\pi_t\}$ are only nonnegative. The derivation of Q using the Δ^- operator permits to see that the reverse time transition mechanism may be viewed as a space-time adjoint to the forward one with respect to the flow of probability distributions $\{\pi_t, t \in \mathbb{Z}\}$. The space-time adjointness relation (11) for Markov chains admits an equivalent, compact formulation.

Proposition 3.1. *The space-time adjointness relation* (11) *holds if and only if the two-time relation*

$$\langle P(t)x, y\rangle_{\pi_t} = \langle x, Q(t)y\rangle_{\pi_{t+1}}, \quad x, y \in \mathbb{R}^n, \quad (14)$$

is satisfied at any t.

Relation (14) will serve as a useful guideline to derive the reverse-time transition mechanism for quantum channels in Section 4, since in that setting we cannot generally relay on conditional probabilities as in (13).

*Of course, Q can be obtained immediately by requiring that the two-time probabilities generated by the forward and backward Markov chains are the same:

$$\mathbb{P}(X(t)=i, X(t+1)=j) = p_{ij}(t)\pi_t(i) = q_{ji}\pi_{t+1}(j). \quad (13)$$

This yields immediately $q_{ji}(t) = p_{ij}(t)\frac{\pi_t(i)}{\pi_{t+1}(j)}$.

4 Time-reversal for quantum Markov channels

Consider an n-level quantum system with associated Hilbert space $\mathcal{H}$ isomorphic to $\mathbb{C}^n$. In its standard statistical description, the role of probability densities is played by density operators, namely by positive, unit-trace matrices $\rho \in \mathcal{D}(\mathcal{H})$. The role of real random variables is taken by Hermitian operators $X \in \mathcal{O}(\mathcal{H})$ representing obervables. Expectations are computed via the trace functional, $\mathbb{E}_\rho(X) = \text{trace}(\rho X)$, and the classical setting may be recovered considering all diagonal matrices. Any linear, Trace Preserving and Completely Positive (TPCP) dynamical map $\mathcal{E}^\dagger$ acting on density operators can be represented by a Kraus operator-sum [Kraus, 1983], i.e.:

$$\rho_{t+1} = \mathcal{E}^\dagger(\rho_t) = \sum_j M_j \rho_t M_j^\dagger, \quad \sum_j M_j^\dagger M_j = I.$$

Following a quite standard quantum information terminology, we refer to linear, completely-positive trace-non-increasing Kraus maps as *quantum operations*. For observables, the *dual* dynamics is given by the *identity-preserving* quantum operation

$$\mathcal{E}(X) = \sum_j M_j^\dagger X M_j. \quad (15)$$

In the remaining of the paper, we consider the discrete-time quantum Markov evolutions associated to an initial density matrix ρ_0 and a sequence of TPCP maps $\{\mathcal{E}_t^\dagger\}_{t\geq 0}$.

In order to find the time-reversal of a given Markovian evolution, rewrite the probability-weighted inner product of the classical case (7) as $\langle x, y\rangle_\pi = \text{trace}(D_x D_\pi D_y)$. Notice that, if we simply drop commutativity, for two observables X, Y and a density matrix ρ, we would obtain $\langle X, Y\rangle_\rho = \text{trace}(X\rho Y)$. This functional is not satisfactory to our scopes, since in general it is neither real nor symmetric, i.e. $\text{trace}(Y\rho X) \neq \text{trace}(X\rho Y)$. It is then convenient to rewrite (7), by using the fact that all matrices commute, in the symmetrized form: $\langle x, y\rangle_\pi = \text{trace}(D_x^{\frac{1}{2}} D_\pi^{\frac{1}{2}} D_y D_\pi^{\frac{1}{2}} D_x^{\frac{1}{2}})$. We shall show that this form of the inner product leads to the correct reverse-time quantum Markov operation. Allowing for a general density operator ρ and observables X, Y, we thus define:

$$\langle X, Y\rangle_\rho = \text{trace}(X^{\frac{1}{2}} \rho^{\frac{1}{2}} Y \rho^{\frac{1}{2}} X^{\frac{1}{2}}).$$

This is a symmetric, real, semi-definite sesquilinear form on Hermitian operators.

