A new parameter estimation method via center manifold theory with application to unknown parameter estimation in a chaos system [1]

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Abstract This report presents a new parameter estimation method for a large class of nonlinear ordinary differential systems based on center manifold theory and nonlinear Luenberger observer. A numerical simulation shows that an unknown parameter in Lorenz system is well-estimated by this estimator. Unlike the approach using chaos synchronization, the estimator design does not depend on particular property of systems.

1 Introduction

In diverse fields of science, engineering and economics, systems of nonlinear ordinary differential equations play roles in describing various phenomena, predicting its future behavior and maximizing benefit from it (or minimizing problems caused by it). Those systems, however, include parameters that are unknown, vary according to environment or involve large amount of uncertainty. Estimation of parameters in ode systems, however, is a challenging problem when they are nonlinear because one cannot rely on analytic solutions.

In this report, we propose a novel approach to estimate parameters in nonlinear dynamical systems based on center manifold theory. In recent years, one of the authors of the present report and others developed a computationally efficient method for approximating center manifolds [6]. Using the computational technique and a modification of Luenberger observer theory [5], an observer is constructed to estimate the states on center part of a dynamical system. The parameters to be estimated are embeded in the center part and the observer states converge to the values of the parameters. This method does not rely on particular structure of the systems to be estimated and can be applied to a large class of dynamical systems.

The organization of this report is as follows. In §2, a recently developed approximation theory for center manifolds is reviewed. §3 presents the main result on the center state estimation based on a nonlinear Luenberger observer. In §4, we demonstrate the proposed method using the Lorenz system. A parameter in the Lorenz system is estimated with scalar measurement output. A relatively detailed construction of the estimator is presented and it can be seen that no special structure of the Lorenz system is employed, which is a crucial difference from earlier works in the parameter estimation methods using chaos synchronization [4, 2].

2 Successive approximation method of center manifolds

In this section, a recursive approximation method proposed in [6] is briefly reviewed. For detail, one refers to the above-mentioned paper. Let us consider the following set of differential equations

$$\begin{cases} \dot{x} = Ax + f(x, y) \\ \dot{y} = By + g(x, y), \end{cases} \tag{1}$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$.

Assumption 1 *A is an $n \times n$ constant real matrix whose eigenvalues have zero real parts. B is an $m \times m$ constant real matrix and its eigenvalues have negative real part.*

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From Assumption 1, it follows that for any constant $a > 0$, there exists a constant $C_1(a) > 0$ such that

$$\left|e^{At}x\right| \leqslant C_1(a)e^{a|t|}|x|,\ ({}^{\forall}t\in\mathbb{R},\ {}^{\forall}x\in\mathbb{R}^n).$$

Also, it follows that there exist constant $b > 0$ and $C_2 > 0$ such that

$$\left|e^{-Bt}y\right| \leqslant C_2e^{bt}|y|,\ (0\geqslant{}^{\forall}t\in\mathbb{R},\ {}^{\forall}y\in\mathbb{R}^m).$$

Assumption 2 *$f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, $g:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$ are C^r functions $(r\geqslant 2)$ and for all $|x|\leqslant\varepsilon$, $|x'|\leqslant\varepsilon$, $|y|\leqslant\varepsilon$, $|y'|\leqslant\varepsilon$, there exist continuous scalar functions $K_1(\varepsilon)$, $K_2(\varepsilon)$ such that*

$$\begin{cases} |f(x,y)|\leqslant\varepsilon K_1(\varepsilon) \\ |g(x,y)|\leqslant\varepsilon K_2(\varepsilon) \\ |f(x,y)-f(x',y')|\leqslant K_1(\varepsilon)(|x-x'|+|y-y'|) \\ |g(x,y)-g(x',y')|\leqslant K_2(\varepsilon)(|x-x'|+|y-y'|) \end{cases}$$

where, $f(0,0)=0$, $g(0,0)=0$, $(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0))=0$, $(\frac{\partial g}{\partial x}(0,0), \frac{\partial g}{\partial y}(0,0))=0$, $K_1(0)=0$, $K_2(0)=0$.

