ESTIMATES OF FREQUENCY OF PERIODIC REGIMES FOR PHASE SYNCHRONIZATION SYSTEMS WITH DISTRIBUTED PARAMETERS

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Abstract

Many systems, arising in electrical and electronic engineering are based on controlled phase synchronization of several periodic processes (“phase synchronization” systems, or PSS). Typically such systems are featured by the *gradient-like behavior*, i.e. any solution of the system converges to one of equilibrium points. If a PSS in not gradient-like it may have periodic regimes which are undesirable for most systems. In the present paper, we address the problem of lack or existence of periodic regimes for phase synchronization systems described by integro-differential Volterra equations. New effective frequency–algebraic estimates for the frequency of possible periodic regimes are obtained by means of Fourier expansions and the tool of Popov functionals destined specially for periodic nonlinearities.

Key words

Phase synchronization systems, frequency-response methods, periodic solutions, infinite–dimensional systems, Fourier series.

1 Introduction

A lot of control systems arising in electrical engineering, electronics, mechanics and telecommunications may be modeled as interconnection of a linear plant, described by differential or integro–differential, equations and a periodic nonlinear feedback. These mathematical models are often referred to as phase synchronization systems (PSS) [Yang and Huang, 2007], being the subject matter of phase synchronization theory [Leonov, 2006], [Lindsey,1972], [Leonov and Kuznetsov, 2014]. This theory gives the opportunity to examine the asymptotic behavior of phase–locked loops (PLL) [Best, 2007], [Leonov, Kuznetsov, Yuldashev, M. V. and Yuldashev, R. V., 2011], [Kudrewicz and Wasowicz, 2007], self–synchronization systems [Blekhman, 1988], electrical and mechanical machines [Leonov and Kondrat’eva, 2008], [Stoker, 1950].

Phase synchronization systems have as a rule an infinite denumerable set of equilibrium points which correspond to synchronous regimes. So the basic problem most of the papers considering PSS deal with is convergence of any trajectory to an equilibrium point. This property is called gradient–like behavior.

The problem of gradient–like behavior has been explored in many published works, see [Leonov, 2006] and reference therein for details. For PSS with lumped parameters Lyapunov direct method turned out to be rather fruitful. But as soon as the two traditional Lyapunov functions, i.e. ”quadratic form” and ”quadratic form plus integral of the nonlinearity” proved to be of no use here, efficient criteria of gradient–like behavior has been obtained by means of several new types of Lyapunov functions [Leonov, 2006], [Leonov, Ponomarenko and Smirnova, 1996].

In particular, periodic Lyapunov functions have been exploited. In the pioneering paper [Bakaev and Guzh, 1965] a periodic Lyapunov function for a third order PSS was considered. Then in [Gelig, Leonov and Yakubovich, 1978] the results of [Bakaev and Guzh, 1965] were extended to a multidimensional PSS, in [Leonov, Ponomarenko and Smirnova, 1996] a further periodic function was introduced and in [Perkin, Shepeljavyi and Smirnova, 2009] a generalized periodic function was offered.

With the help of Kalman–Yakubovich–Popov (KYP) lemma the necessary and sufficient conditions for the existence of a periodic Lyapunov function took the form of frequency–algebraic inequalities with a number of varying parameters. The Popov method of a priori integral indices [Popov, 1973] gave the opportunity to extend the frequency–algebraic criteria of gradient–like behavior to PSS with distributed parameters [Leonov, Ponomarenko and Smirnova, 1996], [Perkin, Shepeljavyi and Smirnova, 2012].

If a PSS is not gradient–like it may have periodic regimes which regimes are undesirable for most systems. So the problem arises if a PSS has a periodic regime of a certain frequency. The problem has been investigated both by approximate calculus [Shakhil'dyan and Lyakhovkin, 1972] and by analytical methods [Evtyanov and Snedkova, 1968].

In paper [Leonov and Speranskaya, 1985] with the help of Fourier series it was shown that the frequency–algebraic conditions of gradient–like behavior can be used to guarantee that multidimensional PSS has no periodic regimes of certain frequencies. The results of [Leonov and Speranskaya, 1985] were extended then to infinite dimensional PSS [Leonov, Ponomarenko and Smirnova, 1996] and to discrete ones [Leonov and Fyodorov, 2011]. On the other hand in paper [Perkin, Shepeljavyi, Smirnova and Utina, 2014] the results of [Leonov and Speranskaya, 1985] were generalized by means of Lyapunov function borrowed from [Perkin, Shepeljavyi and Smirnova, 2009].