By analogy with the classical case, we then define the quantum operation $\mathcal{R}_{\mathcal{E},\rho_t}$ as the space-time $\{\rho_t\}$-adjoint of a quantum operation $\mathcal{E}$ using the quantum version of (14):

$$\langle \mathcal{E}(X), Y\rangle_{\rho_t} = \langle X, \mathcal{R}_{\mathcal{E},\rho_t}(Y)\rangle_{\rho_{t+1}}.$$

Let us assume for now that ρ_{t+1} is full-rank. An explicit Kraus representation is then obtained as follows:

$$\langle \mathcal{E}(X), Y\rangle_{\rho_t} = \sum_j \mathrm{trace}(M_j^\dagger X M_j \rho_t^{\frac{1}{2}} Y \rho_t^{\frac{1}{2}})$$
$$= \sum_j \mathrm{trace}(X \rho_{t+1}^{\frac{1}{2}} \rho_{t+1}^{-\frac{1}{2}} M_j \rho_t^{\frac{1}{2}} Y \rho_t^{\frac{1}{2}} M_j^\dagger \rho_{t+1}^{-\frac{1}{2}} \rho_{t+1}^{\frac{1}{2}})$$
$$= \sum_j \mathrm{trace}(X \rho_{t+1}^{\frac{1}{2}} R_j^\dagger(\mathcal{E},\rho_t) Y R_j(\mathcal{E},\rho_t) \rho_{t+1}^{\frac{1}{2}})$$
$$= \langle X, \mathcal{R}_{\mathcal{E},\rho_t}(Y)\rangle_{\rho_{t+1}},$$

where $\mathcal{R}_{\mathcal{E},\rho_t}$ admits an operator-sum representation with Kraus operators

$$R_j(\mathcal{E},\rho_t) = \rho_{t+1}^{-\frac{1}{2}} M_j \rho_t^{\frac{1}{2}}. \quad (16)$$

Notice that the second equality is non-trivial in the case when ρ_{t+1} is not full-rank and inverses are replaced by the Moore-Penrose pseudoinverse (the latter replacement will be tacitly assumed in the rest of the paper). For any matrix M, the *support* of M, denoted $\mathrm{supp}(M)$, is the orthogonal complement of $\ker(M)$. The following Lemma ensures that the same derivation applies to the general case.

Lemma 4.1. *Let $\rho_{t+1} = \sum_j M_j \rho_t M_j^\dagger$. Let $\Pi_{\rho_{t+1}}$ denote the orthogonal projection onto the support of ρ_{t+1}. Then, for any normal matrix Y:*

$$\Pi_{\rho_{t+1}} \left(\sum_j M_j \rho_t^{\frac{1}{2}} Y \rho_t^{\frac{1}{2}} M_j^\dagger \right) \Pi_{\rho_{t+1}} = \sum_j M_j \rho_t^{\frac{1}{2}} Y \rho_t^{\frac{1}{2}} M_j^\dagger.$$

The proof can be found in [Ticozzi and Pavon, 2008]. It is now natural to define a transformation between Kraus operators. Let $\mathcal{E}^\dagger$ be a quantum operation represented by Kraus operators $\{F_k\}$. For any ρ, define the map $\mathcal{T}_\rho$ from quantum operations to quantum operations

$$\mathcal{T}_\rho : \mathcal{E}^\dagger \mapsto \mathcal{T}_\rho(\mathcal{E}^\dagger), \quad (17)$$

where $\mathcal{T}_\rho(\mathcal{E}^\dagger)$ has Kraus operators $\{\rho^{\frac{1}{2}} F_k^\dagger (\mathcal{E}(\rho))^{-\frac{1}{2}}\}$. The results of [Barnum and Knill, 2002] show that the action of $\mathcal{T}_\rho$ is independent of the particular Kraus representation of $\mathcal{E}^\dagger$. With this definition, we have that $\mathcal{T}_{\rho_t}(\mathcal{E}^\dagger) = \mathcal{R}^\dagger_{\mathcal{E},\rho_t}$.