Note that Assumptions 1 and 2 are satisfied in systems with C^2 smoothness. Next, we define a set of sequences $\{x_k(t,\xi)\}$, $\{h_k(\xi)\}$, $(k{=}0,1,2,\cdots)$ by the following.

$$\begin{cases} x_0(t,\xi) = e^{At}\xi \\ h_0(\xi) = 0 \\ x_{k+1}(t,\xi) = e^{At}\xi + \int_0^t e^{A(t-s)}f(x_k(s,\xi),h_k(x_k(s,\xi)))ds \\ h_{k+1}(\xi) = \int_{-\infty}^0 e^{-Bs}g(x_k(s,\xi),h_k(x_k(s,\xi)))ds. \end{cases} \tag{2}$$

Theorem 3 *Under Assumptions 1 and 2, system* (1) *possesses a local center manifold $y = h(x)$ around the origin and $h_k(x)$ in (2) converges to the local center manifold when $k\to\infty$.*

3 Partial state estimation using center manifold theory

In this subsection, we consider an ode system, in which the eigenvalues of the Jacobian matrix calculated at $x = 0$ have zero real parts, with the following state-space representation.

$$\begin{cases} \dot{x}_c = A_cx_c + f_c(x_c,x_s), \quad \mathrm{Re}(\lambda(A_c)) = 0 \\ \dot{x}_s = A_sx_s + f_s(x_c,x_s), \quad \mathrm{Re}(\lambda(A_s)) < 0 \\ y = h(x_c,x_s) \quad \text{(measurement output).} \end{cases} \tag{3}$$

The design of partial state estimator is based on a dynamical system

$$\dot{w} = Aw + bh(x_c,x_s), \tag{4}$$

where A is any Hurwitz matrix with the same dimension as A_c and the pair (A,b) is controllable.

The composite system (3) and (4) are described as

$$\begin{cases} \dot{x}_c = A_cx_c + f_c(x_c,x_s) \\ \begin{bmatrix}\dot{x}_s \\ \dot{w}\end{bmatrix} = \begin{bmatrix}A_s & 0 \\ 0 & A\end{bmatrix}\begin{bmatrix}x_s \\ w\end{bmatrix} + \begin{bmatrix}f_s(x_c,x_s) \\ bh(x_c,x_s)\end{bmatrix} \end{cases} \tag{5}$$

Center manifold theory states that there exists a center manifold $[x_s\ w]^T = [T_1(x_c)\ T_2(x_c)]^T$ for the system (5) and, locally, all trajectories $x_s(t)$, $x_c(t)$ and $w(t)$ converge to the center manifold as $t \to \infty$ (see, e.g., [1, 3]). Note that the center manifold $w = T_2(x_c)$ satisfies the following pde.

$$\frac{\partial T_2(x_c)}{\partial x_c}(A_c x_c + f_c(x_c, T_1(x_c))) = AT_2(x_c) + bh(x_c, T_1(x_c)), \qquad T_2(0) = 0 \tag{6}$$

Next, we consider the following candidate of a state estimator for x_c.

$$\dot{\hat{x}}_c = A_c\hat{x}_c + f_c(\hat{x}_c, T_1(\hat{x}_c)) + \left[\frac{\partial T_2}{\partial x_c}(\hat{x}_c)\right]^{-1} b(y - h(\hat{x}_c, T_1(\hat{x}_c))), \tag{7}$$

where y is the measurement output of system (3). To see that system (7) estimates x_c, we prove that $T_2(\hat{x}_c) - T_2(x_c)$ goes to 0 as $t \to \infty$. To this end, take its time derivative