In this paper we extend the frequency conditions obtained in [Perkin, Shepeljavyi, Smirnova and Utina, 2014] to infinite dimensional PSS.

2 Problem setup

Consider a system of integro–differential equations as follows

$$\begin{array}{l}\frac{d\sigma(t)}{dt} = \alpha(t) + Rf(\sigma(t-h)) - \\ -\int\limits_0^t \gamma(t-\tau) f(\sigma(\tau))\, d\tau \quad (t \geq 0).\end{array} \tag{1}$$

Here $\sigma(t) = (\sigma_1(t), ..., \sigma_l(t))^T$ and $f(\sigma)$ is an input–output decoupled MIMO nonlinearity: $f(\sigma) = (\varphi_1(\sigma_1), ..., \varphi_l(\sigma_l))^T$. Each map φ_j is assumed to be $\mathbf{C}^1$–smooth and Δ_j–periodic, with simple and isolated roots. It is also assumed that $\int\limits_0^{\Delta_j} \varphi_j(\xi)\, d\xi < 0$. The matrix $R \in \mathbf{R}^{l\times l}$, delay $h \geq 0$, the function $\alpha : [0, +\infty) \to \mathbf{R}^l$, and kernel map $\gamma : [0, +\infty) \to \mathbf{R}^{l\times l}$ in (1) are known, and $\alpha(\cdot)$ is continuous. The solution of (1) is defined by specifying initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t). \tag{2}$$

Assume that $\sigma^0(\cdot)$ is continuous and $\sigma(0+0) = \sigma^0(0)$.

We suppose also that the following restrictions are valid:

$$\begin{array}{l}\alpha(t) \to 0 \text{ as } t \to +\infty; \\ |\alpha(t)| + |\gamma(t)| \in L_1[0, +\infty) \cap L_2[0, +\infty).\end{array} \tag{3}$$

Note that if a vector–function $\sigma(t)$ is a solution of system (1) then an arbitrary function $(\sigma_1(t) + I_1\Delta_1, ..., \sigma_l(t) + I_l\Delta_l)^T$ with $I_j \in \mathbf{Z}$ $(j = 1, ..., l)$ is also a solution of system (1). So (1) can be called a *phase system* by analogy with the phase differential equations [Gelig, Leonov and Yakubovich, 1978].

A phase system (1) has an infinite denumerable set of equilibriums. If every solution of system (1) converges to a certain equilibrium the system is called gradient–like. In case the phase system (1) is not gradient–like it may have periodic solutions which are divided into two classes: solutions of the first kind and solutions of the second kind.

Definition 1. We say that a solution $\sigma(t)$ of (1) is a periodic solution if there exist a number $T > 0$ and integers I_j $(j = 1, ..., l)$ such that

$$\sigma_j(t+T) = \sigma_j(t) + I_j\Delta_j, \ \forall t \quad (j = 1, ..., l). \tag{4}$$

If all $I_j = 0$ $(j = 1, ..., l)$ the solution $\sigma(t)$ is called a periodic solution of the first kind. If $I_1^2 + ... + I_l^2 \neq 0$ it is called a periodic solution of the second kind.

The number T is the period and the number $\omega = \frac{2\Pi}{T}$ is the frequency of a periodic solution. Our goal here is to get some conditions for existence of periodic solutions of system (1). We are going to develop the approach exploited in [Leonov and Speranskaya, 1985], [Leonov, Ponomarenko and Smirnova, 1996]. It combines the idea of Fourier expansions in [Garber, 1967] and the tool of Lyapunov periodic functions or of Popov functionals. The conditions obtained have the form of frequency–algebraic inequalities with varying parameters.

We shall need the transfer matrix of the linear part of (1)

$$K(p) = -Re^{-ph} + \int\limits_0^\infty \gamma(t)e^{-pt}\, dt \ \ (p \in \mathbf{C}). \tag{5}$$

Its real part is defined as follows

$$ReK(p) = \frac{1}{2}(K(p) + K^*(p)), \tag{6}$$

where the symbol * is used for Hermitian conjugation.