We are now in a position to prove the main result of this section, which establishes the role of $\mathcal{R}_{\mathcal{E},\rho_t}(\cdot)$ as the quantum time-reversal of the TPCP map $\mathcal{E}^\dagger$. Augmenting a Kraus map $\mathcal{E}$ with Kraus operators $\{M_k\}_{k=1,\dots,m}$ to a TPCP map means adding a finite number p of Kraus operators $\{M_k\}_{k=m+1,\dots,m+p}$ so that $\sum_k M_k^\dagger M_k = I$.

Theorem 4.2 (Time Reversal of TPCP maps). *Let $\mathcal{E}^\dagger$ be a TPCP map. If $\rho_{t+1} = \mathcal{E}^\dagger(\rho_t)$, then for any $\rho_t \in \mathfrak{D}(\mathcal{H})$, $\mathcal{R}^\dagger_{\mathcal{E},\rho_t} = \mathcal{T}_{\rho_t}(\mathcal{E}^\dagger)$ defined as in* (16) *is the* time-reversal *of $\mathcal{E}^\dagger$ for ρ_t, that is, it satisfies both:*

$$\rho_t = \mathcal{R}^\dagger_{\mathcal{E},\rho_t}(\rho_{t+1}) \quad (18)$$
$$\mathcal{T}_{\rho_{t+1}}(\mathcal{R}^\dagger_{\mathcal{E},\rho_t})(\sigma_t) = \mathcal{E}^\dagger(\sigma_t), \quad (19)$$

for all $\sigma_t \in \mathfrak{D}(\mathcal{H})$ such that $\mathrm{supp}(\sigma_t) \subseteq \mathrm{supp}(\rho_t)$. Morover, it can be augmented to be TPCP without affecting property (18)-(19).

Remark: Property (19) ensure us that among all quantum operations mapping ρ_{t+1} back to ρ_t, $\mathcal{R}^\dagger_{\mathcal{E},\rho_t}$ is the natural *time-reversal* of $\mathcal{E}^\dagger$ with respect to ρ_t. In fact, notice that if ρ_t is full rank, (19) implies that $\mathcal{T}_{\rho_{t+1}} \circ \mathcal{T}_{\rho_t}$ is the the identity map on quantum operations. That is, as one would expect, the time reversal of the time-reversal is the original forward map. While this may seem obvious, notice that property (18) alone is satisfied by any quantum operation of the form $\tilde{\mathcal{R}}^\dagger = \mathcal{T}_{\rho_t}(\mathcal{F}^\dagger)$, with $\mathcal{F}^\dagger$ *any* TPCP map. While studying quantum error correction problems, the same $\mathcal{R}^\dagger_{\mathcal{E},\rho}(\cdot)$ has been suggested by Barnum and Knill as a near-optimal correction operator [Barnum and Knill, 2002] in the full rank case: A more complete discussion on this and other approaches to the time-reversal is given in [Ticozzi and Pavon, 2008].

5 Quantum space-time harmonic processes

While in the framework of quantum probability rigorous extensions of conditional expectations and martingale processes are available for quite some time [Takesaki, 1972; Accardi, Frigerio and Lewis 1982], we show here that some interesting results on entropy dynamics can be derived avoiding most of the related technical machinery. This can be accomplished by introducing a quantum version of space-time harmonic functions. Consider a reference quantum Markov evolution on a finite time interval, generated by an initial density matrix ρ_0 and a sequence of TPCP maps $\{\mathcal{E}_t^\dagger\}_{t\in[0,T-1]}$.

Definition 5.1 (Quantum space-time harmonic process). *A sequence of Hermitian operators $\{Y_t\}_{t\in[0,T-1]}$ is said to be space-time harmonic with respect to the family of identity-preserving maps $\{\mathcal{E}_t\}_{t\in[0,T-1]}$ if:*

$$Y_t = \mathcal{E}_t(Y_{t+1}). \quad (20)$$

In analogy with the classical case, $\{Y_t\}_{t\in[0,T-1]}$ is said to be *space-time harmonic in reverse-time* with respect to the family $\{\mathcal{R}_{\mathcal{E}_T,\rho_t}\}$ if:

$$Y_{t+1} = \mathcal{R}_{\mathcal{E}_t,\rho_t}(Y_t). \quad (21)$$

The sequence is called *space time subharmonic* if $Y_t \leq \mathcal{E}_t(Y_{t+1})$, where we are referring to the natural partial order between Hermitian matrices. In the classical case, space time harmonic functions generate changes of measure through multiplicative functional transformations of the transition mechanism. A similar fact holds in the quantum case. Let Y_t be space time harmonic for $\mathcal{E}_t \sim \{E_k(t)^\dagger\}$ and let N_t be any choice of operator such that $Y_t = N_t N_t^\dagger$. Assume for simplicity Y_t to be full-rank at any t. Then $\mathcal{F}_t \sim \{N_t^{-1} E_k(t)^\dagger N_{t+1}\}$ is an identity-preserving quantum operation. In fact, by using (20), we have $\mathcal{F}_t(I) = \sum_k N_t^{-1} E_k(t)^\dagger N_{t+1} N_{t+1}^\dagger E_k(t) N_t^{-\dagger} = I$. Thus its adjoint is a TPCP map. An analogous result holds for reverse time evolution.

The following result is the quantum counterpart of (3) concerning properties of expectation of (sub)martingales.

Proposition 5.1. *Let $\{Y_t\}_{t\in[0,T-1]}$ be space-time harmonic and let $\{Z_t\}_{t\in[0,T-1]}$ be space-time subharmonic with respect to the reference evolution. Then, for all $t \in [0, T-1]$:*

$$\mathbb{E}_{\rho_0}(Y_0) = \mathbb{E}_{\rho_t}(Y_t), \quad \mathbb{E}_{\rho_t}(Z_t) \leq \mathbb{E}_{\rho_{t+1}}(Z_{t+1}). \tag{22}$$

A function f is called *operator convex* if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$, for any $\lambda \in [0,1]$, and matrices A, B with spectrum in $\mathcal{I}$. Consider now a set of operators $\{M_k\}$, such that $\sum_k M_k^\dagger M_k = I$. Then, for every tuple $\{X_k\}$ of self-adjoint matrices, the operator sum $\sum_k M_k^\dagger X_k M_k$ can be thought as an "operator convex combination" of the $\{X_k\}$. Remarkably, an operator analogue of Jensen's inequality holds (see [Hansen and Pedersen, 2003] and reference therein for a review of the literature on the subject). We give here a reduced statement of Theorem 2.1 in [Hansen and Pedersen, 2003] which is sufficient to our scope.

Theorem 5.2 (Operator Jensen's Inequality). *A function $f : \mathcal{I} \to \mathbb{R}$ is operator convex if and only if for any Hermitian X and set of operators $\{M_k\}$ such that $\sum_k M_k^\dagger M_k = I$ it satisfies*

$$f(\sum_k M_k^\dagger X_k M_k) \leq \sum_k M_k^\dagger f(X_k) M_k. \tag{23}$$

The following Proposition, which is a straightforward application of the result above, gives us a way to derive subharmonic processes from harmonic processes.

Proposition 5.2. *Let Y_t be a space-time harmonic process with respect to $\{\mathcal{E}_t\}_{t\geq 0}$, with eigenvalues $\lambda_{t,i} \in \mathcal{I} \subset \mathbb{R}$ at all times, and $f : \mathcal{I} \to \mathbb{R}$ be operator convex. Then $Z_t := f(Y_t)$ is space-time subharmonic.*

6 Application to information dynamics

The usual definition of quantum relative entropy is due to Umegaki [Umegaki, 1962]. Given two density matrices ρ, σ, the *quantum relative entropy* is defined as: $\mathbb{D}_U(\rho\|\sigma) = \text{trace}(\rho(\log\rho - \log\sigma))$, if $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$, and $+\infty$ otherwise.

As in the classical case, quantum relative entropy has the property of a pseudo-distance (see e.g. [Nielsen and Chuang, 2002]). Moreover, it has been proven by Petz that it is the only functional in a class of quasi-entropies having a certain conditional expectation property [Petz, 1982].