$$\begin{aligned}
\frac{d}{dt}(T_2(\hat{x}_c) - T_2(x_c)) &= \frac{\partial T_2}{\partial x_c}(\hat{x}_c)\dot{\hat{x}}_c - \frac{\partial T_2}{\partial x_c}\dot{x}_c \\
&= \frac{\partial T_2}{\partial x_c}(\hat{x}_c)(A_c\hat{x}_c + f_c(\hat{x}_c, T_1(\hat{x}_c))) + \frac{\partial T_2}{\partial x_c}(\hat{x}_c)\left[\frac{\partial T_2}{\partial x_c}(\hat{x}_c)\right]^{-1} b(y - h(\hat{x}_c, T_1(\hat{x}_c))) \\
&\quad - \frac{\partial T_2}{\partial x_c}(A_c x_c + f_c(x_c, x_s)) \\
&= AT_2(\hat{x}_c) + bh(\hat{x}_c, T_1(\hat{x}_c)) + b(y - h(\hat{x}_c, T_1(\hat{x}_c))) - AT_2(x_c) - bh(x_c, T_1(x_c)) - \frac{\partial T_2}{\partial x_c} f_c(x_c, x_s) \\
&\quad + \frac{\partial T_2}{\partial x_c} f_c(x_c, T_1(x_c)) \\
&= A(T_2(\hat{x}_c) - T_2(x_c)) + \frac{\partial T_2}{\partial x_c}\{f_c(x_c, T_1(x_c)) - f_c(x_c, x_s)\} + B\{h(x_c, x_s) - h(x_c, T_1(x_c))\},
\end{aligned}$$

and then, one can conclude that $T_2(\hat{x}_c) - T_2(x_c) \to 0$ when $t \to \infty$ because the last two terms are vanishing non-homogeneous ones due to the property of center manifold.

Here, attention must be paid to the fact that, from center manifold theory, the Jacobian matrix of T_2 is always singular at the origin, and therefore, the observer (7) always has high-gain. It is possible to numerically circumvent this problem although its detail is omitted.

4 Unknown model parameter estimation in Lorenz system

To demonstrate how proposed method works, we consider the Lorenz system.

$$\begin{cases} \dot{x}_1 = -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 = \beta x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = -\theta x_3 + x_1 x_2 \end{cases} \tag{8}$$

where $(\sigma, \theta) = (10, 8/3)$, but, β is unknown. Suppose that only the measurement of x_2 is available, that is, $y = x_2$. To embed the unknown parameter in the center part, we extend the system with obvious dynamical systems

$$\dot{\beta} = 0 \text{ with } \beta(0) : \text{unknown}, \quad \dot{\alpha} = 0 \text{ with } \alpha(0) = 1 (\text{known}).$$

Then, we have a 5-dimensional system with a measurement

$$\begin{cases} \dot{x}_1 = -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 = -x_1 x_3 + \beta x_1 - \alpha x_2 \\ \dot{x}_3 = -\theta x_3 + x_1 x_2 \\ \dot{\beta} = 0 \\ \dot{\alpha} = 0 \\ y = x_2. \end{cases}$$

The linearization of the system is

$$\begin{pmatrix} -\sigma & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the system has 2-dimensional stable part and 3-dimensional center. Note that nonlinear terms are $x_1(\beta - x_3) - \alpha x_2$ in the second equation and $x_1 x_2$ in the third. The dynamical system (4) takes the form

$$\dot{w} = \begin{pmatrix} -a_1 & & \\ & -a_2 & \\ & & -a_3 \end{pmatrix} w + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_2$$

with arbitrary positive a_1, a_2 and a_3. After coordinate transformation

$$x_{c1} = \beta,\ x_{c2} = \alpha,\ x_{c3} = x_2,$$
$$x_{s1} = x_1 - x_2,\ x_{s2} = x_3,\ x_{s3} = w_1 - x_2/a_1,\ x_{s4} = w_2 - x_2/a_2,\ x_{s5} = w_3 - x_2/a_3,$$

the overall system is diagonalized, corresponding to (1), as follows

$$\dot{x}_c = 0_{3\times 3} x_c + \begin{pmatrix} 0 \\ 0 \\ N_1 \end{pmatrix}, \quad \dot{x}_s = \begin{pmatrix} -\sigma & & & & \\ & -\theta & & & \\ & & -a_1 & & \\ & & & -a_2 & \\ & & & & -a_3 \end{pmatrix} x_s + \begin{pmatrix} -N_1 \\ N_2 \\ -N_1/a_1 \\ -N_1/a_2 \\ -N_1/a_3 \end{pmatrix},$$

where,

$$N_1 = (x_{c3} + x_{s1})(x_{s1} - x_{s2}) - x_{c2} x_{c3}, \quad N_2 = (x_{c3} + x_{s1}) x_{c3}.$$