We shall also need some preliminaries [Leonov and Speranskaya, 1985], [Leonov, Ponomarenko and

Smirnova, 1996]. Suppose that $\sigma(t)$ is a T–periodic solution of system (1). Then $f(\sigma(t))$ is a T–periodic function. Indeed it follows from (4) that

$$\varphi_j(\sigma_j(t+T)) = \varphi_j(\sigma_j(t)+I_j\Delta_j) = \varphi_j(\sigma_j(t)). \quad (7)$$

Then

$$f(\sigma(t)) = \sum_{k=-\infty}^{+\infty} B_k e^{i\omega kt} \quad (i^2=-1), \quad (8)$$

where B_k are l–vectors. By substituting (8) in (1) we have

$$\dot\sigma(t) = \alpha(t) + \beta(t) - \sum_{k=-\infty}^{+\infty} K(i\omega k) B_k e^{i\omega kt}, \quad (9)$$

where

$$\beta(t) = \int_t^{+\infty} \gamma(\tau) f(\sigma(t-\tau))\, d\tau. \quad (10)$$

It follows from the restrictions (3) that $\alpha(t)+\beta(t)\to 0$ as $t\to+\infty$. But since $\dot\sigma(t)$ is T–periodic it follows that $\alpha(t)+\beta(t)=0$ and

$$\dot\sigma(t) = -\sum_{k=-\infty}^{+\infty} K(i\omega k) B_k e^{i\omega kt}. \quad (11)$$

3 Main results

Let $\mu_{1j} = \inf_{\xi\in[0,\Delta_j]} \varphi'(\xi)$, $\mu_{2j} = \sup_{\xi\in[0,\Delta_j]} \varphi'(\xi)$, $(j=1,...,l)$. It is clear that $\mu_{1j}\mu_{2j}<0$. Define the matrices $M_1 = diag\{\mu_{11},\mu_{12},...,\mu_{1l}\}$, $M_2 = diag\{\mu_{21},\mu_{22},...,\mu_{2l}\}$ and introduce constants

$$\nu_j = \frac{\int\limits_0^{\Delta_j} \varphi_j(\xi)\,d\xi}{\int\limits_0^{\Delta_j} |\varphi_j(\xi)|\,d\xi}, \quad \nu_{0j} = \frac{\int\limits_0^{\Delta_j} \varphi_j(\xi)\,d\xi}{\int\limits_0^{\Delta_j} |\varphi_j(\xi)|\Phi_j(\xi)\,d\xi}, \quad (12)$$

where

$$\Phi_j(\xi) = \sqrt{(1-\mu_{1j}^{-1}\varphi_j'(\sigma))(1-\mu_{2j}^{-1}\varphi_j'(\sigma))}. \quad (13)$$

Theorem 1. Suppose there exist $\omega_0>0$, matrix $\varkappa = diag\{\varkappa_1,...,\varkappa_l\}$, positive definite matrices $\tau = diag\{\tau_1,...,\tau_l\}$, $\varepsilon = diag\{\varepsilon_1,...,\varepsilon_l\}$, $\delta = diag\{\delta_1,...,\delta_l\}$ and numbers $a_j\in[0;1]$ $(j=1,...,l)$, such that the following conditions are valid:
1) for $\omega=0$ and all $\omega\ge\omega_0$ the inequality

$$\Omega(\omega) := Re\Big\{\varkappa K(i\omega) - (K(i\omega)+M_1^{-1}i\omega)^*\tau(K(i\omega)+ +M_2^{-1}i\omega) - K^*(i\omega)\varepsilon K(i\omega)\Big\} - \delta > 0; \quad (14)$$

is true;
2) the quadratic forms

$$Q_j(\xi,\eta,\zeta) = \varepsilon_j\xi^2 + \delta_j\eta^2 + \tau_j\zeta^2 + \varkappa_j a_j\nu_j\xi\eta + +\varkappa_j(1-a_j)\nu_{0j}\eta\zeta \quad (j=1,...,l) \quad (15)$$

are positive definite.
Then system (1) has no periodic solutions of the frequency $\omega\ge\omega_0$.
Proof. Let us introduce the functions

$$F_j(\zeta) = \varphi_j(\zeta) - \nu_j|\varphi_j(\zeta)|, \quad (16)$$

$$\Psi_j(\zeta) = \varphi_j(\zeta) - \nu_{0j}\Phi_j(\zeta)|\varphi_j(\zeta)| \quad (17)$$

and vector functions $F(\sigma) = (F_1(\sigma_1),...,F_l(\sigma_l))^T$, $\Psi(\sigma) = (\Psi_1(\sigma_1),...,\Psi_l(\sigma_l))^T$. It is obvious that

$$\int_0^{\Delta_j} F_j(\zeta)\,d\zeta = \int_0^{\Delta_j} \Psi_j(\zeta)\,d\zeta = 0. \quad (18)$$