Nonetheless, here we show how a different quantum extension of classical relative entropy is natural from the viewpoint of space-time harmonic processes and the dynamical structure of Markovian evolutions. In order to do this, we now introduce a special class of space-time harmonic quantum processes. Consider two quantum Markov evolutions, corresponding to different initial conditions $\rho_0 \neq \sigma_0$, but with same family of trace-preserving quantum operations $\{\mathcal{E}_t^\dagger\}$. Define the observable

$$Y_t = \sigma_t^{-\frac{1}{2}} \rho_t \sigma_t^{-\frac{1}{2}}. \tag{24}$$

We thus have that: $\mathcal{R}_{\mathcal{E},\sigma_t}(Y_t) = \sum_k \sigma_{t+1}^{-\frac{1}{2}} M_k \sigma_t^{\frac{1}{2}} \sigma_t^{-\frac{1}{2}} \rho_t \sigma_t^{-\frac{1}{2}} \sigma_t^{\frac{1}{2}} M_k^\dagger \sigma_{t+1}^{-\frac{1}{2}} = Y_{t+1}$. This shows that Y_t evolves in the forward direction with the backward transition mechanism of σ_t, which makes it quantum *space-time harmonic in reverse time* with respect to the transition of σ_t. In view of (24), the natural definition of relative entropy in our setting is thus the Belavkin-Staszewski's relative entropy [Belavkin and Staszewski, 1982]:

$$\mathbb{D}_{BS}(\rho||\sigma) = \text{trace}\left(\sigma\left(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)\log\left(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)\right), \tag{25}$$

where, as usual, $0\log 0 = 0$. As for the Umegaki's version, it enjoys the properties of a pseudo-distance: It is non negative and equal to zero if and only if $\rho = \sigma$. In addition to this, it is clearly consistent with the classical relative entropy, which is recovered by considering commuting matrices, and with the von Neumann entropy, since: $\mathbb{D}_{BS}(\rho||I) = \text{trace}(\rho\log(\rho))$. Another useful property has been proven by Hiai and Petz [Hiai, 1991]: $\mathbb{D}_{BS}(\rho||\sigma) \geq \mathbb{D}_U(\rho||\sigma)$. Hence, convergence in $\mathbb{D}_{BS}(\rho||\sigma)$ ensures convergence in $\mathbb{D}_U(\rho||\sigma)$. The Belavkin-Staszewski's relative entropy has also been shown to be the trace of Fuji-Kamei's operator entropy [Fuji and Kamei, 1989]. As a consequence of the results of Section 5, we have the following Corollary.

Corollary 6.1. *Consider two quantum Markov evolutions associated to the initial conditions $\rho_0 \neq \sigma_0$ and to the same family of TPCP maps $\{\mathcal{E}_t^\dagger\}$. Suppose that ρ_t, σ_t are invertible, for all t's. Let $Y_t = \sigma_t^{-\frac{1}{2}} \rho_t \sigma_t^{-\frac{1}{2}}$ and let $Z_t := g(Y_t)$, with $g(x) = x\log(x)$. Then Z_t*

is a reverse time, space-time subharmonic process with respect to the quantum operations $\{\mathcal{R}_{\mathcal{E},\sigma_t}(\cdot)\}$, i.e.

$$Z_{t+1} = g\left(Y_{t+1}\right) \leq \mathcal{R}_{\mathcal{E},\sigma_t}\left(g\left(Y_t\right)\right) = \mathcal{R}_{\mathcal{E},\sigma_t}\left(Z_t\right). \tag{26}$$

This can be seen as an H-Theorem in operator form: In fact, as in the classical case, the reverse time subharmonic property of $\{Z_t\}$ of Theorem 6.1 implies under expectation a more usual, Lindblad-Araki-Uhlmann-like [Lindblad, 1975; Araki, 1976; Uhlmann, 1977] form of the H-theorem. Namely, we obtain monotonicity for the Belavkin-Staszewski's relative entropy under completely positive, trace-preserving maps. The same result has been derived for conditional expectations in [Hiai, 1991].

Corollary 6.2. *Consider two quantum Markov evolutions associated to the initial conditions $\rho_0 \neq \sigma_0$, and to the same family of TPCP maps $\{\mathcal{E}_t^\dagger\}$. Assume that ρ_t, σ_t are invertible for all t's. Then:*

$$\mathbb{D}_{BS}(\rho_{t+1}||\sigma_{t+1}) \leq \mathbb{D}_{BS}(\rho_t||\sigma_t). \tag{27}$$

If $\bar{\sigma}$ is the unique stationary state of $\{\mathcal{E}_t^\dagger\}$, we get a quantum version of the second law.

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