Now, the center manifold algorithm (2) is applied, with 3 iterations, to get

$$\begin{pmatrix} x_{s1} \\ x_{s2} \end{pmatrix} = T_1(x_c), \quad \begin{pmatrix} x_{s3} \\ x_{s4} \\ x_{s5} \end{pmatrix} = T_2(x_c).$$

The observer (7) is constructed with

$$y = x_{c3}, \quad h(\hat{x}_c, T_1(\hat{x}_c)) = \hat{x}_{c3}, \quad A_c = 0_{3\times 3}, \quad f_c = \begin{pmatrix} 0 & 0 & N_1 \end{pmatrix}^T.$$

Computer simulations show that the proposed method estimates parameter β well. Fig. 1 depicts the time responses for initial condition $\hat{\beta}(0) = 30$, while Fig. 2 depicts those for initial condition $\hat{\beta}(0) = 20$. The real value of β is $\beta = 31$ for which the system (8) exhibits chaotic behavior. In this formulation, $\hat{x}_{c2} = \hat{\alpha}$ also converge to the real value 1. However, since $\alpha = 1$ is known, it is possible to include this knowledge in the observer to get better estimation performance for β.

References

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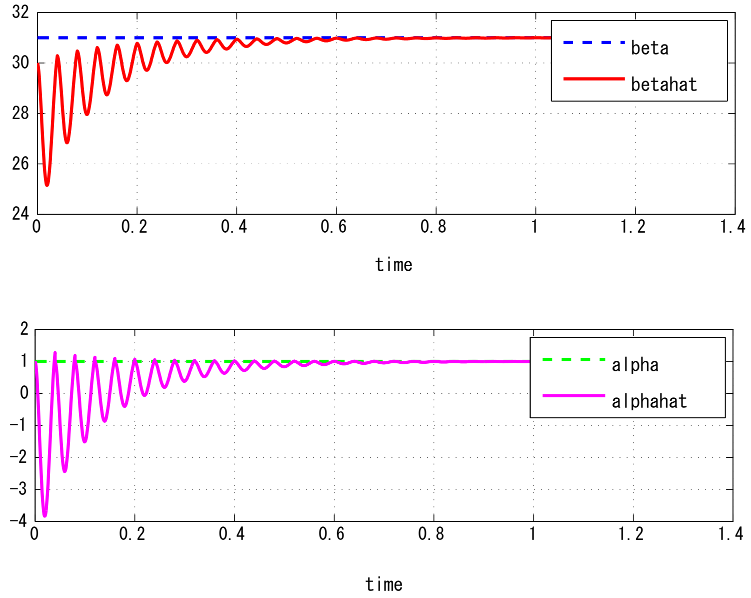

Figure 1: $\beta = 31, \hat{\beta}(0) = 30$

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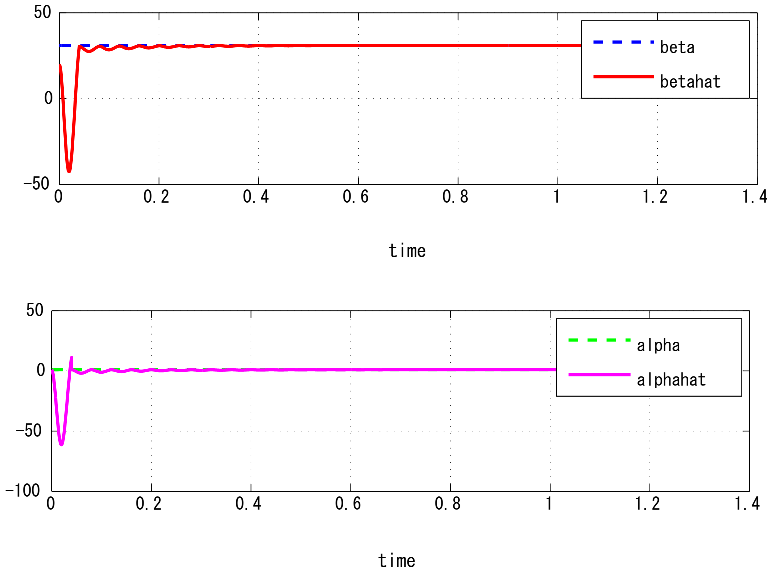

Figure 2: $\beta = 31, \hat{\beta}(0) = 20$