Introduce also the matrices $A = diag\{a_1,...,a_l\}$, $A_0 = diag\{1-a_1,...,1-a_l\}$. Define a function

$$G(t) = \dot\sigma^*(t)\varepsilon\dot\sigma(t) + \dot\sigma^*(t)\varkappa f(\sigma(t)) + +f^*(\sigma(t))\delta f(\sigma(t)) - F^*(\sigma(t))A\varkappa\dot\sigma(t) - -\Psi^*(\sigma(t))A_0\varkappa\dot\sigma(t) + +(\dot\sigma(t) - M_1^{-1}\dot f(\sigma(t)))^*\tau(\dot\sigma(t) - M_2^{-1}\dot f(\sigma(t))) \quad (19)$$

and consider a set of functionals

$$J(\Theta) = \int_0^\Theta G(t)\,dt \quad (\Theta>0). \quad (20)$$

Suppose $\sigma(t)$ is a T–periodic solution of (1). Let us transform the integral of $J(T)$ using (17) and (16):

$$J(T) = \int_0^T \sum_{j=1}^l \Big\{\varepsilon_j\sigma_j^2(t) + \varkappa_j\varphi_j(\sigma_j(t))\dot\sigma_j(t) - -a_j\varkappa_j F_j(\sigma_j(t))\dot\sigma_j(t) + \delta_j\varphi_j^2(\sigma_j(t)) - -(1-a_j)\varkappa_j\Psi_j(\sigma_j(t))\dot\sigma_j(t) + +\tau_j(\dot\sigma_j(t) - \mu_{1j}^{-1}\varphi_j'(\sigma_j(t)))\cdot \cdot(\dot\sigma_j(t) - \mu_{2j}^{-1}\varphi_j'(\sigma_j(t)))\Big\}\,dt = = \int_0^T \sum_{j=1}^l \Big\{\varepsilon_j\dot\sigma_j^2(t) + \delta_j\varphi_j^2(\sigma_j(t)) + +\tau_j\dot\sigma_j^2(t)\Phi_j^2(\sigma_j(t)) + \varkappa_j a_j\nu_j|\varphi(\sigma_j(t))|\dot\sigma_j(t) + +\varkappa_j(1-a_j)\nu_{0j}|\varphi(\sigma_j(t))|\dot\sigma_j(t)\Phi_j(\sigma_j(t))\Big\}\,dt = = \int_0^T \sum_{j=1}^l Q_j(\dot\sigma_j(t), |\varphi(\sigma_j(t))|, \dot\sigma_j(t)\Phi_j(\sigma_j(t))). \quad (21)$$

In virtue of condition 2) of the theorem

$$J(T) > 0. \tag{22}$$

Suppose now that $\sigma(t)$ has the frequency $\omega \geq \omega_0$. Let us transform the functional $J(T)$ using expansions (8) and (11) under the following obvious equalities:

$$B_{-k} = \bar{B}_k \quad (k \in \mathbf{Z}), \tag{23}$$

where the symbol $-$ is used for complex conjugation;

$$\int_0^T e^{i\omega kt} e^{i\omega mt}\, dt = \begin{cases} 0, \text{ if } k \neq -m, \\ T, \text{ if } k = -m, \end{cases} \quad (k, m \in \mathbf{Z}). \tag{24}$$

Notice that in virtue of Definition 1 the following equalities are valid:

$$\int_0^T F_j(\sigma_j(t))\dot{\sigma}_j(t)\, dt = \int_{\sigma_j(0)}^{\sigma_j(T)} F_j(\zeta)\, d\zeta = 0, \tag{25}$$

$$\int_0^T \Psi_j(\sigma_j(t))\dot{\sigma}_j(t)\, dt = 0. \tag{26}$$

We have

$$J(T) = \sum_{k=1}^{4} J_k(T), \tag{27}$$

where

$$\begin{aligned} &J_1(T) = \textstyle\int_0^T \dot{\sigma}^*(t)\varkappa f(\sigma(t))\, dt, \\ &J_2(T) = \textstyle\int_0^T f^*(\sigma(t))\delta f(\sigma(t))\, dt, \\ &J_3(T) = \textstyle\int_0^T \dot{\sigma}^*(t)\varepsilon\dot{\sigma}(t)\, dt, \\ &J_4(T) = \textstyle\int_0^T (\dot{\sigma}(t) - M_1^{-1}\dot{f}(\sigma(t)))^*\tau\cdot \\ &\cdot(\dot{\sigma}(t) - M_2^{-1}\dot{f}(\sigma(t)))\, dt. \end{aligned} \tag{28}$$

Now we may transform each of the integrals $J_j(T)$ using the formulas (8) and (11). We obtain

$$\begin{aligned} &J_1(T) = -\textstyle\int_0^T \Big\{(\sum_{k=-\infty}^{+\infty} B_k^* K^*(i\omega k)e^{-i\omega kt})\varkappa\cdot \\ &\cdot(\textstyle\sum_{r=-\infty}^{+\infty} B_r e^{i\omega rt})\Big\}\, dt = -T\Big\{B_0^* K^*(0)\varkappa B_0 + \\ &+\textstyle\sum_{k=1}^{+\infty}\Big(B_k^* K^*(i\omega k)\varkappa B_k + B_{-k}^* K^*(-i\omega k)\varkappa B_{-k}\Big)\Big\}. \end{aligned} \tag{29}$$

Since $K(-i\omega k) = \bar{K}(i\omega k)$ we have from (23) that

$$\begin{aligned} &J_1(T) = -T\Big\{B_0^* K^*(0)\varkappa B_0 + \\ &+2\textstyle\sum_{k=1}^{+\infty}\Big(B_k^* Re(\varkappa K(i\omega k))B_k\Big)\Big\}. \end{aligned} \tag{30}$$

Further

$$\begin{aligned} &J_2(T) = \textstyle\int_0^T \Big\{(\sum_{k=-\infty}^{+\infty} B_k^* e^{-i\omega kt})\delta\cdot \\ &\cdot(\textstyle\sum_{r=-\infty}^{+\infty} B_r e^{i\omega rt})\Big\}\, dt = \\ &= T\{B_0^*\delta B_0 + 2\textstyle\sum_{k=1}^{+\infty} B_k^*\delta B_k\}. \end{aligned} \tag{31}$$

$$\begin{aligned} &J_3(T) = \textstyle\int_0^T \Big\{(\sum_{k=-\infty}^{+\infty} B_k^* K^*(i\omega k)e^{-i\omega kt})\varepsilon\cdot \\ &\cdot(\textstyle\sum_{r=-\infty}^{+\infty} K^*(i\omega r)B_r e^{i\omega rt})\Big\}\, dt = \\ &= T\{B_0^* K^*(0)\varepsilon K(0)B_0 + \\ &+2\textstyle\sum_{k=1}^{+\infty} B_k^* K^*(i\omega k)\varepsilon K(i\omega k B_k\}. \end{aligned} \tag{32}$$

For integral $J_4(T)$ the following representation is true

$$\begin{aligned} &J_4(T) = \textstyle\int_0^T \dot{\sigma}^*(t)\tau\dot{\sigma}(t)\, dt- \\ &-\textstyle\int_0^T \dot{f}^*(\sigma(t))M_1^{-1}\tau\dot{\sigma}(t)\, dt- \\ &-\textstyle\int_0^T \dot{\sigma}^*(t)\tau M_2^{-1}\dot{f}(\sigma(t))\, dt+ \\ &+\textstyle\int_0^T \dot{f}^*(\sigma(t))M_1^{-1}\tau M_2^{-1}\dot{f}(\sigma(t))\, dt. \end{aligned} \tag{33}$$

We get from (8) that

$$\dot{f}(\sigma(t)) = \textstyle\sum_{k=-\infty}^{+\infty} i\omega k B_k e^{i\omega kt}. \tag{34}$$

Then the following equalities are true:

$$\begin{aligned} &\textstyle\int_0^T \dot{f}^*(\sigma(t))M_1^{-1}\tau\dot{\sigma}(t)\, dt = \\ &= \textstyle\int_0^T \Big\{(\sum_{k=-\infty}^{+\infty}(-i\omega k)B_k^* e^{-i\omega kt})M_1^{-1}\tau\cdot \\ &\cdot(-\textstyle\sum_{r=-\infty}^{+\infty} K(i\omega r)B_r e^{i\omega rt})\Big\}\, dt = \\ &= 2T\textstyle\sum_{k=1}^{+\infty} B_k^* Re(M_1^{-1}\tau(i\omega k K(i\omega k)))B_k; \end{aligned} \tag{35}$$

$$\begin{aligned} &\textstyle\int_0^T \dot{\sigma}^*(t)\tau M_2^{-1}\dot{f}(\sigma(t))\, dt = \\ &= -2T\textstyle\sum_{k=1}^{+\infty} B_k^* Re(i\omega k K^*(i\omega k)\tau M_2^{-1})B_k; \end{aligned} \tag{36}$$

$$\int_0^T \dot{f}^*(\sigma(t))M_1^{-1}\tau M_2^{-1}\dot{f}(\sigma(t))\,dt = \\ = 2T\sum_{k=1}^{+\infty} k^2\omega^2 B_k^* M_1^{-1}\tau M_2^{-1}B_k. \quad (37)$$

From (32)–(37) it follows that

$$J_4(T) = TB_0^*K^*(0)\tau K(0)B_0 + \\ +2T\sum_{k=1}^{+\infty} B_k^* Re\{(K(i\omega k)+M_1^{-1}i\omega k)^*\cdot \\ \cdot\tau(K(i\omega k)+M_2^{-1}i\omega k)\}B_k; \quad (38)$$

From (27)–(31) and (38) we get that

$$J(T) = -TB_0^*\{\varkappa K(0) - K^*(0)(\varepsilon+\tau)K(0) - \\ -\delta\}B_0 - 2T\sum_{k=1}^{+\infty} B_k^*\{Re(\varkappa K(i\omega k) - \\ -(K(i\omega k)+i\omega k M_1^{-1})^*\tau(K(i\omega k)+i\omega k M_2^{-1})) - \\ -\delta - K^*(i\omega k)\varepsilon K(i\omega k)\}B_k. \quad (39)$$

Condition 1) of the Theorem guarantees that all the terms $B_k^*\Omega(\omega k)B_k$ $(k = 0,1,2,...)$ in (39) are non-negative and consequently

$$J(T) \leq 0. \quad (40)$$

This inequality contradicts with (22). The contradiction means that our assumption is wrong and the system (1) has no periodic solution of the frequency $\omega \geq \omega_0$. Theorem 1 is proved.

Theorem 2. Suppose there exist $\omega_0 > 0$, matrix $\varkappa = diag\{\varkappa_1,...,\varkappa_l\}$, positive definite matrices $\tau = diag\{\tau_1,...,\tau_l\}$, $\varepsilon = diag\{\varepsilon_1,...,\varepsilon_l\}$, $\delta = diag\{\delta_1,...,\delta_l\}$, such that condition 1) of Theorem 1 is true and the inequalities

$$4\delta_j\varepsilon_j > \varkappa_j^2\nu_{1j}^2 \quad (j = 1,...,l) \quad (41)$$

with

$$\nu_{1j} = \frac{\int\limits_0^{\Delta_j}\varphi_j(\xi)\,d\xi}{\int\limits_0^{\Delta_j}|\varphi_j(\xi)|\sqrt{1+\frac{\tau_j}{\varepsilon_j}\Phi_j^2(\xi)}\,d\xi} \quad (42)$$

are valid. Then the system (1) has no periodic solution of the frequency $\omega \geq \omega_0$.

Proof. Introduce the functions

$$P_j(\xi) = \sqrt{1+\frac{\tau_j}{\varepsilon_j}\Phi_j^2(\xi)}, \\ Y_j(\xi) = \varphi_j(\xi) - \nu_{1j}|\varphi_j(\xi)|P_j(\xi) \quad (j = 1,...,l). \quad (43)$$

Let

$$Y(\xi) = (Y_1(\xi),...,Y_l(\xi))^T. \quad (44)$$

It is obvious that

$$\int_0^{\Delta_j} Y_j(\xi)\,d\xi = 0. \quad (45)$$

Determine the function

$$G_0(t) = \dot\sigma^*(t)\varepsilon\dot\sigma(t) + f^*(\sigma(t))\varkappa\dot\sigma(t) + \\ +f^*(\sigma(t))\delta f(\sigma(t)) + \\ +(\dot\sigma(t) - M_1^{-1}\dot f(\sigma(t)))^*\tau(\dot\sigma(t) - M_2^{-1}\dot f(\sigma(t))) - \\ -Y^*(\sigma(t))\varkappa\dot\sigma(t) \quad (46)$$

and consider the integral

$$J_0(\Theta) = \int_0^\Theta G_0(t)\,dt \quad (\Theta > 0). \quad (47)$$

Suppose that $\sigma(t)$ is a T–periodic solution of (1). Then

$$J_0(T) = \int_0^T\Big\{\sum_{j=1}^l\Big(\varepsilon_j\dot\sigma_j^2(t) + \delta_j\varphi_j^2(\sigma_j(t)) + \\ +\tau_j\dot\sigma_j^2(t)\Phi_j^2(\sigma_j(t)) + \\ +\varkappa_j\nu_{1j}|\varphi_j(\sigma_j(t))|P_j(\sigma_j(t))\dot\sigma_j(t)\Big)\Big\}\,dt = \\ = \int_0^T\Big\{\sum_{j=1}^l\Big(\varepsilon_j(\dot\sigma_j P_j(\sigma_j(t)))^2 + \delta_j\varphi_j^2(\sigma_j(t)) + \\ +\varkappa_j\nu_{1j}|\varphi_j(\sigma_j(t))|P_j(\sigma_j(t))\dot\sigma_j(t)\Big)\Big\}\,dt. \quad (48)$$

In virtue of (41) we have

$$J_0(T) > 0. \quad (49)$$

Let the T–periodic solution of (1) has the frequency $\omega \geq \omega_0$. Since

$$\int_0^T Y_j(\sigma_j(t))\dot\sigma_j(t)\,dt = \int_{\sigma_j(0)}^{\sigma_j(T)} Y_j(\xi)\,d\xi = 0, \quad (50)$$

we make a conclusion that $J_0(T) = J(T)$ with $J(\Theta)$ defined by formula (20). It has already been proved at the proof of the Theorem 1 that if inequality (14) is true for all $\omega \geq \omega_0$ then

$$J_0(T) = J(T) \leq 0, \quad (51)$$

which contradicts with (49). So the assumption that there exists a T–periodic solution of (1) with frequency $\frac{2\Pi}{T} \geq \omega_0$ is wrong. Theorem 2 is proved.

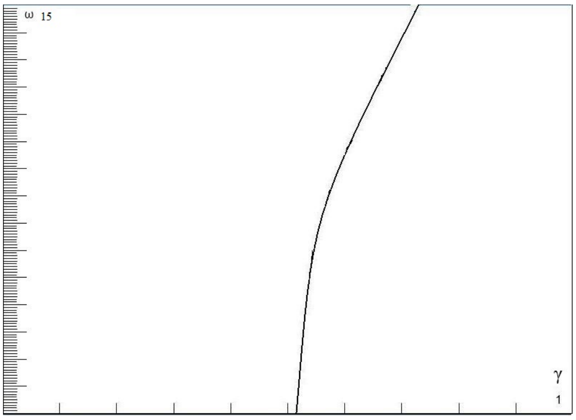

Figure 1. The region of the absence of the beat mode for PLL.

4 Example

Theorem 1 was applied to a second order phase-locked loop (PLL) with a proportional-integrating filter and time delay in the loop: $l = 1,\ f(\sigma) = \varphi_1(\sigma_1) = = \sin(\sigma_1) - \gamma,\ \gamma \in (0, 1)$,

$$K(p) = 100\frac{1 + 0.2p}{1 + p}e^{-0.1p}. \tag{52}$$

The results are presented in the figure 1. The region of the absence of the beat mode obtained by using Theorem 1 is on the left of the line.

5 Conclusion

The paper is devoted to the problem of lack or existence of periodic regimes in phase synchronization systems with distributed parameters. The PSSs described by integro-differential Volterra equations are addressed. The case of differentiable periodic nonlinearities is considered. The problem is investigated with the help of Fourier expansions and Popov functionals destined to periodic nonlinearities. New frequency-algebraic conditions for the lack or existence of periodic regimes of certain frequencies are obtained.

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